Closed Smooth 4-Manifolds **Spring 2025**

Part I

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Overview

Lecture 1: January 14^{th}

All our manifolds will be 4-dimensional, compact, oriented, and smooth unless stated otherwise. Dimension four is the phase boundary, in some sense, for the problem of classifying manifolds. In dimensions at most three, the problem is solved (though not without difficulty), and in dimensions at least five, there is a sense in which the problem is not solvable (tied to the word problem for groups), but there are tools such as surgery theory and the s-cobordism theorem which give partial results (e.g., simply-connected closed 5-manifolds are classified). In dimension four, even in the simply-connected case, there is no conjectural classification, and the tools that work in higher dimensions generally fail.

How can we come up with a conjectural classification? The basic strategy is to look for a set of algebraic invariants these manifolds have, and check whether they completely determine the manifolds in question. The first such feature to consider is the fundamental group. If we restrict to $\pi_1 = 1$, we may ask whether this is complete; evidently, it is not, as H_2 and its intersection form Q_X are not detected by π_1. Per Freedman, these additional data are sufficient for classifying simply-connected closed 4-manifolds up to homeomorphism; per Donaldson, these data are insufficient in the smooth category via gauge/Floer theoretic invariants. These gauge and Floer theoretic data are also insufficient to complete our classification, since, for example, exotic copies of $\mathbb{CP}^2 \#_k \overline{\mathbb{CP}^2}$ with metrics of positive scalar curvature have been constructed,[1] and all known gauge theoretic invariants vanish in the presence of a metric of positive scalar curvature.

If we allow for manifolds with boundary, conjecturing a classification is more difficult as, in addition to the features of the manifold itself, the features of the boundary and the features of $\iota : \partial X \hookrightarrow X$ are also essential to consider, which results in complication that we will seek for the moment to avoid.

[1] Răsdeaconu and Şuvaina, *Smooth structures and Einstein metrics on* $\mathbb{CP}^2\#5, 6, 7\overline{\mathbb{CP}^2}$.

Returning for a moment to Donaldson's results, which give multiple smooth structures corresponding to *some* intersection forms Q_X; it is then also interesting to ask whether Q_X is insufficient to determine X (smoothly) for *all* Q_X. That is, are there exotic copies of all closed, smooth 4-manifolds? This will be one of our guiding questions in this course.

How To Construct Exotica

The procedure for building exotica is as follows:

Step I Build candidate manifolds X, X' (one often gets X for free)

Step II Show that $X \underset{\text{top}}{\cong} X'$

Step III Show that $X \underset{\text{sm}}{\ncong} X'$

Step 2. usually involves calculating the classical algebraic invariants of the manifolds, step 3. usually involves calculating some non-classical gauge/Floer/lasagna-theoretic invariants. We will primarily be focused on step 1. in this class, though all three can be quite difficult.

The outline for the course is as follows:

Part I We will begin with constructive basics, in the setting of manifolds with boundary. We will introduce Floer TQFT methods invariants on a formal level and develop enough tools to do some nontrivial calculations.

Part II We will apply our invariants from Part I to build some exotic homotopy S^2s (i.e., knot traces), homotopy B^4s, and, finally, exotic closed manifolds. We will develop the theory of Lefschetz fibrations (these give Donaldson's original exotic construction of $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$). We will consider various surgery operations such as torus surgeries and knot surgeries, and cover the relevant TQFT gluing formulae relevant to these surgeries. We will sketch many many examples of exotica (down to $\mathbb{CP}^2\#_5\overline{\mathbb{CP}^2}$), and discuss Lefschetz fibration geography.

Part III We will sketch the record-setting construction of an exotic $\mathbb{CP}^2\#_3\overline{\mathbb{CP}^2}$ due to Akhmedov-Park,[2] and the construction of small $\sigma = 0$ exotica due to Baykur-Hamada.[3] More, as time permits.

[2] Akhmedov and B Doug Park, *Exotic smooth structures on small 4-manifolds*.

[3] R. I. Baykur and Hamada, *Exotic 4-manifolds with signature zero*.

We didn't end up making it to Part III.

Framings

An n-dimensional k-handle is D^n thought of as $D^k \times D^{n-k}$ (with corners appropriately smoothed), which we attach to some manifold X via a smooth embedding $f : \partial D^k \times D^{n-k} \to \partial X$.

Lemma 1.2.1

$X \cup_f$ handle is well-defined up to isotopy of f.

The essential data of a handle attachment is where $\partial D^k \times \{0\}$ goes (called the *attaching sphere*) in ∂X. The attaching map turns out to constitute only a mild decoration of this data, corresponding to a choice of trivialization for the tubular neighborhood of the attaching sphere.

Definition 1.2.2: Framings

For a $k-1$-sphere $S \hookrightarrow Y^{n-1}$ with trivial normal bundle, a *framing* is a choice of diffeomorphism (or bundle isomorphism) $f : S^{k-1} \times D^{n-k} \to \nu(S)$.

A framed sphere is enough info to attach a handle. With the title of our class in mind, we don't need to be coy and work in generality; set $n = 4$ and $k = 2$, so we are interested in embeddings $K : S^1 \hookrightarrow Y^3$ and framings thereof.

From a handle decomposition, by deformation retracting every handle onto its core, we obtain a CW complex homotopy equivalent to it; this process is insensitive to framings i.e. is determined by the data of the attaching spheres alone.

Lemma 1.2.3

The set of framings in this setting (up to isotopy of the framing map) is *non-canonically* in bijection with $\mathbb{Z} = \pi_1(\mathrm{SO}(2))$.

PROOF : Choose any framing f and declare $\varphi(f) = 0$; for any other framing g, consider $f^{-1} \circ g : S^1 \times D^2 \to S^1 \times D^2$. Using the fact that the mapping class group $\mathrm{Diffeo}^+(S^1 \times D^2)/\text{isotopy}$ is isomorphic to $\mathbb{Z}$ and generated by the Dehn twist element τ, set $\varphi(g) = [f^{-1} \circ g] \in \mathbb{Z}$. φ is our desired bijection. ■

In general, the set of framings for an n-dimensional k-handle are a $\pi_{k-1}\,\mathrm{O}(n-k)$ torsor.

Exercise 1.2.4

1. Show that φ as above is a bijection
2. Show that f is determined by $f(S^1 \times \{1\})$

Bonus

3. Prove that handle attachment is determined by the isotopy class of the attaching map
4. Exhibit the relationship between framings in this setting and $\pi_1(\mathrm{SO}(2))$

This is not a particularly satisfactory situation (e.g., 3 is non-canonically 7 in torsor land); fortunately, we can do a little better if we assume that our attaching circles are nullhomologous.

Definition 1.2.5: Meridians and Longitudes

For a (classical) knot $K : S^1 \to Y^3$, the *meridian* of K is $\mu_K : S^1 \hookrightarrow \partial\nu(K)$ the boundary of a disc fiber of the normal bundle. A *longitude* is $\lambda_K \to S^1 \to \partial\nu(K)$ such that $[\lambda_K]$ generates $H_1(\nu(K))$ (equivalently, λ_K intersects μ_K geometrically once).

Corollary 1.2.6

Framings are determined by a choice of longitude.

Lisa says that this is the only foolproof notion of framing, and what you should return to if/when you become confused.

Definition-Proposition 1.2.7

If K is nullhomologous in Y, then there is a unique longitude λ_0 in $\partial\nu(K)$ s.t. $[\lambda_0] = 0 \in H_1(Y \setminus \nu(K))$, called the *Seifert longitude*. Thus, for K nullhomologous, we have a canonical bijection for framings.

Exercise 1.2.8

Prove the above proposition.

Bonus

Show that if $\Gamma^2 \hookrightarrow Y$ is a surface with $\partial\Gamma = Y$, show that $\lambda_n \cap \Gamma$ in n points (where λ_n is the longitude with $\varphi(\lambda_n) = n$).

Lecture 2: January 16$^{\text{th}}$

Proof : We will invoke without proof a theorem of Thom:

Theorem 1.2.9: Thom

If $n \leq 4$, then any class in $H_*(X^n)$ can be represented by a submanifold. Moreover, if $Y^m \hookrightarrow X^n$ with $m \leq n-1$, with $[Y] = 0 \in H_m(X)$ then $Y = \partial Z^{m+1} \hookrightarrow X$.

To see the result from this, since $[K] = 0$, there exists a Seifert surface $\Sigma^2 \hookrightarrow Y$ with $\partial\Sigma = K$. A pushoff of K into Σ gives λ with $[\lambda] = 0 \in H_1(Y \setminus \nu(K))$.

To prove uniqueness of λ, we claim that $[\mu_K]$ is infinite order in $H_1(Y \setminus \nu(K))$. Suppose there exists λ' with $[\lambda'] = 0$, then $[\lambda - \lambda'] = [0]$, and $[\lambda - \lambda'] = n[\mu_K] \in H_1(Y \setminus \nu(K))$ so $n = 0$ and $\lambda = \lambda'$.

To see that $[\mu_K]$ is non-torsion, note that 3-manifolds have a bilinear intersection pairing $H_1(Y) \times H_2(Y) \to \mathbb{Z}$ given by counting the number of intersections; this implies that meridians are infinite order. To see this, note that if $n[\mu_K] = 0 \in H_1(Y \setminus \nu(K))$, then $n[\mu_K]$ bounds a surface $\Gamma \hookrightarrow Y \setminus \nu(K)$, then in Y we can obtain a new surface Γ' by gluing in the obvious disc bounded by each of the n parallel copies of μ_K. In Y, $|\Gamma' \cap K| = n$ but $[K] = 0 \in H_1(Y)$ so $n = 0$. ■

One can also prove this result using Mayer-Vietoris. Lisa remarks that in other settings, there might be a different preferred longitude given by some additional data (e.g. a surface the knot lies on).

Definition-Proposition 1.2.10: Linking Pairing

For $K, K' \subseteq Y$ a pair of nullhomologous knots, we may consider $K' \subseteq Y \setminus \nu(K)$. We claim that $H_1(Y) = H_1(Y \setminus \nu(K))/\langle\mu_K\rangle$. Since $[K'] = 0 \in H_1(Y)$, $[K'] = n[\mu_K] \in H_1(Y \setminus \nu(K))$. Thus, we may define the *linking pairing* $\mathrm{lk}(K, K') := n$.

Exercise 1.2.11

Show that $H_1(Y) = H_1(Y \setminus \nu(K))/\langle \mu_K \rangle$.

Bonus

Give a diagrammatic definition of lk for knots in S^3 or arbitrary 3-manifolds; prove that $\mathrm{lk}(K, K') = \mathrm{lk}(K', K)$.

One can also do this with Mayer-Vietoris or van Kampen.

PROOF : $Y = Y \setminus \nu(K) \cup (S^1 \times D^2)$. We may decompose $S^1 \times D^2$ as a 3-ball (A) with one 1-handle attached (B), so, instead, we can write

$$Y = (Y \setminus \nu(K) \cup_g A) \cup B$$

Gluing in A along g is essentially just gluing in a disc along its boundary, and B gets glued in along its entire S^2 boundary (which does not affect H_1). Gluing in a disc picks us up a relator in homology, so, $H_1(Y) = H_1(Y \setminus \nu(K))/\langle g(\gamma) \rangle$ where γ is the boundary of the disc, and $g(\gamma) = \mu_K$ by construction. ■

By convention X denotes a 4-manifold, Y a 3-manifold, Σ and Γ 2-manifolds. The mnemonic for this is to count the number of leaves of the graph defined by each letter; X has 4 leaves, Y, 3, etc.

