

Let $X = \bigsqcup_{i=1}^n X_i$ where each X_i is closed, connected, & has a spin^c structure

Furthermore, choose separating necks $N(L)_i \cong [-L, L] \times S^3$ s.t. $X_i = X_i^- \cup X_i^+$ and $\partial X_i^\pm \cong \{0\} \times S^3$

For $\tau \in A_n$, let $X^\tau = \bigsqcup_{i=1}^n X_i^- \cup X_{\tau(i)}^+$

For a given spin^c -structure ξ on X we get a spinor bundle $W = W^+ \oplus W^-$ and a line bundle $\det(\xi)$

~~For any spin~~ Let A represent a flat ~~spin~~ connection on L that induces ∇_A on W

Fix all above choices

The gauge group is $\mathcal{G} = C^\infty(X, S^1)$, let $\mathcal{G}_r \subset \mathcal{G}$ be the subgroup of trivial transformations over $N(1) \subset N(L) \cong [-L, L] \times \prod_i S^3$

$\mathcal{G}_r$ then splits as a product of groups for every $X_i^\pm$

The orbit of the $\mathcal{G}_r$ -action on spin^c -connections is then $\cong A + i \ker d_r$ where $\ker d_r \subset \ker d$ s.t. the form vanishes on $N(1)$

Notice that $A + i \ker d_r \cong A^\tau + i \ker d_r$ and so after Sobolev completion

$A + i \ker d_r / \mathcal{G}_r$ identifies w/ $\text{Pic}^0(X)$ and $\text{Pic}^0(X^\tau)$

Define, $\mathcal{A} := (A + i \ker d_r) \times (\Gamma(W^+) \oplus \Omega^1(X)) / \mathcal{G}_r$

$\mathcal{C} := (A + i \ker d_r) \times (\Gamma(W^-) \oplus \Omega_+^2(X) \oplus \mathcal{H}^1(X; \mathbb{R}) \oplus \Omega^0(X) / \mathbb{R}) / \mathcal{G}_r$

and $\mu := \tilde{\mu} / \mathcal{G}_0 : \mathcal{A} \rightarrow \mathcal{C} \quad (A, \psi, a) \mapsto (A, D_{A+ia} \psi, F_{A+ia}^+ - q(\psi), a_0, d^* a)$ as before

We would like to compare $\mu : \mathcal{A} \rightarrow \mathcal{C}$ and $\mu^\tau : \mathcal{A}^\tau \rightarrow \mathcal{C}^\tau$

To this end, define $R : [0, 1] \rightarrow SO(n)$ s.t. $R_0 = \text{id}$ $R_1 = (\delta_{ij} \tau(j))_{ij}$

and $\sigma : [-L, L] \times S^3 \rightarrow [0, 1]$ to be 0 $[-L, -1]$, 1 $[1, L]$, and sigmoid w/no S^3 dependence

If $E \rightarrow X$ is a bundle w/ restrictions $E_i^\pm$, build $E^\tau \rightarrow X^\tau$ by identifying $E_{N(L)_i} \cong F$ $\downarrow$ $N(L)_i : [-L, L] \times S^3$ $\downarrow$

which identifies the n -sections of E w/ $\bigoplus F$ as a vector $\vec{e} = [e_1, \dots, e_n]^T$

leave E unchanged in $X \setminus N(L)$ and let $\vec{e}^\tau = (R \circ \sigma) \vec{e}$ on $N(L)$

Doing this gives a linear map $V : (\psi, a) \mapsto (\psi^\tau, a^\tau)$

So we'd hope that $\mathcal{A} \xrightarrow{\mu} \mathcal{C}$ commutes

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mu} & \mathcal{C} \\ \downarrow V & & \downarrow V \\ \mathcal{A}^\tau & \xrightarrow{\mu^\tau} & \mathcal{C}^\tau \end{array}$$

To find ~~such~~ a homotopy $\mu_t = l_t + c_t: A \rightarrow \mathcal{C}$ such that $\mu_0 = \mu$ & $\mu_1 = V^{-1} \mu^\tau V$ is then the goal. However, we don't need the boundedness condition anymore $\forall t$, instead we just require that there is a bounded disk bundle that contains the solutions at $t=0,1$ & no solution escapes for $t \in [0,1]$.

Let's try to get some bounds: let (ϕ, a) be such that $\mu(A, \phi, a) = 0$ for fixed A , $A' = A + a$.

$$1) \text{ As before } \Delta_M |\Psi|^2 = 2 \langle \nabla_{A'}^* \nabla_{A'} \Psi, \Psi \rangle - 2 \langle \nabla_{A'} \Psi, \nabla_{A'} \Psi \rangle \\ \leq 2 \langle \nabla_{A'}^* \nabla_{A'} \Psi - \frac{s}{4} \Psi + \frac{1}{2} F_{A'}^+ \Psi, \Psi \rangle, \quad \nabla_{A'} \Psi = 0, (F_{A'}^+) = i q(\Psi)$$

