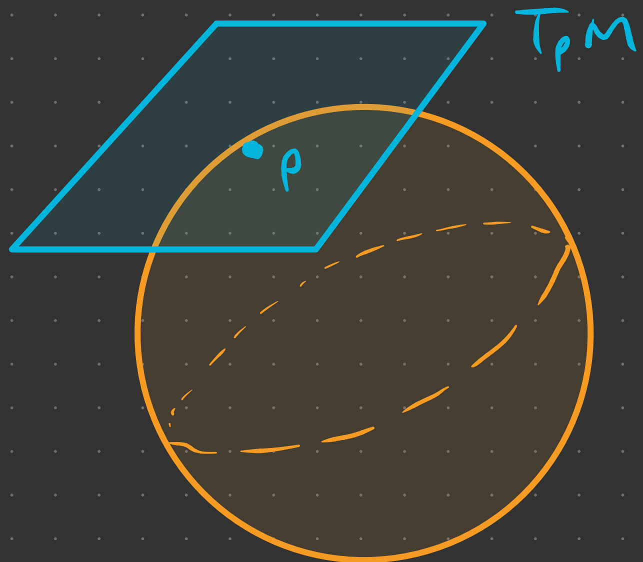
# almost no* spheres are PARALLELIZABLE

* almost = all but 4




# Outline

1. Pushing my veggies around
2. Def's + Tools
3. A MAJOR KEY $^{\sigma\pi}$ lemma

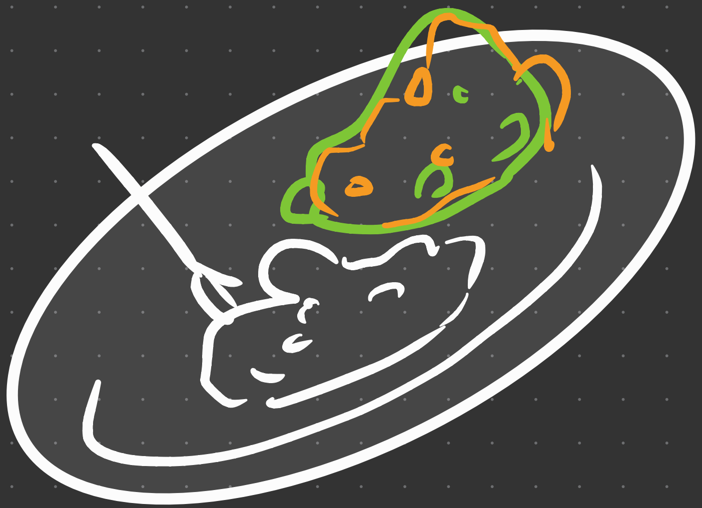
\+ wild punchline

CASANDRA MONROE | ⌜K-Theory Sem⌟

10/10/23

# 7 Pushing my veggies around

(aka vamping)



# GOAL

We want to show that the only parallelizable spheres are

$$S^0,\ S^1,\ S^3,\ \&\ S^7$$

# That is...

almost no spheres are parallelizable

$n \in \mathbb{N}$ $\longrightarrow$

$S^0 \; S^1 \; S^2 \; S^3 \; S^4 \; S^5 \; S^6 \; S^7 \; S^8 \; S^9 \; S^{10} \; S^{11} \; S^{12} \; S^{13} \; S^{14} \; S^{15} \; S^{16} \; S^{17} \; S^{18} \; S^{19} \; S^{20} \ldots$

This is it

Recall:

**Def$^n$**: A mfld $M$ is said to be **parallelizable**

- if $TM$ is a trivial bundle

(OR equiv.)

- if $\exists$ $n$ smooth vector fields $\{V_1, \dots, V_n\}$ on $M$, $n = \dim M$, s.t. $\forall p \in M$

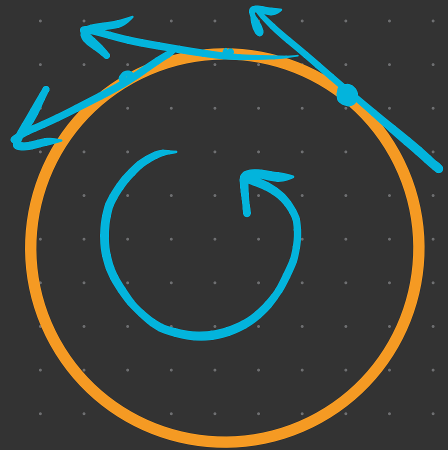
$\{V_1(p), \dots, V_n(p)\}$ are a basis for $T_pM$