Suppose X is a 2-handlebody (which, for our purposes, means that it is built from a single 0-handle and some collection of 2-handles) described as (K_i, f_i).

Question 1.2.12

What are the λ_{f_i}? Why do we care, if we have this apparently simpler description in terms of an integer framed link diagram?

Exercise 1.2.13

Suppose X is a 2-handlebody with two 2-handles (K_1, f_1) and (K_2, f_2). Suppose further that we do some slide of the K_2 handle over K_1. What is the resulting attaching circle for the new handle diagram?

Bonus

Prove that the effect on the boundary of 2-handle attachment is λ-framed Dehn surgery.

Note that λ_{f_1} bounds a disc in the *new* boundary created by attaching h_1, i.e., in

$$\partial(X \cup h_1) = (\partial X \setminus S^1 \times D^2) \cup D^2 \times S^1$$

The moral of this story is that it pays to know where the non-obvious discs are; this will come up repeatedly in this course.

PROOF : K_2' is a band sum of K_2 with λ_{f_1}. The point is that a handle slide is just an isotopy, so we're taking a bight of K_2 and sliding it over some disc in the boundary. Since λ_{f_1} is the image of $\partial D^2 \times \{1\}$ under the attaching map for the 2-handle, it bounds the embedded disc $D^2 \times \{1\} \hookrightarrow \partial(X \cup h_1)$; this is the disc we are sliding over. The effect of this isotopy is a band sum. ■

Note that, in general, our attaching circles may not appear nullhomologous in our handle diagram (and so our framing numbers *a priori* do not make sense); the convention for 2-handlebodies is that we are considering the framing number for $K_i \subseteq S^3$ (i.e., ignore everything but the one knot in question).

Lecture 3: January 23rd

Recall *Seifert's algorithm*, which from a diagram of a knot K in S^3 produces an orientable surface $\Sigma \hookrightarrow S^3$ with $\partial\Sigma = K$.

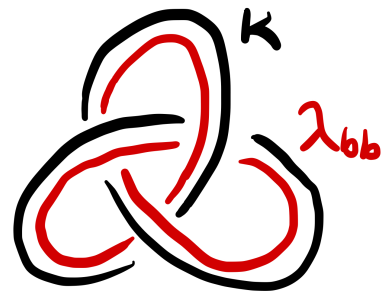


Figure 1.1: The blackboard framing for the standard drawing of the right-handed trefoil.

Definition 1.2.14: Blackboard Framing

For a knot diagram, the blackboard framing λ_{bb} is the longitude that lies in the blackboard.

Exercise 1.2.15

Pick any nontrivial knot, and draw both λ_{bb} and λ_0 (by running Seifert's algorithm). Prove that $\lambda_0 = \lambda_{\text{bb}} - \omega$ where ω is the *writhe* of the diagram, i.e., the number of positive minus the number of negative crossings.

The right hand rule is useful for remembering the signs of crossings.

Bonus

Consider how integer framings change under handle slides (the relevant players are f_1, f_2 and $\text{lk}(K_1, K_2)$).

Proof : To convince yourself of the formula, it suffices to consider the various framings at a single crossing. The Seifert longitude is obtained by pushing λ_{bb}

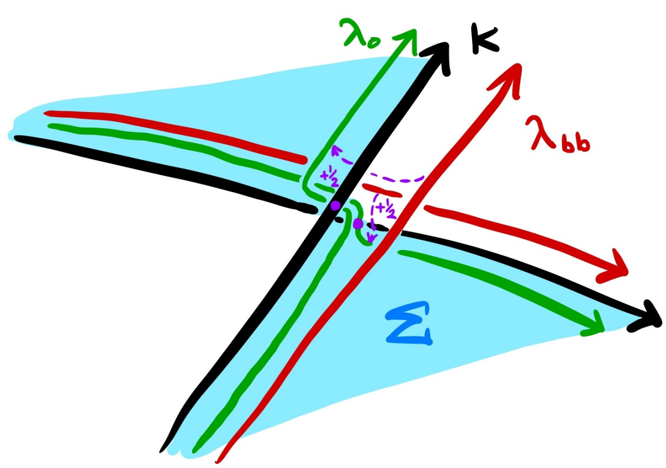


Figure 1.2: Pushing λ_{bb} onto the Seifert surface.

onto the Seifert surface Σ whose boundary is K; each $\pm$ crossing of the diagram of K induces two $\mp\frac{1}{2}$ twists, which gives us the desired formula after summing over all crossings. ∎

Dotted Circles

A 4-dimensional 1-handle is $D^1 \times D^3$ attached along $S^0 \times D^3$ which is just a pair of 3-balls. One way to draw a 1-handle attachment, then, is just to draw a pair of balls in the handle diagram. This has some downsides: for one, this is just cumbersome when there's multiple 1-handles since they need to be labeled; for another, this does not automatically give us a Dehn surgery presentation of the boundary which we had for 2-handlebodies. Thankfully, there is a remedy.

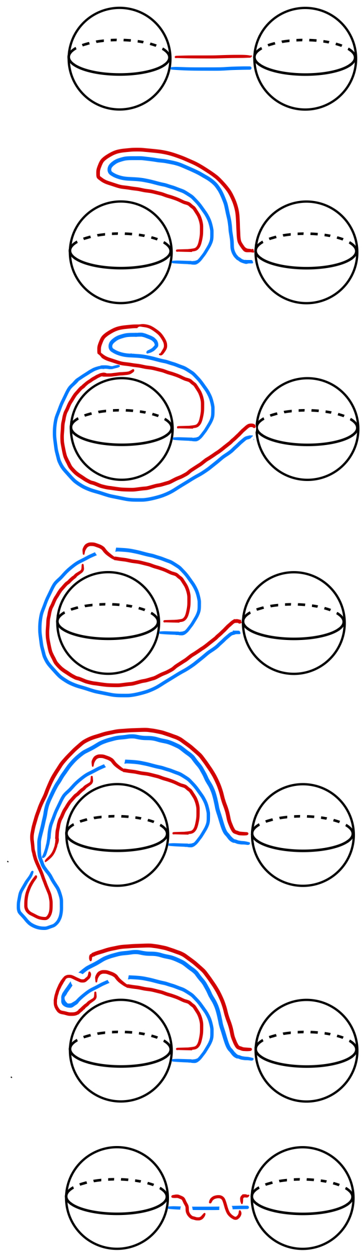


Figure 1.3: Planar isotopy that changes 2-handle framing by 2. Since this is a picture of a cancelling pair, this shouldn't be too surprising; as we will discuss below, 1-handle attachment does 0-framed Dehn surgery on the boundary, so this is essentially a drawing of the standard fact that the $(0, 0)$-framed Hopf link is the same (as a handle diagram) as the $(0, 2n)$-framed Hopf link.

Remark 1.3.1

One objection to the "pair of 3-balls" notation is that 2-handle framings are not well-defined as in Figure 1.3, where the 2-handle framing can change by ± 2 via planar isotopy. In fact, this is only a mental problem, and it is perfectly fine for framings to change along a planar isotopy.

Proposition 1.3.2

If a 2-handle h_2 runs over a 1-handle h_1, such that the attaching sphere of h_2 *geometrically* intersects the belt sphere of h_1 in a single point, then h_1 and h_2 are a *cancelling pair*, i.e., $X \cup h_1 \cup h_2 = X$.

One mental model for this is just a thickened disc filling in a genus, but this is too simple for our 4-dimensional needs. The point will of course be that a dotted circle meridian to any 2-handle (with any framing) forms a cancelling pair.

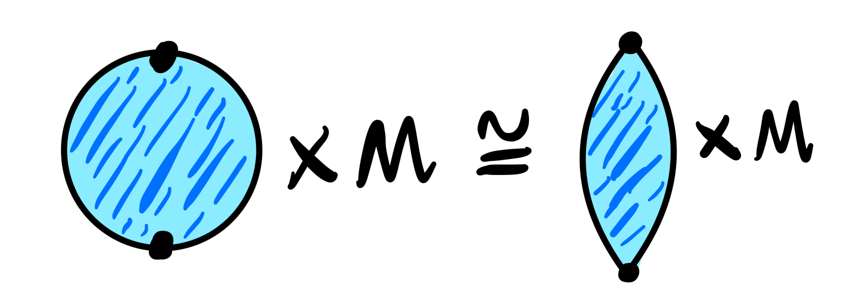
Lemma 1.3.3

Let $Y^3 \hookrightarrow \partial Z^4$, then $Z \cup_\tau (Y \times I) \underset{\text{sm}}{\cong} Z$ where $Y \times \{0\} \xrightarrow{\tau} \partial Z$ is the embedding we already had.

PROOF : Essentially, we are just attaching reverse collar neighborhoods only to $Y \subseteq \partial Z$. Taking an existing collar neighborhood of Y, we can build a diffeomorphism from Z that is the identity away from this collar and just stretches $Y \times I$ out by a factor of 2 on the collar. ■

Lemma 1.3.4: Skinny Discs are Discs

For any manifold M,



Exercise 1.3.5

Prove the above proposition using the above two lemmas.

Bonus

Using your proof, give a handle diagram for the diagram in the margin after cancelling the appropriate handles.

This is barely a lemma and does not deserve a proof.

Proof : $h_2 = D^2 \times D^2$ attached along a thickened S^1; the point of the second lemma above is that we should (here) think of a disc bounded by a circle as guiding an interpolation between two paths (given by splitting the circle into two arcs) rather than as guiding a nullhomotopy.

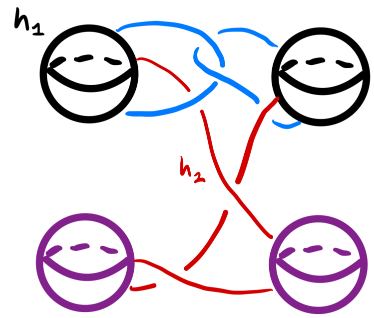


Figure 1.4: A tricky handle diagram.

In our case, we want to partition our attaching circle for our 2-handle into two arcs, one of which is visible in the handle diagram, and the other runs over the 1-handle (see below). Then we use the non-obvious disc between these two arcs (the core disc of the 2-handle) to guide an isotopy that places the two arcs almost parallel, and makes the 1-handle into a bulge on the boundary.

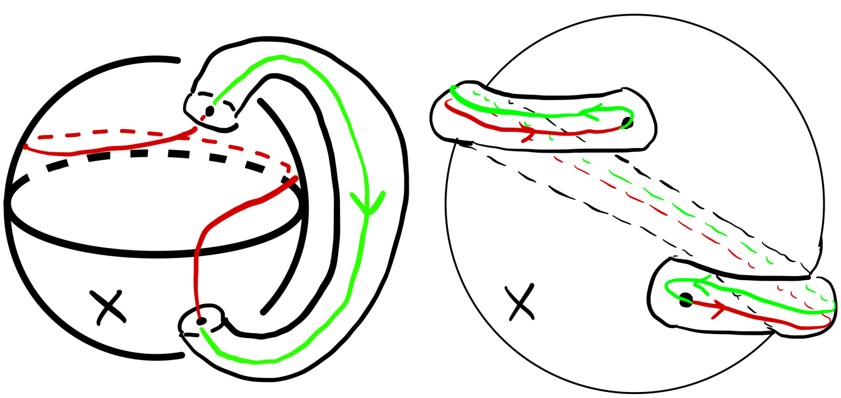


Figure 1.5: Guiding an isotopy using a hidden disc.

