Dehn Twist on $K3 \# K3$.

Today: Use BF to detect exotic diffeomorphisms

$$f : X \to X, \qquad f \underset{\text{DIFF}}{\overset{\text{Top}}{\simeq \not\simeq}} id.$$

Ruberman '90's: parametrized/families gauge theory.

$$(X, \mathfrak{s}, g) \overset{f}{\longmapsto} (X, (f^{-1})^* \mathfrak{s}, f_* g)$$

Suppose
- $f^*(\mathfrak{s}) \cong \mathfrak{s}$
- $\dim(\mathcal{M}(X, \mathfrak{s}, g)) = -1$.

Pick a path of metrics

$$g \xrightarrow{g_t} f_* g.$$

$$\mathcal{M}(X, \mathfrak{s}, g, f) := \bigcup_{t \in [0,1]} \mathcal{M}(X, \mathfrak{s}, g_t)$$

ind. of g

signed count of points

$= 0$

$\Longleftarrow \; f \sim_{\text{DIFF}} id$

FBF: Let $f: X \to X$ be as above

$M_f :=$ mapping torus of f

$$\begin{array}{ccc} X & \hookrightarrow & M_f \\ & & \downarrow \pi \\ & & S^1 \end{array}$$

Obsv: $M_f \cong M_{id}$ as bundles over S^1

$\Longleftrightarrow f \sim_{DIFF} id$.

Refinement for SW/BF setup:

Def:

$\tilde{\mathfrak{s}} :=$ fiberwise $spin^c$ structure on M_f

$$\left(\tilde{\mathfrak{s}}|_{\pi^{-1}(b)} \cong \mathfrak{s} \quad \forall\, b \in S^1 \right)$$

Obsv': $f \sim_{DIFF} id \Longleftrightarrow \exists$ a bijection

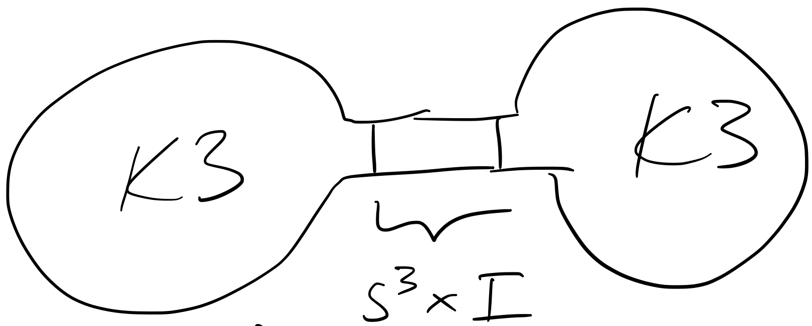
$$\text{f.w. } Spin^c(M_f) \xrightarrow[\cong]{g} \text{f.w. } Spin^c(M_{id}) \text{ s.t.}$$

Idea: Construct BF invariants for

$$\begin{array}{ccc} (X, \mathfrak{s}) & \hookrightarrow & (M_f, \tilde{\mathfrak{s}}) \\ & & \downarrow \\ & & S^1 \end{array}$$

KM '20: Consider $X = K3 \# K3$.

Dehn Twist along connect sum sphere:



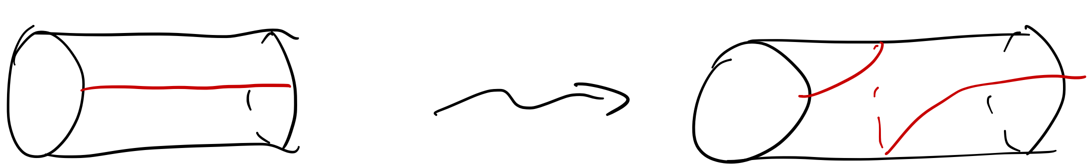
$[\varphi] = 1 \in \pi_1(SO(4)) \cong \mathbb{Z}/2$

$$S^1 \xrightarrow{\varphi} SO(4) \longrightarrow \mathrm{Diff}^+(S^3)$$

with the composite $S^1 \to \mathrm{Diff}^+(S^3)$ also labelled φ.

$$\tilde{\delta} : S^3 \times I \longrightarrow S^3 \times I$$

$$(x, t) \longmapsto (\varphi(t)(x), t)$$



$\delta : X \to X$ extend $\tilde{\delta}$ by id outside $S^3 \times I$.

Thm. (K-M '20)

$$\delta \underset{SM}{\overset{TOP}{\underset{\not\simeq}{\simeq}}} id$$

follows from

Thm. Quinn-Perron

X^4 s.c.: $\delta \overset{TOP}{\sim} id$

$\iff \delta \curvearrowright H_*(X)$ trivially.

Thm (Lin '20)

$$\delta \# id_{S^2 \times S^2} \underset{SM}{\overset{TOP}{\underset{\not\simeq}{\simeq}}} id$$

on $K3 \# K3 \# S^2 \times S^2$

Families Monopole Map.