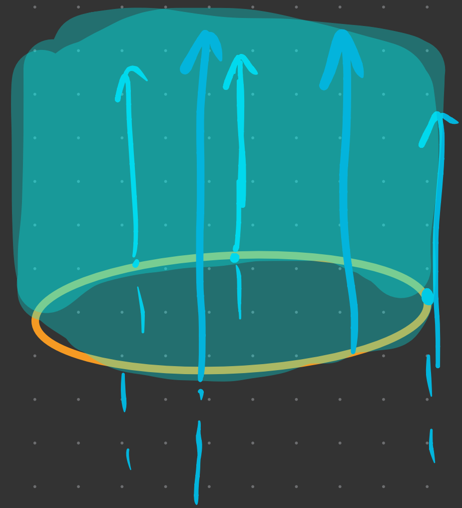
1st Quest: Why are these parallelizable?


$$S^0 = \cdot \qquad \cdot \qquad \boxed{\checkmark}$$

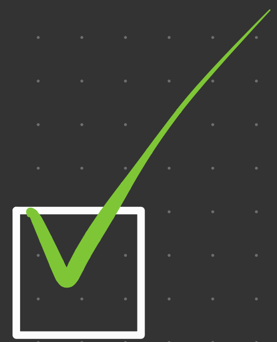
$S^1 =$ 

$TS^1 \cong S^1 \times \mathbb{R}$

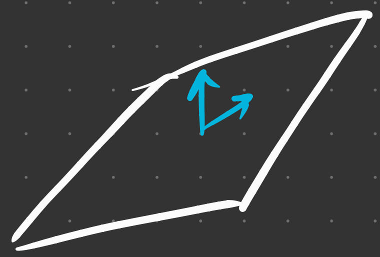


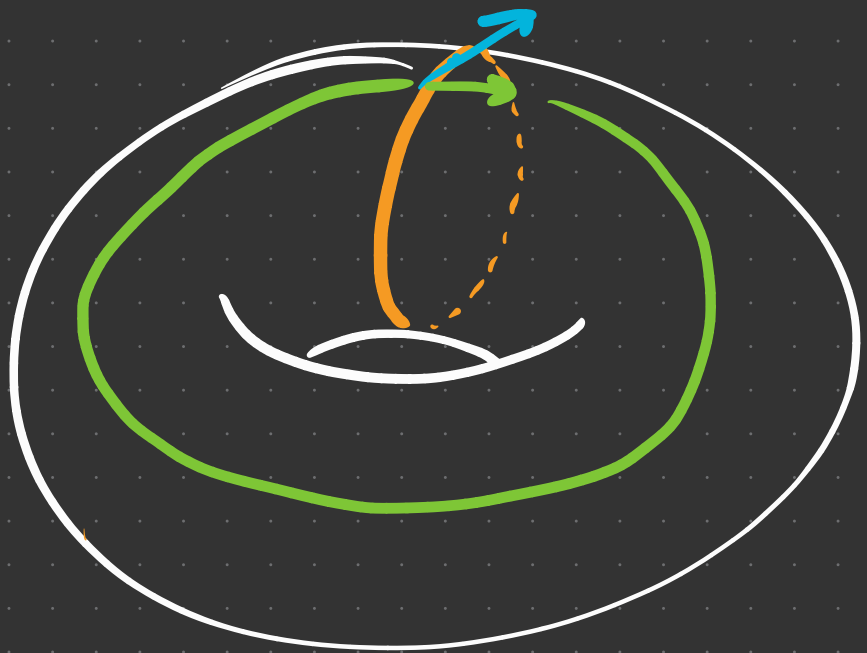


note!