We may then apply the first of the two above lemmas (bulges don't matter) to finish. ■

Another example of it paying to know where your hidden discs are.

Thus $X \cong X \cup h_1 \cup h_2 \cong X \cup (D^3 \times D^1)$; if we want to recover $X \cup h_1$ from this, we need to remove a thickened disc. This is the key observation that leads to dotted circle notation for 1-handles: that 1-handle attachment is the same as deleting a disc.

Lecture 4: January 28th

To make this explicit, we consider $X \cong X \cup (D^3 \times D^1)$ with a cancelling pair attached, thought of as a 1-handle "lying down" as in the above picture. In particular, thinking of $D^3 \times D^1$ as a 1-parameter family (indexed by time) of 3-balls, glued to ∂X along $D^2 \times D^1$ (the $D^2 \subseteq D^3$ being one half of the boundary sphere), we may obtain $X \cup h_2$ by deleting a judiciously chosen disc (see Figure 1.4). In this setup, the evident disc to delete

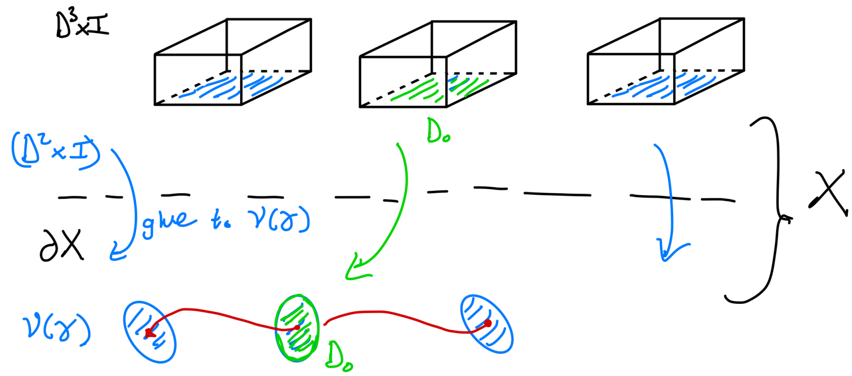


Figure 1.6: We can produce the correct dimensional "hole" (as is created by attaching a 1-handle) by deleting an appropriate neighborhood of D_0.

is $D_0 = D^2 \times \{0\} \subseteq D^3 \times [-1, 1]$. Note that $\partial D_0 \subseteq \partial X$ is unknotted (i.e., bounds a disc in ∂X) by construction, and this disc is boundary parallel. ∂D_0 in fact bounds D_0, so this belabors the point a little since D_0 automatically lies in ∂X but these are precisely the criteria for a circle in the boundary to define a unique disc, and therefore prescribe a 1-handle attachment.

Recall that a submanifold is boundary parallel if there is an ambient isotopy (fixing the boundary) taking the submanifold to the boundary.

Exercise 1.3.6

We know that $X \cup h_1 \cong X \natural S^1 \times B^3$. Where is $S^1 \times \{\bullet\}$ in the dotted circle notation (where $\bullet \in \partial B^3$)? How can we convert from the dotted circle notation back to standard notation?

Proposition 1.3.7

The effect on the boundary of 1-handle attachment is 0-framed Dehn surgery on the dotted circle (*not* the attaching circle, which is S^0).

Bonus

Prove this using $Q_{S^4} = (0)$.

My answer to this was that $S^1 \times \{\bullet\}$ is where we would attach the 2-handle that cancels the 1-handle, so it must be a meridian of the dotted circle. This is a little circular, and you can come up with the same answer by somewhat carefully studying the above diagram. The standard fact that the boundary of $(D^2, S^1) \cap (D^2, S^1) \subseteq (D^4, S^3)$ intersecting transversely in a single point has boundary the Hopf link may be relevant.

PROOF : Since the effect of attaching a handle is local, we will study the effect of 1-handle attachment on B^4. $\partial(B^4 \cup h_1) = \partial(B^4 \setminus \nu(D_0)) = (S^3 \setminus \nu(U)) \cup D_0 \times S^1$ so deleting a disc evidently results in Dehn surgery, and it remains only to determine the surgery coefficient. Observe that we know that $\partial(B^4 \cup h_1) = S^1 \times S^2$, so $H_1(B^4 \cup h_1) = \mathbb{Z}$. This will imply that the surgery coefficient is 0.

Recall that $H_1(S^3) = 0 \cong H_1(S^3 \setminus \nu(U))/\langle \mu_0 \rangle$ and μ_0 is infinite order; the key point is that $H_1(S^3_{\frac{p}{q}}(U)) = \langle \mu_0 \rangle / \langle p\mu_0 + q\lambda_0 \rangle = \mathbb{Z}/p\mathbb{Z}$, so we know that $p = 0$. Then $q = 1$ since other values of q prescribe attaching S^1 to disjoint copies of circles ($q = -1$ is disallowed to preserve orientation). ■

For the bonus, note that we may split S^4 into two copies of B^4 glued along their common S^3 boundary. In one of these copies of B^4, we may remove $D_0 \times D^2$, i.e., attach a 1-handle, and glue this $D_0 \times D^2$ to the other B^4. Taking the standard surface associated to this 2-handle (core disc union Seifert surface), this surface must have self-intersection 0 since $Q_{S^4} = (0)$, so the framing of the 2-handle curve is 0. But since $\partial(B^4 \setminus D_0 \times D^2) = \partial(B^4 \cup D_0 \times D^2)$, the surgery coefficient for 1-handle attachment is 0.

This also gives an argument that no 4-manifolds (with boundary) with nonzero intersection form can embed in S^4.

Proposition 1.3.8

We can slide 2-handles "over" (some say "under") 1-handle dotted circles.

Since the effect on the boundary of a dotted circle is 0-framed Dehn surgery, sliding over a 1-handle has to be identical to sliding over a 0-framed 2-handle. The effect on framing under such a slide is also given by pretending dotted circles are 0-framed 2-handles.

Remark 1.3.9

There are many proscriptions for dotted circle notation. You can only dot unknots (a knot, even a slice knot, will not prescribe a unique disc). You cannot slide a dotted circle over over another dotted circle in general as the result may no longer bound a boundary parallel disc (i.e. it may no longer be unknotted). Dotted circles may not be linked (this screws up the discs).

Relative Handle Calculus

Lecture 5: January 30th

Exercise 1.4.1

Build simply-connected closed 4-manifolds out of at most (say) six handles (including the 0 and 4-handles) that are not obviously $\#_k S^2 \times S^2$ or $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$. Can you do it with $b_2 = 0$? $b_2 = 4$?

The point of this exercise is that constructions are quite hard, and this is a big problem in the field. The two dual problems are that it's hard to close things up, and that when we can, closing tends to collapse everything back to one of our standard examples.

For intersection forms of size less than 8, we also know from number theory that they are a direct sum of a ± 1 diagonal matrix plus some hyperbolic forms $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which would correspond topologically to $\#_k S^2 \times S^2 \# m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$, so this exercise was to produce small exotica explicitly from handles.

Relative handle decompositions allow us to build interesting closed manifolds out of handles by eliminating the problem of closing things up. The idea is to build two 4-manifolds X and X' with the same boundary and then glue them together to get a closed 4-manifold. The tricky part is that we have to turn X' "upside down" in order to have a unified handle diagram for $X \cup_\partial X'$. An alternative way to think of these data is to consider the handle diagram of X' relative to that of X.

A relative handle diagram starts from some 3-manifold Y (the common boundary of X and X' in the above setup), which we stretch to a 4-manifold $X = Y \times I$, which we then add more handles to. The left boundary is $\partial^- X = Y$, and $\partial^+ X$ depends on the handles we add. To describe X we have to describe the attaching regions of handles in Y. For example, if we have a Heegaard splitting for Y, we may draw our 2-handle attaching curves on the pair of handlebodies. Integer framings are tricky in this setting, so we need to provide an explicit longitude (see Figure 1.4 below). We may also draw 1-handles as dotted circles provided they bound a disc in Y (i.e. they cannot wrap the topology of the handlebodies).

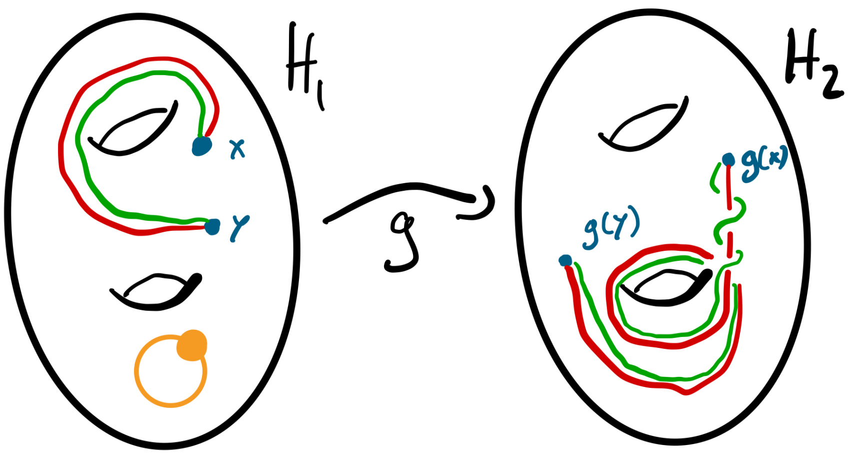


Figure 1.7: A relative handle diagram starting from a genus 2 Heegaard splitting of Y. The red curve is the attaching curve for a 2-handle, the green curve is its framing longitude. The dotted circle is appropriately contained in a 3-ball.

We may also start with a Dehn surgery presentation for Y and attach 4-dimensional handles from there. The Dehn surgery curves are distinguished from the handle attaching curves by bracketed framings. We can slide the 4-manifold 2-handles over the bracketed handles but not vice versa, as the bracketed handles give us Y, and Y cannot be modified in this setting. We may also slide the bracketed handles over each other to modify our Dehn surgery presentation for Y.