And $\Delta_M |\Psi|^2 \geq 0$ at maximum so $0 \leq -\frac{s}{2} |\Psi|^2 + i \langle i \langle \Psi, \Psi \rangle \Psi - i/2 |\Psi|^2 \Psi, \Psi \rangle$
 $0 \geq \frac{s}{2} |\Psi|^2 + \frac{1}{2} |\Psi|^4$ or simply $|\Psi|^2 \leq \max(0, s)$

2) Take Sobolev & elliptic estimates ($p > 4$)

$$\|a\|_{L^\infty} \leq C_s \|a\|_{L^p_1} \leq C_e (\|d^* a\|_{L^p_0} + \|d^+ a\|_{L^p_0} + \|a_0\|_{L^p_0})$$

Which along w/ the identity that $d^+ a = i q(\Psi) \frac{\rho(F_{A'}^+)}{\rho(F_A^+)}$, A not A' , gives

$$\|a\|_{L^\infty} \leq C_s C_e (\|F_A^+\|_{L^p_0} + \|q(\Psi)\|_{L^p_0}) \text{ since } d^* a = 0 = a_0 \text{ by } \mu = 0$$

3) Recall L^2_k -norm constructed by $(-, -)_i := (-, \epsilon)_0 + (\nabla_A + d^+, \nabla_A + d^+)_{i-1}$ inductively

$$\text{Thus, } \|(\Psi, a)\|_{L^p_i}^p = \|(\Psi, a)\|_{L^p_0}^p = \|(\nabla_A \Psi, d^+ a)\|_{L^p_{i-1}}^p$$

We now try to gain control of these bounds

For $x \in X$ we may choose β_x bump functions w/ norms bounded by M and let C be some Sobolev estimate that works for a given L . Then,

$$\|a\|_{L^\infty} = \text{ess sup } \|\beta_x a\| \leq \text{ess sup } C \|\beta_x a\|_{L^p} \leq MC \|a\|_{L^p} \Rightarrow MC = C_s \text{ independent of } L$$

To gain control of C_s we need to attach tubular ends as in "Floer Homology Groups in Yang-Mills Theory" Donaldson

Outline:

$$1) X = X^- \cup X^+ \rightsquigarrow Y^- = X^- \cup [0, \infty) \times S^3 \quad Y^+ = X^+ \cup (-\infty, 0]$$

2) Weight functions $f_\alpha^\pm$ that vanish outside $X^\pm \setminus N(L) \subset Y^\pm$ and grows ~~exponentially~~ ^{w/ slope α} towards end

$$3) \text{ Weighted Sobolev spaces } \|g\|_{L_k^{p,\alpha}(Y^\pm)} := \|e^{f_\alpha^\pm} g\|_{L_k^p(Y^\pm)}$$

4) Identify $\Omega^1(\mathbb{R} \times S^3) \cong \Omega^0(\mathbb{R} \times S^3) \oplus \Omega^2_+(\mathbb{R} \times S^3)$ so that $d^* + d^+ \mapsto \frac{\partial}{\partial t} + L$, $L = \begin{pmatrix} 0 & d^+ \\ d & 0 \end{pmatrix}$ elliptic on S^3

5) $\frac{\partial}{\partial t} + L$ is Fredholm $L_1^{p,\alpha}(Y^\pm) \rightarrow L_0^{p,\alpha}(Y^\pm)$ if α not in spectrum of L

and if $\alpha < 0$ but greater than maximal negative eigenvalue of L then $\ker \cong H^1(Y^\pm; \mathbb{R})$ & $\text{coker} \cong H^0(Y^\pm; \mathbb{R}) \oplus H^2_+(Y^\pm; \mathbb{R})$

$$6) \text{ Apply estimate } \|b\|_{L_1^{p,\alpha}(Y^\pm)} \leq C^\pm \|(d^* + d^+)b\|_{L_0^{p,\alpha}(Y^\pm)} + \tilde{C} \|b\|_{L_0^{p,\alpha}} \text{ for}$$

compactly supported forms $b = \phi_\pm a$ on $Y^\pm$ from a on X $\phi_\pm: 0 \rightarrow 1$ on $N(L-1)$ $1 \rightarrow 0$

Important fact, you may always choose α such that it avoids the spectrum & where it lies to get the isomorphism

$$\text{Altogether, } \|a\|_{L^\infty} \leq MC \max(C^\pm) (\| (d^* + d^+) \phi_+ a \|_{L_0^p} + \| (d^* + d^+) \phi_- a \|_{L_0^p}) + MC \tilde{C} \|a\|_{L_0^p} = C$$

since $\|b\| = \|a\|$ on $X \setminus N(L-1)$ and on $N(L-1)$ bounded w/ $\max(C^\pm)$

The first homotopy: $\mu_t: \mathcal{A} \rightarrow \mathcal{C}$ $\mu_t(A, \Psi, a) = (A, D_A \Psi, F_A^+ - \rho_{L,t} q(\Psi), a_0, d^* a)$

where $\rho_{L,t} := (1-t) + t\rho_L$ and ρ_L vanishes on $N(R-1)$ independent of S^3