# Can we extend this reasoning?





yes :)

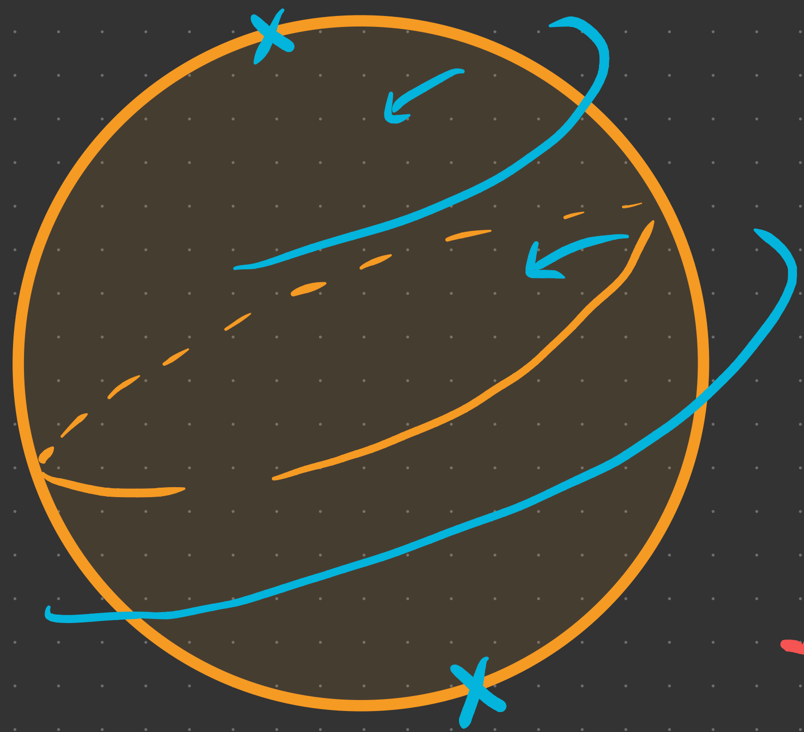
for $T^2 = S^1 \times S^1$

Two ways to see it:

① $S^1 \times S^1$ is parallelizable b/c the factors are

② $\exists$ a nonvanishing vector field and some notion of "group transport" at play

# Can we extend this reasoning?



No ☹

for $S^2$

due to

♥ Poincare-Hopf ♥ (Best Thm)

$\Rightarrow$ P.H. tells us that $\chi(M) = \#$ zeroes in a vector field

$\Rightarrow$ Need $\chi(M) = 0$ to even have a hope

That's about half of them gone!

By either P.D. or C.W.-cell computations

That is...

almost no spheres are parallelizable

$n \in \mathbb{N} \longrightarrow$

$S^0 \; S^1 \; \cancel{S^2} \; S^3 \; \cancel{S^4} \; S^5 \; \cancel{S^6} \; S^7 \; \cancel{S^8} \; S^9 \; \cancel{S^{10}} \; S^{11} \; \cancel{S^{12}} \; S^{13} \; \cancel{S^{14}} \cdots$

This is it

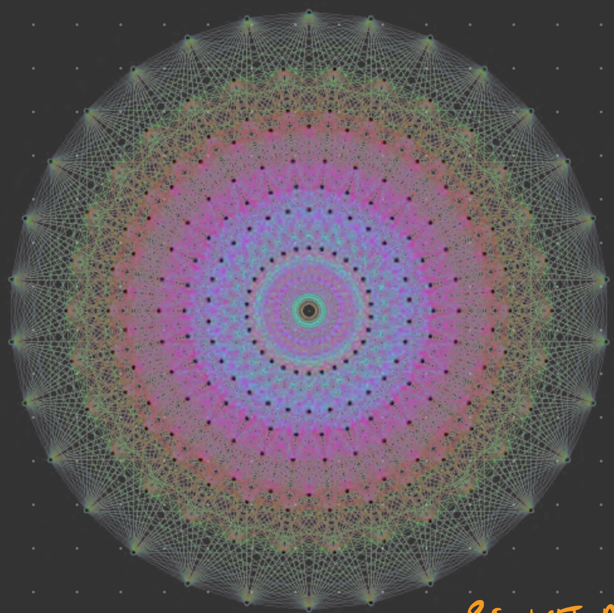
(You choose!)