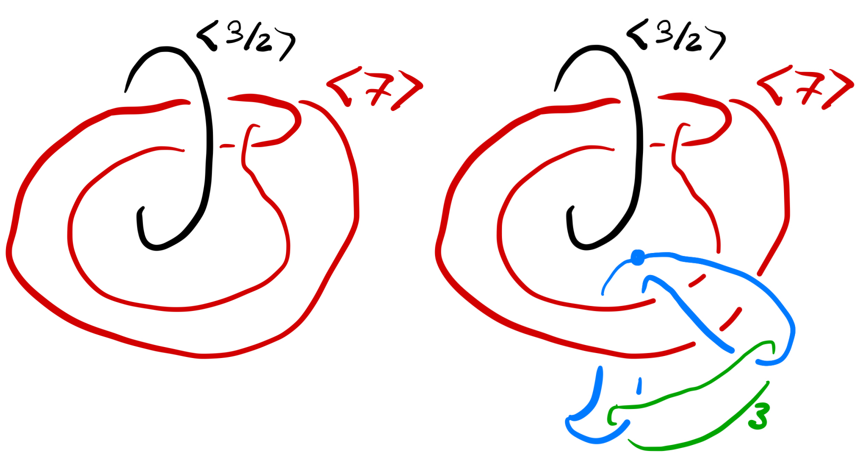


Figure 1.8: A relative handle diagram from a Dehn surgery presentation. Y is on the left, before the handle attachments, and X is on the right.

Exercise 1.4.2

Let X be the 4-manifold described by the relative handle diagram in Figure 1.9. Turn X upside down, and then draw a simpler relative handle diagram for X.

Bonus

Build three other manifolds X_i with $\partial X_i = \partial X$.

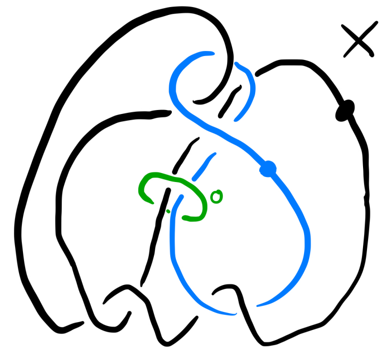


Figure 1.9: X, which will imminently be upside down

The 1-handles upside down become 3-handles so we don't need to draw them (using Laudenbach-Poénaru — 3-handle attachment is unique). A 2-handle upside down becomes a 0-framed meridian to its attaching curve, so a relative handle diagram for X upside down is the first of the following diagrams:

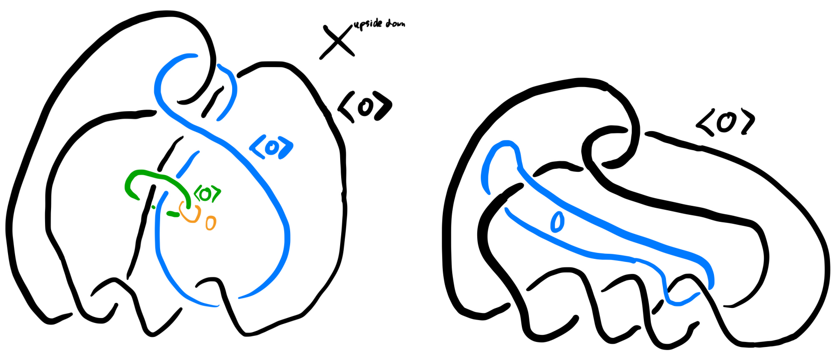
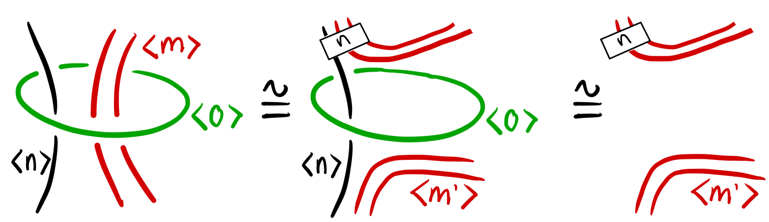


Figure 1.10: Two relative handle diagrams for X turned upside down, 3 and 4-handles not depicted.

Our original handles for X are now surgery curves for $\partial X = Y = \partial X^{\text{upside down}}$. In order to draw a simpler diagram for X, we need the following fact:

One can also derive this lemma using the slam-dunk move from Kirby calculus.

Lemma 1.4.3



This holds for 3-manifold surgery presentations (as indicated by the bracketed framings), not for 4-manifold handle diagrams, and follows by sliding the red strands over the black strand and then cancelling black with green (this makes sense if we think of our 3-manifold as the boundary of a 4-manifold, where the black strand is a 2-handle and the green circle is a cancelling 1-handle). We apply this lemma to the above diagram by sliding black, then yellow over blue (so that neither links green), then cancelling blue and yellow. The result of these slides is the second diagram in Figure 1.8 (where the yellow 2-handle has been redrawn in blue).

For the bonus, one option is to swap $\langle 0 \rangle$ and 0 in our simplified diagram. I'm not quite sure how to come up with others.

Now, if we can find X' and $f : \partial X \xrightarrow{\sim} \partial X'$, we can form $Z = X \cup_f X'$, a closed 4-manifold using the following lemma:

Lemma 1.4.4

Given $\varphi : Y \xrightarrow{\sim} Y'$,

$$(Y \times I) \cup_A \text{handles} \cong (Y' \times I) \cup_{\varphi(A)} \text{handles}$$

where A denotes the attaching regions.

This lemma just ensures that our gluing operation is actually well-defined, and that a diffeomorphism of the boundary is enough to glue our data together (a possible counterfactual would be dependence on the specific surgery presentation of the common boundary Y).

Lecture 6: February 4th

Note that ∂X as drawn in the right-hand side of Figure 1.8 is just 0-surgery on a single knot, which turns out to be 6_1. This is because, for relative handlebodies, $\partial^- X$ is the 3-manifold prescribed (here) by the Dehn surgery curves, before any 2, 3, and 4-handles (in this case, $\partial^+ = \emptyset$); see Figure 1.11.

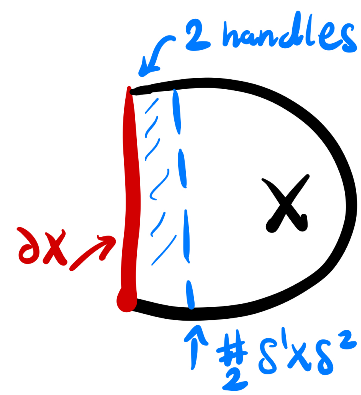


Figure 1.11: $\partial^- X$ is never affected by handle attachments.

In particular, ∂X has the same boundary as the 0-*trace* of 6_1:

Definition 1.4.5: n-Trace of a Knot

Given a knot $K \subseteq S^3$, its *n-trace* $X_n(K)$ is the 4-manifold given by attaching an n-framed 2-handle to B^4 along K.

Thus, we set $Z = X \cup_\partial X_0(6_1)$ and obtain a closed 4-manifold. Z is simply-connected since there are no 1-handles — we can combine the natural handle decomposition for $X_0(6_1)$ with the above relative handle decomposition for X to obtain an absolute handle decomposition for Z with no 1-handles. The data of the map identifying ∂X with $\partial(X_0(6_1))$ tells us (in principle) where the relative handle attachment curves go. In this case, since we have manipulated the relative handle diagram for X to look exactly like the natural handle diagram for $X_0(6_1)$, the identification is superimposition, and thus, the handle diagram for Z we obtain in Figure 1.12 is just the relative handle diagram for X with $\langle 0 \rangle$ replaced by 0.

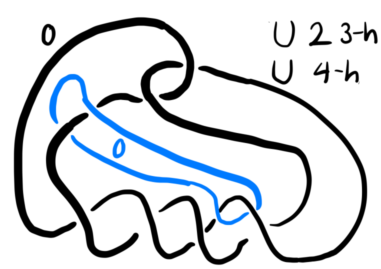


Figure 1.12: The handle diagram for Z obtained from the relative handle diagram for X (upside down).

Exercise 1.4.6

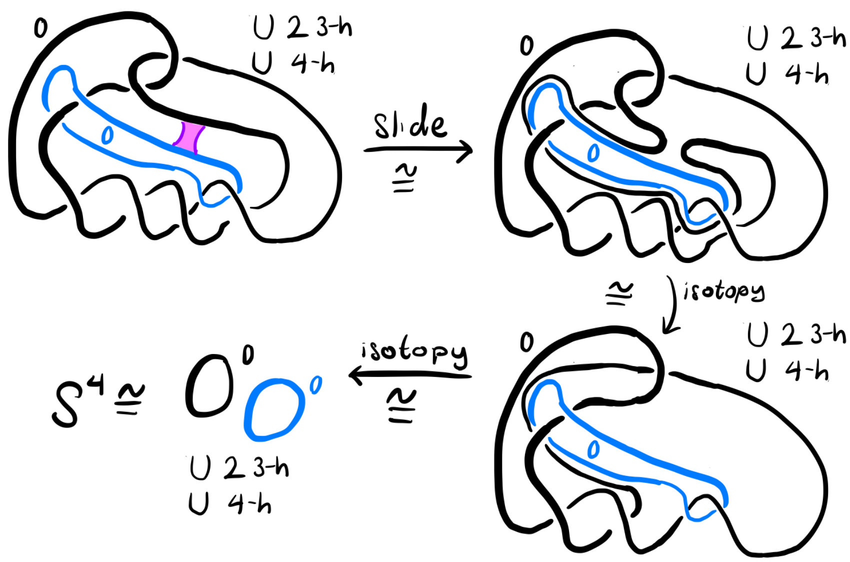
Show that $Z \underset{\text{sm}}{\cong} S^4$.

Bonus

For any $f' : \partial X \to \partial X_0(6_1)$, $\pi_1(Z_{f'}) = 1$ where $Z_{f'}$ is obtained by gluing X to $X_0(6_1)$ along f'. Moreover, for any such f', $H_*(Z_{f'}) \cong H_*(S^4)$ so $Z_{f'}$ is a homotopy 4-sphere.

When saying that $\pi_1 = 1$ and an isomorphism on H_* are enough to imply homotopy equivalence, we are invoking a mildly nontrivial application of Whitehead's theorem; Freedman then implies that homotopy 4-spheres are homeomorphic to S^4.

Proof : Sliding black over blue in the above handle diagram produces two unlinked 0-framed unknots which are then cancelled by the 3-handles:



I wonder if this example has anything to do with sliceness of 6_1, this almost looks like a Kirby diagram for a ribbon disc exterior.

For the first bonus, note that the above argument still suffices; there are no 1-handles. Any gluing also produces a homology 4-sphere by counting handles; the problem is that, in principle, f' might produce a very complicated handle diagram for Z. However, since we know $\pi_1(Z_{f'}) = 1$, we know that H_2 is free and $H_3 = 0$, so it follows that the 2 and 3-handles must homologically cancel, as otherwise the 3-handles would generate nontrivial H_3 (one can also see this by calculating $\chi(X) = 2 + b_2(X)$ from the given handle presentation and noting that there can be no torsion in H_2). ■

In general, we use MCG to denote the orientation-preserving mapping class group unless stated otherwise.

Since homotopy 4-spheres are exciting (pending resolution of the smooth Poincaré conjecture in dimension 4), we should be interested in finding $f' \in \mathrm{MCG}(S^3_0(6_1))$ different from our f above. To that end, we have the following definition:

Definition 1.4.7: Generalized Dehn Twists

Suppose we have $\iota : W^{n-1} \hookrightarrow Y^n$ and $\varphi : S^1 \to \text{Diff}^+(W)$ based at the identity. Then define the (ι, φ) Dehn twist of Y to be $\Phi : Y \to Y$ supported on $\nu(W) \cong W \times I$ given by

$$\Phi|_{W \times I}(w, t) = (\varphi(t)(w), t)$$

where $t \in S^1 = [0, 1]/(0 \sim 1)$. Then Φ is a self-diffeomorphism of Y.