Lemma

The preimage $\mu_t^{-1}(0)$ is bounded

$s|\Psi|^2 + |\Psi|^4 \leq 0 \leadsto s|\Psi|^2 + \rho_{L,t} |\Psi|^4 \leq 0 \Rightarrow |\Psi|^2$ is bounded by s outside $N(L)$ where it is ≥ 0

and $\|\rho_{L,t} q\| \leq \|q\|$ so nothing else breaks

The 2nd homotopy: $\mu_{t+1}(A, \Psi, a) = (A, D_{A+\rho_{2,t}a} \Psi, F_{A+\rho_{2,t}a}^+ - \rho_L q(\Psi), a_0, d^* a)$

Notice that μ_2 is linear on $N(1)$

To bound (Ψ, a) notice that $|\Psi|^2 \leq \max(0, -s) = S$ is no issue but C_e from

$\|a\|_{C^0} \leq C_s C_e (\|F_A^+\|_{L^6} + \|\rho_L q(\Psi)\|_{L^6}) \leq C'_s C'_e (\|F_A^+\|_{L^\infty} + S) \text{Vol}(X')$ $F_A = 0$ on $N(L)$ as does ρ_L on $N(L-1)$

This is bad since it depends on global geometry of X

Lemma

$\exists$ constants U & L_0 such that $\forall$ solution (Ψ, a) to μ_1 on $X(L)$ w/ neck $L \geq L_0$, the 1-form a satisfies $\|a\|_{L^\infty} \leq U$

To see this note 1) a is harmonic on $N(L-1)$ 2) Ricci curvature nonnegative along neck $\xrightarrow{\text{together}}$ Maximum principle

Now let $\phi_\pm$ be as before w/ roughly constant slope $|d\phi_\pm| < L^{-1}$

Estimate $\|a\|_{L^\infty(X')} \leq C_1 (\|(d^* + d^+) \phi_+ a\|_{L^6_0} + \|(d^* + d^+) \phi_- a\|_{L^6_0}) + C \|a_0\|_{L^6_0}$ but $\|a_0\| = 0$

~~so that $\|a\|_{L^\infty(X')} \leq C_1 (\|(d^* + d^+) \phi_\pm a\|_{L^6_0} + \|(d^* + d^+) \phi_\pm a\|_{L^6_0}) + C \|a_0\|_{L^6_0}$~~ and $\|(d^* + d^+) \phi_\pm a\|_{L^6_0} \leq \text{notice change order} \|\phi_\pm (d^* + d^+) a\|_{L^6_0} + \|d\phi_\pm\|_{L^6_0} \|a\|_{L^\infty(N(L-1))}$

by Maximum principle, $\|a\|_{L^\infty(N(L-1))} \leq \|a\|_{L^\infty(X')} = \|a\|_{L^\infty(X)}$

$\Rightarrow \|a\|_{L^\infty(X)} \leq C_1 (\|F_A^+\|_{L^\infty} + S) \text{Vol}(X') + C_1 L^{\frac{1}{p}-1} \text{Vol}(S^3) \|a\|_{L^\infty(X)}$

So we solve for $L_0 = [2C_1 \text{Vol}(S^3)]^{4/3}$ $U = 2C_1 (\|F_A^+\|_{L^\infty} + S) \text{Vol}(X')$

Lemma 3.4

If $L_1 \geq L_0$ then $\forall (\psi, a) \in \mu_t^{-1}(0)$ $t \in [1, 2]$ such that $|\psi|^2 \leq 2S$ and $|a| \leq 2U$ also satisfies $|\psi|^2 \leq S$ & $|a| \leq U$

a again harmonic on $N(L-1)$ and splits into $a = a_i + a_s$ by splitting of $T^*N(L-1)$

Sphere component is such that $\Delta |a_s|^2 \leq -2 \langle \text{Ric}(a_s), a_s \rangle \leq -\delta^2 |a_s|^2$ $\delta > 0$

Define $\alpha = \sum |a_{s,i}|^2$ so $\frac{d^2 \alpha}{dt^2} \geq -\Delta_X \alpha \geq \delta \alpha$ & $|a_s|^2 \leq \frac{4_n U^2 \cosh(\delta t)}{\cosh(\delta(L-1))} \rightarrow$ 

so $|a_s|$ decays exponentially toward middle

Next, if $|\psi|^2$ attains max outside $N(L-1)$ we're done so assume it doesn't, can it?

$$\Delta |\psi|^2 \leq -\frac{s}{2} |\psi|^2 + \langle (dp_{2,t} \wedge a)^+ \psi, \psi \rangle \text{ but } dp_{2,t} \wedge a = dp_{2,t} \wedge a_s$$

so we may stretch $N(L)$ to make $-\frac{s}{2}$ term dominate and $|\psi|^2 \stackrel{=0}{\text{on } N(L-1)}$

for bounding $|a|$ do what we just did since $L_1 \geq L_0$

From here, note the two homotopies get us $\mu \simeq P$ and $V^{-1} \mu^\pi V \simeq V^{-1} P^\pi V$