$f : X^4 \circlearrowleft \quad \rightsquigarrow \quad X \hookrightarrow M_f \twoheadrightarrow S^1$

equipped w/ f.w. spin structure

$$\mathrm{sw}: \begin{tikzcd} \mathcal{U} \oplus \mathcal{V} \arrow[rr] \arrow[dr] & & \mathcal{U}' \oplus \mathcal{V}' \arrow[dl] \\ & S^1 & \end{tikzcd}$$

Hilbert bundles $/S^1$

f.d. approx:

$$F: \begin{tikzcd} U \oplus V \arrow[rr] \arrow[dr] & & U' \oplus V' \arrow[dl] \\ & S^1 & \end{tikzcd}$$

f.d. vector bundles $/S^1$

Kuiper's Thm:

$$\begin{tikzcd} F: \underline{\widetilde{\mathbb{R}}}^M \oplus \underline{\mathbb{H}}^N \oplus \mathrm{ind}_{\mathbb{H}}(\cancel{D}_M^+) \arrow[rr] \arrow[dr] & & \underline{\widetilde{\mathbb{R}}}^M \oplus H_+^2(M) \oplus \underline{\mathbb{H}}^N \arrow[dl] \\ & S^1 & \end{tikzcd}$$

ind bundle of $\cancel{D}_X^+$ as a family of operators over S^1

bundle of $\mathcal{H}_+^2(X)$ as a family of v.s.'s over S^1.

- $H_+^2(M) = \underline{\widetilde{\mathbb{R}}}^{b_2^+(X)}$ trivial bundle since $f_*: H_+^2(X) \circlearrowleft = id$
- $\mathrm{ind}_{\mathbb{H}}(\cancel{D}_M^+) = \underline{\mathbb{H}}^{\mathrm{ind}(\cancel{D}_X^+)} = \underline{\mathbb{H}}^{\frac{-\sigma(X)}{16}}$ trivial bundle since $\pi_1(Sp(N)) = 1$ for $N \gg 0$

$$\rightsquigarrow \begin{tikzcd} F: \underline{\widetilde{\mathbb{R}}}^M \oplus \underline{\mathbb{H}}^{N - \frac{\sigma(X)}{16}} \arrow[rr] \arrow[dr] & & \underline{\widetilde{\mathbb{R}}}^{M + b_+^2(X)} \oplus \underline{\mathbb{H}}^N \arrow[dl] \\ & S^1 & \end{tikzcd}$$

one-pt compact.:

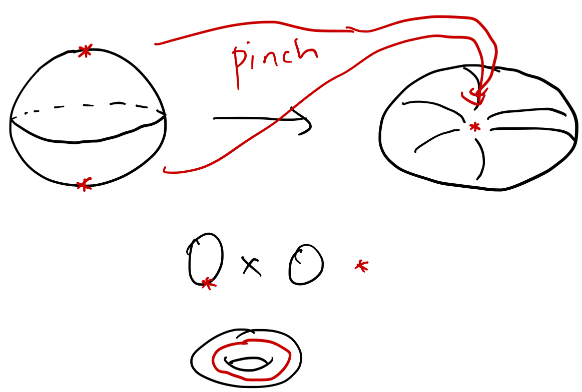
$$F: S^{M\tilde{\mathbb{R}}+(N-\frac{\sigma(X)}{16})\mathbb{H}} \wedge S^1_+ \longrightarrow S^{(M+b_+^2(X))\tilde{\mathbb{R}}+N\mathbb{H}} \wedge S^1_+$$

- We have a canonical stable map

$$S^{\mathbb{R}+M\tilde{\mathbb{R}}+(N-\frac{\sigma(X)}{16})\mathbb{H}} \xrightarrow{P} S^{M\tilde{\mathbb{R}}+(N-\frac{\sigma(X)}{16})\mathbb{H}} \wedge S^1_+$$

induced stably by the map

$$S^{2\mathbb{R}} \longrightarrow S^{\mathbb{R}} \wedge S^1_+$$



$\rightsquigarrow FBF^{Pin(2)}(X,f)$ is given by the composition

$$S^{\mathbb{R}+M\tilde{\mathbb{R}}+(N-\frac{\sigma(X)}{16})\mathbb{H}} \xrightarrow{P} S^{M\tilde{\mathbb{R}}+(N-\frac{\sigma(X)}{16})\mathbb{H}} \wedge S^1_+$$

$$\xrightarrow{F} S^{(M+b_+^2(X))\tilde{\mathbb{R}}+N\mathbb{H}} \wedge S^1_+ \xrightarrow{pj.} S^{(M+b_+^2(X))\tilde{\mathbb{R}}+N\mathbb{H}}$$

$$\rightsquigarrow FBF^{Pin(2)}(X,f) \in \left\{ S^{\mathbb{R} - \frac{\sigma(X)}{16}\mathbb{H}},\ S^{b_2^+(X)\widetilde{\mathbb{R}}} \right\}.$$