I suppose the point(s) are that 1) the mapping class group in general is hard to understand and 2) having such a concrete element of an automorphism in terms of a codimension 1 submanifold allows one to (hopefully) apply some kind of gluing formula to compute Floer invariants.

Example 1.4.8: Ungeneralized Dehn Twists

Consider a circle in a genus 2 surface which splits it into two genus 1 surfaces, and $S^1 \to \text{Diff}^+(S^1)$ given by θ mapping to rotation through θ. This is an ordinary Dehn twist.

Exercise 1.4.9

Find $T^2 \hookrightarrow S^3_0(6_1)$. Use that T^2 to define a Dehn twist homeomorphism f' of $S^3_0(6_1)$.

Bonus

Give a handle diagram for $Z_{f'}$.

Floer Homology

spin^c structures

Lecture 7: February 6th

Our coverage of Floer homology will be very high-level, and we will mostly engage with the formal properties of the invariants which are often enough to do nontrivial computations with them. In particular, we don't really even need to know what a spin^c structure *means*, so we won't try to motivate them for the umpteenth time.

An orientation is extra structure on a manifold; one may or may not exist, and if one exists, there are several. spin^c structures can be thought of analogously, with the same properties. For 4-manifolds X, if $H_1(X)$ has no 2-torsion, then

$$\text{Spin}^c(X) = \{\text{spin}^c \text{ structures on } X\} = \\ \{\alpha \in H_2(X, \partial) : \forall \beta \in H_2(X, \partial) \quad \alpha \cdot \beta \cong \beta \cdot \beta \pmod 2\}$$

i.e. spin^c structures are in bijection with the set of characteristic elements of second homology.

To see that a characteristic vector/element always exists, note that 0 is characteristic in the even case. In the odd case, pick an odd orthogonal basis for H_2 (one can always do this by starting with any orthogonal basis then adding any odd basis element (one must exist) to the even basis elements) and set α to be the sum of all basis elements. Then $\alpha \cdot \beta$ is the sum of the coefficients of β in this basis, and $\beta \cdot \beta$ is the sum of the squares of these coefficients. Since $x \equiv x^2 \pmod 2$, $\alpha \cdot \beta \equiv \beta \cdot \beta \pmod 2$.

For 3-manifolds Y, if $H_1(Y)$ has no 2-torsion, and Y is closed,

$$\text{Spin}^c(Y) = \{\alpha \in H_1(Y) : \alpha = 2\beta \text{ for some } \beta \in H_1(Y)\}$$

A spin^c manifold is then a pair $(M, \mathfrak{s})$ of a manifold and a spin^c structure on it.

Note that we are not interested *per se* in spin^c 4-manifolds, it just turns out that the invariants we want to study are invariants of spin^c manifolds. Thus, if we want to show that $M \underset{\text{sm}}{\ncong} M'$ using an invariant σ of spin^c manifolds, then we need to show that $\sigma(M, \mathfrak{s}) \neq \sigma(M', \mathfrak{s}')$ for any $\mathfrak{s}'$ which could arise as $f_*(s)$.

Note that we are interested in 4-manifolds with boundary and closed 3-manifolds which will arise as boundaries in our cases of interest; in particular, a spin^c structure on X will induce a spin^c structure on $\partial X = Y$ via the long exact sequence

$$\cdots \to H_2(X) \to H_2(X, Y) \xrightarrow{\partial} H_1(Y) \to \cdots$$

Exercise 1.5.1

Show that a characteristic element is sent to a 2-divisible element by the boundary map.

Example 1.5.2

If X is B^4 with a single 2-handle (i.e. a knot trace) then the cocore disc D of the 2-handle gives a class $[D] \in H_2(X, \partial)$, hence a spin^c structure on X ($[D]$ is a generator hence vacuously characteristic in a rank 1 lattice); the corresponding spin^c structure on $H_1(\partial)$ is given by $[\partial D]$.

Exercise 1.5.3

Show that there can be boundary spin^c structures which don't extend over X. Show that there can be boundary spin^c structures with non-unique extensions.

Bonus

For $X_n(K)$, compute $\partial : H_2(X_2(K), \partial) \to H_1(\partial)$.

Proof : Working purely on the level of homology, for the first problem, we just want the boundary to have more H_1 than the manifold itself has H_2. Set $X = S^1 \times D^3$, where $H_1(Y) = \mathbb{Z}$, and inspect the long exact sequence.

For the second problem, set $X = (\mathbb{CP}^2)^\circ$ where M° denotes the punctured manifold $M \setminus \text{int}(B^4)$; $\partial X = Y = S^3$, and once again, inspection of the long exact sequence provides the result. ∎

A Taxonomy of Floer H_* Theories

There are roughly three main types of Floer homology:

For the purposes of this class, symplectic manifolds don't exist.

Instanton (I_*) due to Floer

Monopole (HM) due to Kronheimer-Mrowka

Heegaard (HF) due to Ozsváth-Szabó

Fundamentally, all these homology theories come from some kind of functorial association taking $(Y^3, \mathfrak{s})$ to an infinite dimensional Lie group G equipped with a Morse function $f : G \to \mathbb{R}$, and an infinite-dimensional analogue of Morse homology in that setting. The various Floer theories give us chain complexes C_* over a field $\mathbb{F}$ or a polynomial ring $\mathbb{F}[U]$ for some formal variable U (more than one formal variable is irrelevant to us in this class), generically denoted R. The Floer theories also capture essentially the same data as various gauge theoretic invariants: monopole Floer homology is equivalent to Seiberg-Witten theory, instanton Floer to Donaldson theory, and Heegaard-Floer (conjecturally) to Seiberg-Witten theory again.

The chain complexes C_* will be finitely generated, graded, and the chain homotopy type of C_* will be an invariant of $(Y, \mathfrak{s})$. spin^c-cobordisms $(W, \mathfrak{s}) : (Y_1, \mathfrak{s}_1) \to (Y_2, \mathfrak{s}_2)$ will induce an R-linear map $F_{(W,\mathfrak{s})} : C_*(Y_1, \mathfrak{s}_1) \to C_*(Y_2, \mathfrak{s}_2)$ (i.e, our chain complexes are TQFTs). $H_*(W)$ and $\mathfrak{s}$ govern the grading shifts in the chain complex, so the classical topology determines the grading shift (via the Atiyah-Singer index theorem).

We can extract invariants of closed spin^c 4-manifolds from Floer homologies as cobordisms between the empty manifold. For reasons we will see, these will always vanish when $b_2^+ + b_1 \equiv 1 \pmod 2$. Thus, for simply-connected 4-manifolds, b_2^+ must be odd for us to extract invariants from Floer theories, so the smooth $4D$ Poincaré conjecture is out of reach.

I only know one incarnation of this parity problem for invariants of smooth structures on 4-manifolds, the version for Seiberg-Witten invariants; there, it arises from having our moduli space sitting inside of $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$, and we get our invariants by evaluating the homology class of the moduli space against $[\mathbb{CP}^1]^k = [\mathbb{CP}^k]$ which is an even-dimensional cocycle (at least this is how it works in the simply-connected case). It's very bizarre to me that this fact that seems very specific to the setup of Seiberg-Witten theory is apparently a core limitation of all gauge/Floer-theoretic invariants of 4-manifolds.

For what follows, we will first develop the theory for 4-manifolds with boundary, then for closed 4-manifolds, and, finally, we will discuss some gluing formulae.

Heegaard Floer Homology

There are two flavors of Heegaard-Floer chain complexes of interest to us: $\widehat{\mathrm{CF}}$ (over $\mathbb{F}$), and CF^- (over $\mathbb{F}[U]$). For our purposes, $\mathbb{F} = \mathbb{F}_2$; much of what we will discuss is also known over $\mathbb{Z}$, but we leave those extensions to the Floer theorists. The Floer chain complexes are $\mathbb{Q}$-graded for torsion spin^c structures. Note that $\mathrm{CF}^- \neq \widehat{\mathrm{CF}} \otimes_{\mathbb{F}} \mathbb{F}[U]$.

We will write CF° and $\mathrm{HF}^\circ(Y, \mathfrak{s}) = H_*(\mathrm{CF}^\circ(Y, \mathfrak{s}))$ when we are talking about either flavor. Also note that if the spin^c structure is not defined, then $\mathrm{HF}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \mathrm{HF}^\circ(Y, \mathfrak{s})$ by convention.

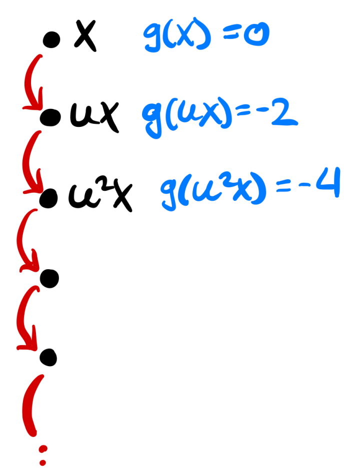


Figure 1.13: The HF^- tower for S^3, whose top grading is 0 and rank is 1 (this is all of the salient information but this tower serves as a good example for more complicated towers to come).

Example 1.6.1: $\mathrm{HF}^\circ(S^3)$

We begin with the simplest 3-manifold, S^3; $H_1(S^3) = 1$ and therefore there is a unique spin^c structure (by convention, for integer homology 3-spheres, the spin^c structure is unique hence omitted in the notation). $\widehat{\mathrm{HF}}(S^3) = \mathbb{F}$ and $\mathrm{HF}^-(S^3) = \mathbb{F}[U]$; U has grade -2, so multiplication by U lowers the grading by 2, which is drawn as a tower, see Figure 1.13 in the margin.

Example 1.6.2: $\mathrm{HF}^\circ(S^1 \times S^2)$

Next, we consider $S^1 \times S^2$; $H_1(S^1 \times S^2) = \mathbb{Z}$, so $\mathrm{Spin}^c(S^1 \times S^2) = 2\mathbb{Z}$. $\widehat{\mathrm{HF}}(S^1 \times S^2, \mathfrak{s}_{2i}) = 0$ for $i \neq 0$ and $\mathbb{F} \oplus \mathbb{F}$ for $i = 0$. $\mathrm{HF}^-(S^1 \times S^2, \mathfrak{s}_{2i}) = 0$ for $i \neq 0$ and $\mathbb{F}[U] \oplus \mathbb{F}[U]$ for $i = 0$. The absolute grading of the two generators for HF^- are $\pm\frac{1}{2}$, see Figure 1.14.

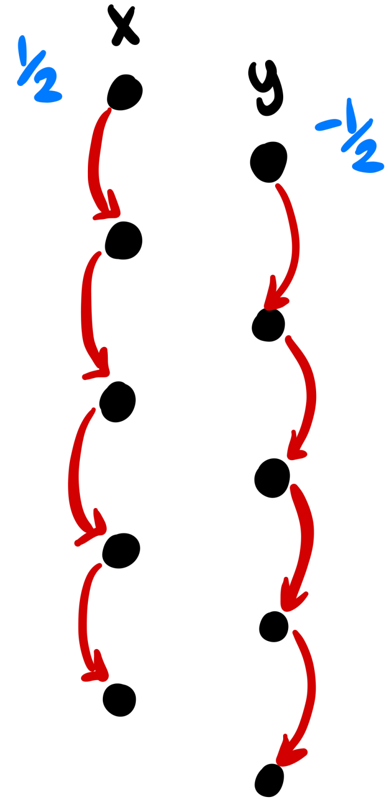


Figure 1.14: The HF^- tower for $S^1 \times S^2$ with two infinite towers and no torsion.

