Some structural properties:

From now on supp. $b_1(X) = 0$.

Universe $\mathcal{U} = \mathbb{R}^\infty \oplus \mathbb{C}^\infty$

Observation 1:

$$BF^{S^1}(X) \in \pi^{b_2^+(X)}_{S^1,\mathcal{U},st}(\mathrm{Pic}^0(X); \mathrm{ind}(D)) = \pi^{b_2^+(X)}_{S^1,\mathcal{U},st}(Th(\mathrm{ind}(D)))$$

$$\cong \pi^{b_2^+(X)}_{S^1,\mathcal{U},st}(S^{\mathrm{ind}(\not{D}_X^+)\mathbb{C}})$$

$$\cong \{S^{\mathrm{ind}(\not{D}_X^+)\mathbb{C}}, S^{b_2^+(X)\mathbb{R}}\}_{S^1,\mathcal{U}}$$

$$\overset{\text{Atiyah-Singer}}{=} \{S^{\frac{c_1(\mathfrak{s})^2-\sigma(X)}{8}\mathbb{C}}, S^{b_2^+(X)\mathbb{R}}\}_{S^1,\mathcal{U}}$$

$\Rightarrow BF^{S^1}(X)$ is represented by some map

$$(*)\quad f: S^{M\mathbb{R}} \wedge S^{(N+\frac{c_1(\mathfrak{s})^2-\sigma(X)}{8})\mathbb{C}} \longrightarrow S^{(M+b_2^+(X))\mathbb{R}} \oplus S^{N\mathbb{C}}$$

Observation 2:

Given f as in $(*)$, the map on S^1-fixed pts

$$f^{S^1}: S^{M\mathbb{R}} \longrightarrow S^{(M+b_2^+(X))\mathbb{R}}$$

is homotopic to the inclusion map

$$S^M \approx S^M \wedge S^0 \hookrightarrow S^M \wedge S^{b_2^+(X)} \approx S^{M+b_2^+(X)}$$

$$\Rightarrow f^{S^1} \simeq \begin{cases} \mathrm{id} & \text{if } b_2^+(X) = 0, \\ 0 & \text{if } b_2^+(X) \geq 1. \end{cases}$$

<u>Pf</u>. The monopole map restricted to S^1-fixed pts (forms) is given by

$$[A', a, 0] \longmapsto [A', \not{D}^+_{A'+a}(0), F^+_{A'} + d^+a - \sigma(0), pr(a), d^*a]$$

$$= [A', 0, F^+_{A'} + d^+a, pr(a), d^*a]$$

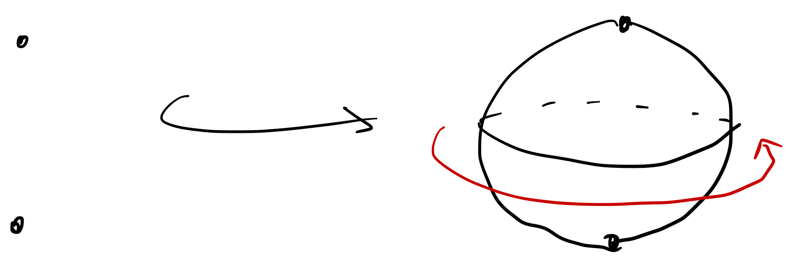
Prop. Let X be neg. def. Then

$$d := \operatorname{ind}(\not{D}_X^+) = \frac{c_1(\mathfrak{s})^2 - \sigma(X)}{8} \leq 0,$$

and $BF^{S^1}(X)$ is represented by the map

$$f : S^0 \longrightarrow S^{d\mathbb{C}}$$

obtained as the d-fold smash product of the inclusion $S^0 \hookrightarrow S^{\mathbb{C}}$:



Pf. We have that $BF^{S^1}(X)$ is repr. by

$$f : S^{M\mathbb{R}} \wedge S^{\left(N + \frac{c_1(\mathfrak{s})^2 - \sigma(X)}{8}\right)\mathbb{C}} \longrightarrow S^{M\mathbb{R}} \wedge S^{N\mathbb{C}}$$

for some $M, N \geq 0$.

Lemma. Let $f : S^{a\mathbb{R}} \wedge S^{b\mathbb{C}} \longrightarrow S^{a\mathbb{R}} \wedge S^{c\mathbb{C}}$ be s.t. $f^{S^1} \simeq id$. Then $b \leq c$ and f is homotopic to the inclusion

$$S^{a\mathbb{R}} \wedge S^{b\mathbb{C}} \approx S^{a\mathbb{R}} \wedge S^{b\mathbb{C}} \wedge S^0$$
$$\hookrightarrow S^{a\mathbb{R}} \wedge S^{b\mathbb{C}} \wedge S^{c-b\mathbb{C}} \approx S^{a\mathbb{R}} \wedge S^{c\mathbb{C}}.$$

Pf. Consider the following diagram:

$$\begin{tikzcd}
S^{a\mathbb{R}} \wedge S^{b\mathbb{C}} \arrow[r, "f"] & S^{a\mathbb{R}} \wedge S^{c\mathbb{C}} \\
S^{a\mathbb{R}} \arrow[u, hook, "\iota_1"] \arrow[r, "f^{S^1} = id"] & S^{a\mathbb{R}} \arrow[u, hook, "\iota_2"']
\end{tikzcd}$$

Induced map on red. S^1-equiv. top. K-theory:

$$\begin{tikzcd}
\widetilde{K}_{S^1}(S^{a\mathbb{R}} \wedge S^{b\mathbb{C}}) \arrow[d, "\iota_1^*"] & \widetilde{K}_{S^1}(S^{a\mathbb{R}} \wedge S^{c\mathbb{C}}) \arrow[l, "f^*"'] \arrow[d, "\iota_2^*"] \\
\widetilde{K}_{S^1}(S^{a\mathbb{R}}) & \widetilde{K}_{S^1}(S^{a\mathbb{R}}) \arrow[l, "(f^{S^1})^* = id"']
\end{tikzcd}$$

all maps are $R(S^1)$-module homomorphisms.

<u>Fact</u>: (equiv. Bott periodicity)

Let V be a complex G-rep. then

$$\widetilde{K}_G(X) \xrightarrow{\cong} \widetilde{K}_G(\Sigma^V X) \longrightarrow \widetilde{K}_G(X)$$

is given by multiplication with

$$\lambda_{-1}(V) := \sum_{d=0}^{\dim(V)} (-1)^d [\Lambda^i V] \in R(G)$$

$$\Rightarrow \quad (1-\theta)^c x = (1-\theta)^b \cdot f^*(x)$$

Cor. (Donaldson's Diagonalization Thm).

Any closed smooth 4-mfld w def. intersection form is diagonalizable.

Pf. Can assume X is neg. def.

Prop $\Rightarrow$ $c_1(\mathfrak{s})^2 \leq \sigma(X)$.

Apply Elkies' Thm.