But for $S^3$, we can appeal to this

$\{$ group ? $\cap$ non.van. vector field $\}$ story

bc $S^3$ ... is a LIE GROUP

lie group

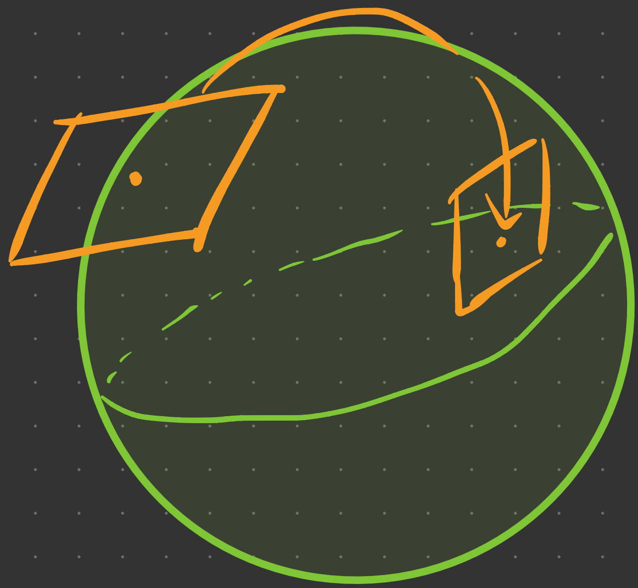


BE NOT AFRAID

In particular, $S^3 = SU(2)$ $\boxed{\checkmark}$

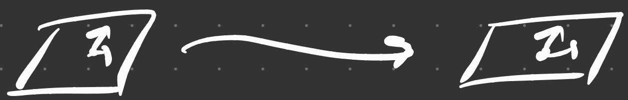
$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}$$

$$\left.\begin{aligned} &\hookrightarrow \alpha = a + bi \\ &\quad \beta = c + di \end{aligned}\right\} \Rightarrow a^2 + b^2 + c^2 + d^2 = 1 \quad \subset \mathbb{C}^2$$



RMK: Lie groups are always parallelizable

$\Rightarrow$ algebraic structure is giving you a way to move



In fact: 

= a silly lie group :)

$= U(1)$ or $SO(2)$ hook 'em!

$= SU(2)$

Can Lie Theory take it all? 

?

[spoiler: is Texas back?]


NO :(

But!

Hook 'em!!


↳ Sadly, not a Lie group...

It turns out, a different thing is still true!

+ ↳ from now on we're ignoring $S^0$

In fact: 

$= \text{unit sphere in } \mathbb{C}$

$= \text{unit sphere in } \mathbb{H}$

$= \text{unit sphere in } \mathbb{O}$

* the other slide nearly follows from this but $\mathbb{O}$ is <u>NOT</u> associative

# ② Def$^n$s & Tools

Q: What do these things all have in common?

$\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are all

$\mathbb{R}$-normed division algebras

**Def$^n$**: A **division algebra** over $\mathbb{R}^n$ is a multiplication map $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

s.t. the maps

$$x \longmapsto ax$$

$$x \longmapsto xa$$

are linear for each $a \in \mathbb{R}^n$, and invertible when $a \neq 0$

Def$^n$: division algebra over $\mathbb{R}^n$

$\Updownarrow$

$x$ having trivial kernel

$\Updownarrow$

$x$ has no zero divisors

**Def$^n$:** We say $S^{n-1}$ is an **H-Space** if there is a continuous multiplication map

$$S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$$

having a two-sided identity $e \in S^{n-1}$

**RMK:** This is weaker than being a Lie group b/c assoc. and inverses are not assumed

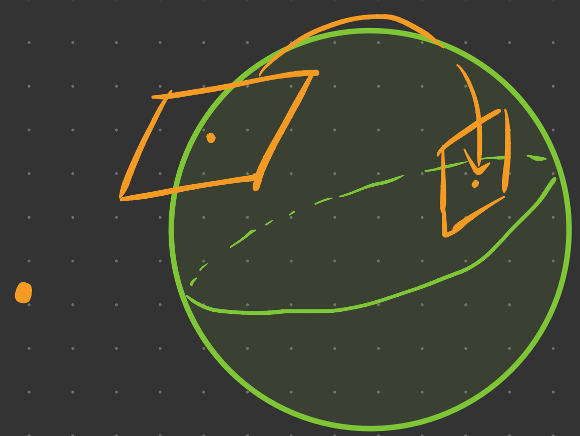
$\rightarrow$ (1) is covered now!