Example 1.6.3: $\mathrm{HF}^-(S^3_{-1}(Q))$

Let Q be the connected sum of the right and left-handed trefoils (i.e. the square knot), and consider $S^3_{-1}(Q)$ which is a homology 3-sphere and therefore has a unique spin^c structure; its HF^- tower is depicted in Figure 1.15.

It turns out that $\widehat{\mathrm{HF}}(Y, \mathfrak{s}) = \bigoplus_i \mathbb{F}_{g_i}$ so the data of $\widehat{\mathrm{HF}}$ is the number of nonzero elements and their gradings. Similarly, $\mathrm{HF}^-(Y, \mathfrak{s}) = \bigoplus_j \mathbb{F}[U] \bigoplus_k \mathbb{F}[U]/U^{n_k}$ so HF^- has infinite towers and some torsion; the torsion part is denoted $\mathrm{HF}_{\mathrm{red}}$. By Figure 1.15, $\mathrm{HF}_{\mathrm{red}}(S^3_{-1}(Q)) = \langle x + y, y + z \rangle$.

Lecture 8: February 11th

When $b_1(Y) = 0$, it turns out that there is a unique tower, whose grading is denoted $d(Y)$ (the d-invariant). We say that Y is an *L-space* if $\mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{s})$ is trivial for all spin^c structures $\mathfrak{s}$.

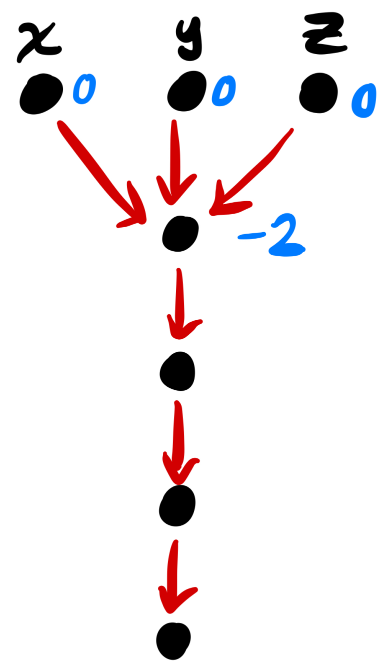


Figure 1.15: The HF^- tower for $S^3_{-1}(Q)$ with one infinite tower and nontrivial torsion.

To obtain 4-manifold invariants, we want to study the induced map on HF° by a spin^c cobordism $(W, \mathfrak{s}) : (Y, \mathfrak{s}) \to (Y', \mathfrak{s}')$ denoted $F^\circ_{W,\mathfrak{s}} : \mathrm{HF}^\circ(Y, \mathfrak{s}) \to \mathrm{HF}^\circ(Y', \mathfrak{s}')$. Note that HF is functorial so if W is the composition of two cobordisms W_1, W_2, then $F_{W,\mathfrak{s}} = F_{W_2,\mathfrak{s}_2} \circ F_{w_1,\mathfrak{s}_1}$. $F^\circ_{W,\mathfrak{s}}$ is an invariant of $(W, \mathfrak{s})$ in the sense that if there exists a diffeomorphism $f : (W, \mathfrak{s}) \to (W', \mathfrak{s}')$ then there exist isomorphisms g, g' such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{HF}^\circ(\partial^- W, \mathfrak{s}) & \xrightarrow{F^\circ_{W,\mathfrak{s}}} & \mathrm{HF}^\circ(\partial^+ W, \mathfrak{s}') \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathrm{HF}^\circ(\partial^- W', \mathfrak{s}) & \xrightarrow{F^\circ_{W,\mathfrak{s}}} & \mathrm{HF}^\circ(\partial^+ W', \mathfrak{s}') \end{array}$$

Note that these are not yet invariants of closed 4-manifolds, but a first attempt at turning them into closed invariants uses the following lemma:

Lemma 1.6.4

If X and X' are closed 4-manifolds, $X \underset{\text{sm}}{\cong} X'$ if and only if $X^{\circ\circ} \underset{\text{sm}}{\cong} X'^{\circ\circ}$ where $X^{\circ\circ}$ denotes X twice punctured.

Unfortunately, we are immediately foiled:

Theorem 1.6.5: Ozsváth-Szabó

For X closed, $F^{\circ}_{X^{\circ\circ},\mathfrak{s}}$ is determined by $(H_*(X), \mathfrak{s})$. In particular, $H_*(X)$ alone determines the grading shift.

So the most naïve approach to produce closed invariants fails as the cobordism data is determined by the classical topology. However, if X has one boundary component, then it turns out that the map $F_{X^{\circ},\mathfrak{s}} : \mathrm{HF}^{\circ}(S^3) \to \mathrm{HF}^{\circ}(\partial X, \mathfrak{s})$ can be interesting. Recall that $\mathrm{HF}^{\circ}(S^3)$ has a unique generator α, so $F_{X^{\circ},\mathfrak{s}}(\alpha)$ is a relative invariant of $(X, \mathfrak{s})$. Over the rest of the course, we will broadly cover three techniques for computing such invariants: adjunction inequalities, surgery exact triangles, and knot Floer homology, but first, we will give our first nontrivial example of a cobordism map.

Lemma 1.6.6

For $X_n(K)$, the boundary long exact sequence is

$$\cdots \to H_2(X_n(K)) = \mathbb{Z} \xrightarrow{\times n} H_2(X_n(K), \partial) \xrightarrow{\partial} H_1(S^3_n(K)) = \mathbb{Z}/n \to 0$$

Allegedly we calculated this in a previous exercise.

Corollary 1.6.7

If $n = 0$, ∂ is an isomorphism and therefore spin^c structures on $S^3_0(K)$ in $2\mathbb{Z}$ have unique extensions and we may regard $\mathrm{Spin}^c(X_0(K))$ as a subset of $\mathbb{Z}$ via $\mathfrak{s}_{2i} \mapsto 2i$.

Example 1.6.8: Explicit Cobordism Maps

$S^2 \times D^2 = X_0(U)$, so we have cobordism maps

$$F^{\circ}_{(S^2\times D^2)^{\circ},\mathfrak{s}_{2i}} : \mathrm{HF}^{\circ}(S^3) \to \mathrm{HF}^{\circ}(S^1 \times S^2, \mathfrak{s}_{2i})$$

If we take our flavor to be $-$ and $i \neq 0$, $\mathrm{HF}^-(S^1 \times S^2, \mathfrak{s}_{2i}) = 0$ so F° is trivial. At $i = 0$, we computed $\mathrm{HF}^-(S^1 \times S^2)$ above, and it turns out that the cobordism map in this case sends the generator for the only tower in $\mathrm{HF}^-(S^3)$ to the generator of the $\frac{1}{2}$-tower in $\mathrm{HF}^-(S^1 \times S^2, \mathfrak{s}_0)$; see Figure 1.16.

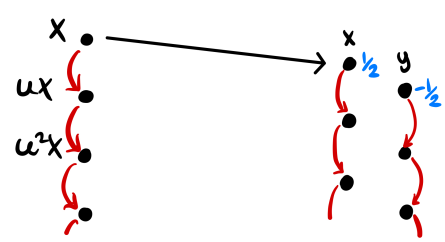


Figure 1.16: The map $F^-_{(S^2\times D^2)^{\circ},\mathfrak{s}_0}$.

Adjunction

Let $\Sigma \hookrightarrow X$ be an embedded surface with $[\Sigma]^2 = n$. Then, we may associate an integer $s \in \mathbb{Z}$ to all spin^c structures $\mathfrak{s}$ on X via the following map: $r : H_2(X, \partial X) \to H_2(\nu(\Sigma), \partial\nu) = \mathbb{Z}$. Geometrically, we can think about this as taking an embedded surface representative for $H_2(X, \partial X)$ and intersecting it with $\nu(\Sigma)$; on the level of spin^c structures, which are trivializations of an almost-complex structure on the 2-skeleton which extend over the 3-skeleton, we can think of this as just literally restricting this trivialization. In terms of $r(\mathfrak{s}) = s \in \mathbb{Z}$, we have the following:

Theorem 1.7.1: Adjunction

If $F^\circ_{X^\circ,\mathfrak{s}} \neq 0$, then $|s| + n \leq 2g(\Sigma)$. If in addition $n > -2g(\Sigma)$, then $|s| + n \leq 2g(\Sigma) - 2$.

An immersed version of this inequality also exists, taking into account the actual double points in addition to the homological self-intersection.

Thus, nonzero maps on Floer homology give lower bounds on genera for representatives of H_2. Contrapositively, the existence of low genus surfaces representing some homology class imply that $F^\circ_{X^\circ,\mathfrak{s}} = 0$; this is how we will more often be applying adjunction.

Proposition 1.7.2

If $\Sigma \hookrightarrow X_n(K)$ generates $H_2(X_n(K), \partial)$, then $r : H_2(X_n(K), \partial) \to H_2(\nu(\Sigma), \partial)$ is an isomorphism inducing a bijection on the set of spin^c structures.

Exercise 1.7.3

Let K be the connect sum of two right-handed trefoils; it turns out that $\hat{F}_{X_0(K)^\circ,\mathfrak{s}_2} \neq 0$. What can you say about the slice genus of K?

Recall that the *slice genus* of K, sometimes denoted $g_4(K)$, is the minimal genus of a surface inside B^4 whose boundary in S^3 is K. Slice knots K are those with $g_4(K) = 0$. Note that $g_4(K) \leq u(K)$ where $u(K)$ is the unknotting number; to see this, note that we may pick some nullhomotopy of K into B^4, which will generically have only double point singularities. We may then apply the standard trick of resolving each self-intersection by increasing the genus by 1 (by deleting a neighborhood of the self-intersection, whose boundary in the surface is a Hopf link, and gluing in the annulus bounded by the Hopf link).

Proof : The slice surface union the core disc of the 2-handle give a generator $[\Sigma]$ for the second homology of $X_0(K)$, and $[\Sigma]^2 = 0$ since the framing is 0. Adjunction then implies that $2 \leq 2g(\Sigma)$ so $g(\Sigma) \geq 1$. But $n = 0 > -2$ so we may bootstrap to the stronger form of adjunction, which implies that $2g(\Sigma) - 2 \geq 2 \implies g(\Sigma) \geq 2$, so the slice genus of K is at least 2 (in fact, since the slice genus of the trefoil is 1, this gives us that the slice genus of K is exactly 2). ∎

Classical invariants can tell us that the trefoil is not slice (e.g. the knot determinant via the Fox-Milnor condition).

Exercise 1.7.4

Let K be a slice knot. For what $\mathfrak{s}_{2i}$ is $F^\circ_{X_0(K)^\circ,\mathfrak{s}_{2i}} \neq 0$?

Bonus

Sketch a proof of adjunction when $n = 0$ and $g = 0$.

PROOF : Sliceness of K implies that we have an embedded sphere generating H_2, so (assuming $F^\circ_{X^\circ,\mathfrak{s}} \neq 0$) $|s| \leq 0$ for all $\mathfrak{s}$, so for $s \neq 0$, $F^\circ_{X^\circ,\mathfrak{s}} = 0$.

For the bonus, the $n = g = 0$ case of adjunction states that $F^\circ_{X^\circ,\mathfrak{s}} \neq 0 \implies |s| \leq 0 \implies s = 0$. Suppose there is a 0-sphere (an embedded sphere with self-intersection 0) in X, then we factor the cobordism X° from S^3 to ∂X as a sequence of cobordisms $S^3 \to S^2 \times S^1 \to \partial X$ by puncturing the interior of $S^2 \times D^2$ (the normal bundle of our $S^2 \hookrightarrow X$ is trivial since the self-intersection is 0). From the above calculations of $\mathrm{HF}^\circ(S^3)$ and $\mathrm{HF}^\circ(S^1 \times S^2)$, this immediately implies adjunction since $\mathrm{HF}^\circ(S^2 \times S^1, \mathfrak{s}_{2i}) = 0$ for $i \neq 0$ so the cobordism maps have no choice but to vanish. ■

Adjunction can only tell us when maps are zero; to prove that they're nonzero, we need further tools.

Knot Floer Homology

Knot Floer homology was developed independently by Ozsváth-Szabó and Rasmussen.

For our purposes, we will consider knots $K \subseteq S^3$ though the theory can be developed far more generally. As above, we have two flavors: $\hat{}$ and $^-$. The additional data of $K \subseteq S^3$ gives an additional filtration on $\mathrm{CF}^\circ(S^3)$ which gives rise to $\mathrm{CFK}^\circ(K)$, a bigraded chain complex over $\mathbb{F}$ or $\mathbb{F}[U]$. The two $\mathbb{Z}$-gradings are m, the Maslov grading, which is inherited from S^3, and s, the Alexander grading, related to K. As above, $\mathrm{HFK}^\circ(K) := H_*(\mathrm{CFK}^\circ(K))$.

Theorem 1.8.1: Ozsváth-Szabó

Knot Floer homology categorifies the Alexander polynomial, i.e.,

$$\Delta_K(t) = \sum_{m,s} (-1)^m \dim(\widehat{\mathrm{HFK}}_m(K, s)) t^s$$

RHT denotes the right-handed trefoil.

Example 1.8.2

$$\widehat{\mathrm{HFK}}_m(\mathrm{RHT}, s) = \begin{cases} \mathbb{F} & (m, s) = (0, -1), (1, 0), (2, 1) \\ 0 & \text{otherwise} \end{cases}$$

There are extremely quick computer programs to calculate HFK due to Zoltan Szabó.

Exercise 1.8.3

Recover $\Delta_{\mathrm{RHT}}(t)$ from the above.

There are also a handful of useful $\mathbb{Z}$-valued invariants that we get from from HFK^-:

- $\tau, \nu : \{\text{knots}\} \to \mathbb{Z}$

- $\epsilon : \{\text{knots}\} \to \{\pm 1, 0\}$

These invariants are *very* effectively computable (e.g. SnapPy can handle ~ 100 crossing knots), and give us useful information (for example, ν can tell us when $\widehat{F}_{X_n(K)^\circ, \mathfrak{s}_i}$ is 0). These invariants are highly related, and two out of three determine the third.

Lecture 9: February 13th

Proposition 1.8.4

$\epsilon(K) = 0 \implies \nu(K) = 0.$

Lemma 1.8.5: ν Trichotomy

Consider $\widehat{F}_{X_n(K), \mathfrak{s}_s} : \widehat{\mathrm{HF}}(S^3) \to \widehat{\mathrm{HF}}(S^3_n(K)i, \mathfrak{s}_s)$. Then we have the following trichotomy:

- $|s| + n > 2\nu(K)$ implies that $\widehat{F}_{X_n(K)^\circ, \mathfrak{s}_s} = 0$
- $|s| + n < 2\nu(K)$ implies that $\widehat{F}_{X_n(K)^\circ, \mathfrak{s}_s} \neq 0$
- $|s| + n = 2\nu(K)$ implies that $\widehat{F}_{X_n(K)^\circ, \mathfrak{s}_s} \neq 0$ if and only if $\epsilon(K) = 0$ and $n \leq 0$

Exercise 1.8.6

Take $n = 0$, then $\nu(\mathrm{RHT}) = 0$, $\epsilon(\mathrm{RHT}) = -1$, $\nu(\mathrm{LHT}) = 1$, and $\epsilon(\mathrm{LHT}) = 1$. Both $X_0(\mathrm{LHT})$ and $X_0(\mathrm{RHT})$ have a genus one surface generating their second homology. Compute which spin^c structures on $X_0(K)$ have nonzero maps for $K \in \{\mathrm{LHT}, \mathrm{RHT}\}$ and compare the result to the conclusion from adjunction.

Bonus

Use SnapPy to find other knots where knowing (ν, ϵ) gives strictly more information than the adjunction inequality.

Proof : Adjunction tells us that for $F^\circ_{X_0(K)^\circ, \mathfrak{s}}$ to be possibly nonzero, $s = 0$. With ϵ and ν, we can do a little better:

RHT We fall into the third case of our trichotomy above, and $\epsilon(\mathrm{RHT}) \neq 0 \implies \hat{F}_{X_n(K)^\circ, \mathfrak{s}_0} = 0$ so the last remaining map also vanishes,

LHT Here we fall into the second case of the above, so we know that $\hat{F}_{X_n(K)^\circ, \mathfrak{s}_0} \neq 0$ (this is still information, as adjunction does not give an if and only if statement). ■

Remark 1.8.7

Note that $X_0(\mathrm{LHT}) = -X_0(\mathrm{RHT})$, but their relative invariants are very different.

In fact, one can show formally that TQFT invariants that don't assign different quantities to X and $-X$ *cannot* detect exotica, so it is good that the relative invariants strongly depend on orientation.

Lemma 1.8.8: Twist Inequality

For knots $K, J \subseteq S^3$ related by either of the transformations in Figure 1.17, $\nu(J) \geq \nu(K)$. More generally, the insertion of a negative full twist into algebraically 0 or 1 parallel strands (counted with their orientations) results in the same inequality.

Corollary 1.8.9

$\nu(4_1) = 0$.

This follows since 4_1 is a twist knot, and can be drawn with either a positive or negative full twist, so $\nu(U) \geq \nu(4_1) \geq \nu(U)$ and (I assume) $\nu(U) = 0$ so $\nu(4_1) = 0$.

The invariants and results we have developed in this section will suffice to do some nontrivial calculations in the following chapters.

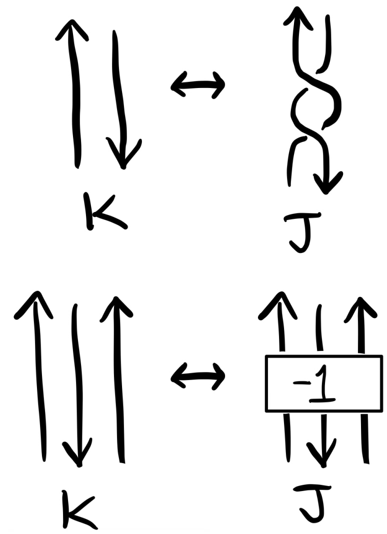


Figure 1.17: Twists which induce an inequality on ν.

Closed Smooth 4-Manifolds **Spring 2025**

Part II

Professor Lisa Piccirillo
Abhishek Shivkumar
Illustrations by Elise Brod

Relative Exotica

Recall that an *exotic pair* (X, X') of 4-manifolds with boundary satisfies $X \underset{\text{top}}{\cong} X'$ and $X \underset{\text{sm}}{\not\cong} X'$. However, this is not the only notion of exoticness for manifolds with boundary; in fact, it is not even historically the primary notion that people considered.

Definition 2.1.1: Relatively Exotic Pairs

A *relatively exotic pair* is a triple (X, X', f), where X and X' are smooth manifolds with boundary, and $f : \partial X \to \partial X'$ a diffeomorphism such that there exists a homeomorphism $F : X \to X'$ extending f but no such diffeomorphism extending f.

We say F extends f (as above) if $F|_\partial = f$.

Lemma 2.1.2

If (X, X', f) is a relatively exotic pair with $\text{MCG}(\partial X) = 1$, then (X, X') are an (absolute) exotic pair.

Recall that $\text{MCG}(X)$ consists of the (for our purposes, orientation preserving) self-diffeomorphisms of X up to isotopy.

Remark 2.1.3

A generic 3-manifold Y (without defining generic) has $\text{MCG}(Y) = 1$. For example, a generic 3-manifold is hyperbolic, and hyperbolic 3-manifolds all have finite, usually trivial mapping class group. As is often the case in math, the generic situation does not particularly inform what you should expect for your favorite small-ish 3-manifold built by hand, which will usually have nontrivial mapping class group.

We use $\underset{\text{h.e.}}{\cong}$ to denote homotopy equivalence.

Theorem 2.1.4: Akbulut-Ruberman '14[1]

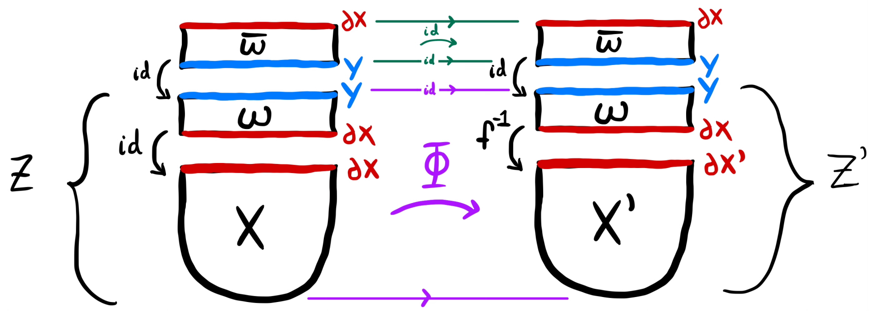
Given any relatively exotic pair (X, X', f), there exists an exotic pair (Z, Z') with $X \underset{\text{h.e.}}{\cong} Z$.

[1] Akbulut and Ruberman, *Absolutely exotic compact 4-manifolds*

Sketch : The main technical step (which we omit) is to build an invertible homology cobordism $W : \partial X \to Y$ for some Y with $\text{MCG}(Y) = 1$. A cobordism W is *invertible* if W glued to $\overline{W}$ along id_{Y_1} is diffeomorphic rel. boundary to $Y_0 \times I$ and similarly for $\overline{W}$ glued to W along id_{Y_0} *mutatis mutandis.*

Recall that a *homology cobordism* $W : Y_0 \to Y_1$ is a cobordism W with boundary components Y_0, Y_1 such that $\iota_* : H_*(Y_i) \to H_*(W)$ is an isomorphism. Morally, W is a cylinder as far as homology can see.