**Lemma**: If $\mathbb{R}^n$ is a division algebra,

or $S^{n-1}$ is parallelizable,

then $S^{n-1}$ is an H-space.

"pf": $(x,y) \longmapsto \frac{xy}{|xy|}$ keeps H-space structure b/c no zero d.v.



$\square$

(Adams)

**THM** $(\star)$: The following statements are true only for $n = 1, 2, 4$, and $8$:

- $\mathbb{R}^n$ is a division algebra
- $S^{n-1}$ is parallelizable,

Q: What do these #s have in common?

$\sigma\pi$

**Big Idea:** Gonna show $S^{n-1}$ is an H-space

iff $n = 1, 2, 4,$ or $8$

↳ So far, we know $n \neq 2k+1$, $k \geq 1$

( ↳ we used $\chi(S^{n-1})$ but we could have used $K$-theory lol )

↳ now NTS $n \neq 2k$ unless it's

$n = 2, 4,$ or $8$

First step towards proof of THM(*):

- We will associate to a map

$$g : S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}$$

a map

$$\hat{g} : S^{2n-1} \longrightarrow S^n$$

$$S^{2n-1} = \partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n$$

$$\hookrightarrow S^n = D^n_+ \cup_\partial D^n_-$$

$\square$

$$\hat{g} : S^{2n-1} \longrightarrow S^n$$

$$\hat{g}\big|_{\partial D^n \times D^n} : (x,y) \longmapsto |y|\, g\left(x, \frac{y}{|y|}\right) \in D^n_+$$

$$\big|_{D^n \times \partial D^n} : (x,y) \longmapsto |x|\, g\left(\frac{x}{|x|}, y\right) \in D^n_-$$

$$\hat{g}\big|_{S^{n-1} \times S^{n-1}} = g$$

Since $n$ is even, we replace $n$ with $2n$ throughout.

If $f: S^{4n-1} \longrightarrow S^{2n}$,

let $C_f$ be $S^{2n}$ with a cell $e^{4n}$ attached by $f$

$\Rightarrow$ then $C_f / S^{2n} = S^{4n}$ &

Think: Aru's talk

since $\tilde{K}^1(S^{4n}) = \tilde{K}^1(S^{2n}) = 0$,

the exact seq of $(C_f, S^{2n})$ becomes the SES

$$0 \longrightarrow \tilde{K}(S^{4n}) \longrightarrow \tilde{K}(C_f) \rightarrow \tilde{K}(S^{2n}) \rightarrow 0$$

let $\alpha \in \tilde{K}(C_f)$ be $\mathrm{im}\left(\left((H-1)*\cdots*(H-1)\right) \text{ of } \tilde{K}(S^{4n})\right)$

and let $\beta \in \tilde{K}(C_f) \longmapsto \left((H-1)*\cdots*(H-1)\right)$ in $\tilde{K}(S^{2n})$

$$\beta^2 \longmapsto 0$$

& exactness

$$0 \longrightarrow \tilde{K}(S^{4n}) \longrightarrow \tilde{K}(C_f) \longrightarrow \tilde{K}(S^{2n}) \longrightarrow 0$$

$$\alpha, \beta$$

$\Longrightarrow$ $\beta^2 = h\alpha$ for some $h \in \mathbb{Z}$

$h$ is called the* Hopf invariant of $f$

* [well def'd, !?]

$(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta$ since $\alpha^2 = 0$; NTS $\alpha\beta = 0$.

$\alpha \longmapsto 0$, $\alpha\beta \longmapsto 0$; $\alpha\beta = k\alpha$ for some $k$; $\underset{0}{k\alpha\beta} = \alpha\beta^2 = \alpha(h\alpha) = h\alpha^2 = 0$ $\Rightarrow \alpha\beta = 0$

**Lemma**: If $g: S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ is an H-space multiplication, then the assoc. map $\hat{g}: S^{4n-1} \to S^{2n}$ has Hopf invariant $= \pm 1$

"Sketch":

oops! almost all isos!

$$
\begin{tikzcd}
K(C_f) \otimes \tilde{K}(C_f) \arrow[r] & K(C_f) \\
K(C_f, D^{2n}) \otimes K(C_f, D^{2n}_{+}) \arrow[r] \arrow[u, "\approx"'] \arrow[d, "\Phi^* \otimes \Phi^*"'] & \tilde{K}(C_f, S^{2n}) \arrow[u] \\
\tilde{K}(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \otimes \tilde{K}(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n}) \arrow[r] \arrow[d, "\cong"'] & \tilde{K}(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \\
\tilde{K}(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}) \otimes \tilde{K}(\{e\} \times D^{2n}, \{e\} \times \partial D^{2n}) \arrow[ur, "\cong"'] &
\end{tikzcd}
$$

# A New THM

<u>THM</u> (Adams):

$\exists$ a map $f : S^{4n-1} \to S^{2n}$ of Hopf Invariant $\pm 1$

only when $n = 1, 2,$ or $4$

See also:
cross products

# Cool New Tool : We demand nicer maps on $K(X)$

forgalisions

**Def$^n$ / Thm**: $\exists$ ring homs $\psi^k : K(X) \to K(X)$ $\forall k \geq 0$ satisfying

$X$ compact Hdf

s.t.

(1) $\psi^k f^* = f^* \psi^k$ $\forall f : X \to Y$

(2) $\psi^k(L) = L^k$ if $L$ is a line bundle

(3) $\psi^k \circ \psi^l = \psi^{kl}$

Adam's ops

(4) $\psi^p(\alpha) \equiv \alpha^p \mod p$, $p$ prime

$\hookrightarrow$ $\psi^p(\alpha) - \alpha^p = p\beta$ for some $\beta \in K(X)$

Idea: These maps $\psi^k$ behave like

$$\psi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k + \cdots + L_n^k$$

when $L_1 \oplus \cdots \oplus L_n = E$ is a sum of line bundles

↪ What happens generally?

Idea$^2$: Use $\lambda^k(E)$ exterior powers

In fact

**Thm (Splitting Principle):** Given a vector bundle $E \to X$, $X$ cmpct Hdf, $\exists$ a compact Hdf space $F(E)$ and a map $p: F(E) \to X$ s.t. the induced map $p^*: K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles

So Adams (Pr) ops + Splitting Principle

$$\Rightarrow \psi^k(E) = s_k(\lambda^1(E), \ldots, \lambda^k(E))$$

$s_k$ a poly w/ $\mathbb{Z}$-coeff

And In Partic: $\psi_k$ on $\tilde{K}(S^{2n}) \cong \mathbb{Z}$ acts like mult by $k^n$.

So now, Behold!!

$$\psi^2\psi^3 = \psi^6 = \psi^3\psi^2$$

So what?

Rememember $\alpha, \beta \in K(C_f)$?

Well,

$$\psi^k(\alpha) = k^{2n}\alpha$$

$$\psi^k(\beta) = k^n\beta + \mu_k\alpha \quad \text{, some } \alpha \in \mathbb{Z}$$

Therefore:

very illuminating

$$\psi^k\psi^l(\beta) = \psi^k(l^n\beta + \mu_l\alpha) = k^n l^n\beta + (k^{2n}\mu_l + l^n\mu_k)\alpha$$

Also: $\psi^k\psi^l = \psi^l\psi^k$, so

$$k^{2n}\mu_l + l^n\mu_k = l^{2n}\mu_k + k^n\mu_l$$

$$k^{2n}\mu_\ell + \ell^n \mu_k = \ell^{2n}\mu_k + k^n \mu_\ell$$

$$\Updownarrow$$

$$(k^{2n} - k^n)\mu_\ell = (\ell^{2n} - \ell^n)\mu_k$$

$\Rightarrow$ Since $\beta^2 = h\alpha$, $\psi^2(\beta) = 2^n\beta + \mu_2\alpha$

$\Rightarrow$ $\mu_2 \equiv h \bmod 2$ $\Rightarrow$ $\mu_2$ odd when $h = \pm 1$

$\Rightarrow$ need

$$(2^{2n} - 2^n)\mu_3 = (3^{2n} - 3^n)\mu_2$$

Basically — —

it's all just:

③ A MAJOR Key π

Lemma

σπ

<u>**Lemma**</u>: If $2^n \mid 3^n - 1$

then $n = 1, 2,$ or $4$

$\Rightarrow \quad S^{2n-1} = S^1, S^3,$ or $S^7$

... yeah!

That's why!!

(lol)