Assuming we may build such a W, we will then build $Z = X \cup_{\text{id}} W$ and $Z' = X' \cup_{f^{-1}} W$ which will be our (absolute) exotic pair, with common boundary Y. To finish from here, suppose $\Phi : Z \to Z'$ is a diffeomorphism; since $\text{MCG}(Y) = 1$ by assumption, we may assume $\Phi|_Y = \text{id}$, so we may glue $\overline{W}$ onto Z and Z' by the identity map:



But then

$$Z \cup_{\text{id}} \overline{W} \cong_{\partial} X \cup_{\text{id}} (\partial X \times I) \cong_{\partial} X \qquad Z' \cup_{\text{id}} \overline{W} \cong_{\partial} X' \cup_{f^{-1}} (\partial X' \times I) \cong_{f} X'$$

Thus, in composition, we obtain a diffeomorphism from X to X' inducing f on the boundary, which is a contradiction, so we are done.

Note that it is evident that Z is *homology* equivalent to X, but to show that they are homotopy equivalent (in the non-simply-connected case) is a little trickier, and we've omitted this step. ■

Note that the ends Y_0, Y_1 of an invertible cobordism W satisfy $Y_0 \times I \underset{\text{sm}}{\cong} Y_1 \times I$, but this does not imply that Y_0 and Y_1 are diffeomorphic. An example of Y_0, Y_1 which are diffeomorphic after $\times I$ are the pair of pants and a punctured torus (they both become a genus 2 handlebody) which are evidently not diffeomorphic, and we can realize these as ends of an invertible cobordism by attaching a $3D$ 1-handle to two of the boundary components of the pair of pants. This cobordism upside down attaches a 2-handle to the meridian of the torus which takes us back to a pair of pants.

I'm not sure what the rules are for absolute cobordisms of manifolds with boundary (we're attaching a 1-handle along a pair of discs that don't exist in the pair of pants), but this gives us the right answer so it should be correct.

Exercise 2.1.5

Build any nontrivial invertible homology cobordism from S^3.

$+1$ surgery on the right handed trefoil (or -1 on the left handed trefoil) results in the Poincaré (integral) homology sphere so attaching a 2-handle to B^4 along either knot with corresponding framing gives a cobordism from S^3. This is not a homology cobordism since the cobordism itself now has H_2, so probably one has to do some 3-manifold Kirby calculus wizardry to make this example work.

Exotic Homotopy 2-Spheres

Theorem 2.1.6: Akbulut '91, Yasui '15[2]

There exist relatively exotic pairs (X, X', f) with X homotopy equivalent to S^2 and $Q_X = (n)$.

The $n \neq 0$ case was handled by Akbulut, and $n = 0$ by Yasui. We will explore a version of their construction starting with the following exercise:

[2] Akbulut, *An exotic 4-manifold*; Yasui, *Corks, exotic 4-manifolds and knot concordance*

Lecture 10: February 18th

Exercise 2.1.7

Build a pair of simply-connected manifolds such that $\partial X \cong \partial X'$and $Q_X = Q_{X'}$.

Bonus

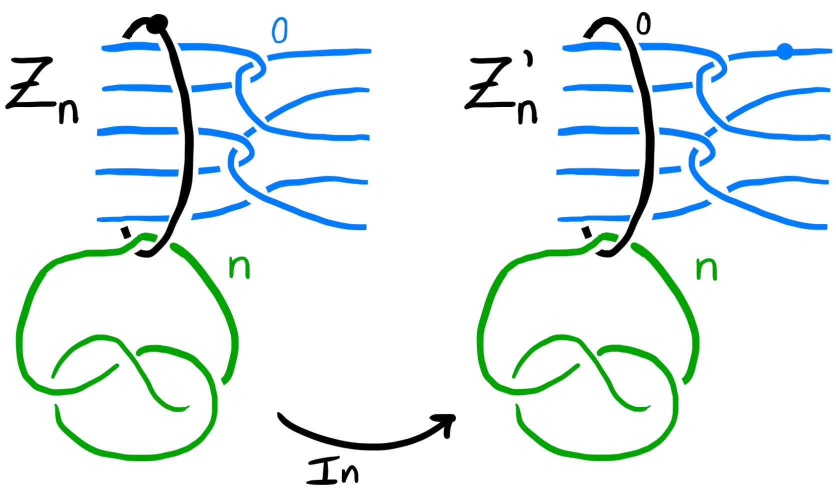
1. Build X, X' such that both are homotopy equivalent to S^2.
2. Build X, X' such that $X' = X_n(K)$ for some knot $K \subseteq S^3$.

3. Build X, X' such that both are knot traces.

Proof : Recalling that both 1-handle and 0-framed 2-handle attachment have the same effect on the boundary, the first thought is to start with some link of two unknots where one component is 0-framed and the other is dotted; to get X', swap the zero and the dot. Evidently, $\partial X = \partial X'$. If the linking number of the two unknots is 1, then $\pi_1 = 1$ and the intersection forms don't exist since H_2 vanishes (for other linking numbers, we get nontrivial π_1). In fact, these manifolds are contractible by Whitehead's theorem since their homology and fundamental group vanish.

This trick, of building exotica starting from links of unknots, is due to Mazur (the number theorist).

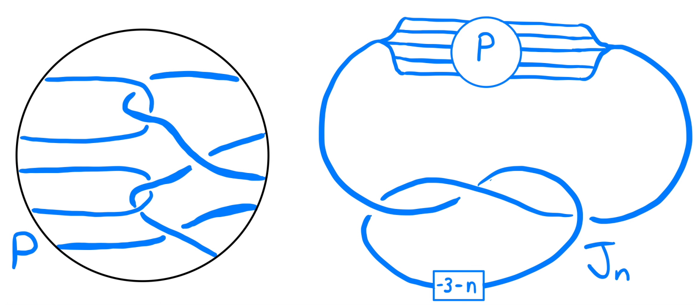
We can upgrade this answer to an answer for the first bonus by adding in another 2-handle; if we choose our attaching curve carefully, this will also be an answer to the second bonus. The third bonus is a little harder, and we'll leave it alone for now.



Ignoring the n-framed green trefoil on both sides, we have a link of unknots with algebraic linking number 1 but geometric linking number 5 (i.e. the minimum number of intersections between one of the unknots and the disc bounded by the other). Note that, in Z_n, the green and blue attaching curves both represent the homology class of the 1-handle (this is just a linking number calculation), so their difference in H_1 is a boundary. Alternatively, and as we will see below, Z_n is a knot trace, and slice $\cup$ core of that knot generates the second homology as usual.

In Z_n', the blue and black handles cancel homologically, so slice $\cup$ core of the trefoil generates H_2.

Z_n and Z_n' are both homotopy S^2s since we have added a single 2 handle to a contractible manifold. $Q_X = Q_{X'} = (n)$. and the boundary identification I_n is the obvious one given by superimposing the Kirby diagrams. Moreover, Z_n is a knot trace, which we can see by sliding all 5 strands of the blue curve over the green curve, and then cancelling green with the dotted circle, which results in the following knot:



We have drawn the five parallel strands of J_n as a single strand away from the "clasping region" represented by the pattern P.

Thus, $Z_n = X_n(J_n)$, where the framing is still n after five slides, since the strands of the blue curve will be oriented in opposing directions so that the result is a single "algebraic slide".

It's not obvious in either direction whether Z'_n is a knot trace. There are obstructions (beyond $H_2 = \mathbb{Z}$) to being a knot trace; for example, Levine-Lidman[3] use the d-invariants from Floer homology to obstruct the existence of a *spine*, i.e., a piecewise-linear embedding of S^2 into our candidate knot trace which induces a homotopy equivalence. All knot traces admit spines given by the union of the core disc and a cone on the knot (cones are PL but not smooth).

[3] Levine and Lidman, *Simply connected, spineless 4-manifolds*

So far, all we've built is a pair of 4-manifolds with the same boundary; to show that Z_n and Z'_n are an exotic pair, we first want a homeomorphism $F_n : Z_n \to Z'_n$ such that $F_n|_\partial = I_n$.

Lemma 2.1.8

For $X^4 \underset{\text{h.e.}}{\cong} S^2$, $Q_X = (n)$, the boundary long exact sequence is

$$\cdots \to \overset{\mathbb{Z}}{H_2(X)} \xrightarrow{\times n} \overset{\mathbb{Z}}{H_2(X,\partial)} \xrightarrow{\partial} \overset{\mathbb{Z}/n}{H_1(\partial)} \to 0$$

$H_2(X,\partial)$ is generated by any $\Gamma^2 \hookrightarrow X$ such that for $\Sigma \hookrightarrow X$ with Σ generating $H_2(X)$, $|\Gamma \cap \Sigma| = 1$ (i.e., an algebraic dual).

Note that Γ is not generally a closed surface, but a properly embedded one (meaning that its boundary sits in the ambient boundary).

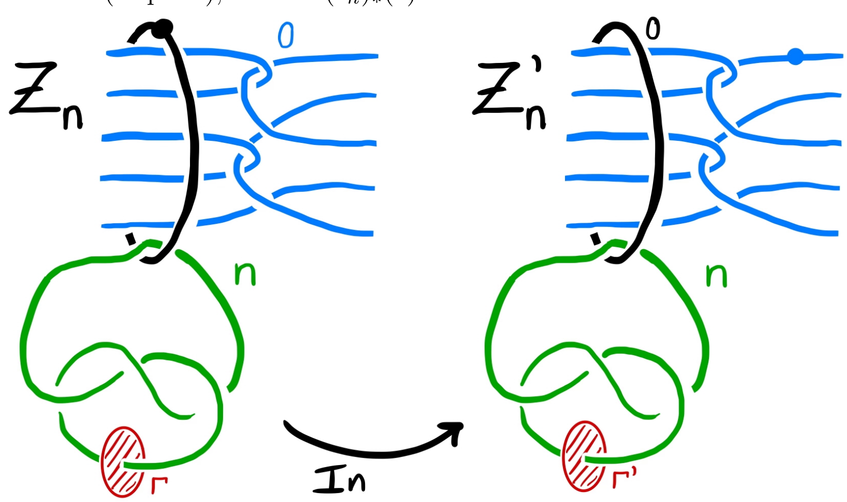
For $X \underset{\text{h.e.}}{\cong} S^2$ as above, there exists a preferred element of homology $\delta \in H_1(\partial)$ given by $\delta = [\partial(\Gamma)]$.

Theorem 2.1.9: Boyer '95

For $X, X' \underset{\text{h.e.}}{\cong} S^2$, $Q_X = Q'_X = (n)$, and $f : \partial X \to \partial X'$, for $n \neq 0$, f extends to a homeomorphism $F : X \to X'$ if and only if $f_*(\delta) = \delta'$. If $n = 0$, f extends iff $f_*(\delta) = \delta'$ and the (mod 2) surface framing on $K \subseteq \partial X$ representing δ is also preserved by f.

$f_*(\delta) = \delta'$ is necessary to make the diagram resulting from the map of (boundary) long exact sequences induced by F commute.

With Boyer's result in hand, we can check that I_n extends to a homeomorphism by selecting a meridional disc to the trefoil in each diagram as our surface Γ (resp. Γ'), whence $(I_n)_*(\delta) = \delta'$ is immediate:



It's mildly nontrivial that Γ as drawn is algebraically dual to the generator of $H_2(Z_n)$ (given by the knot J_n) — the key point is that the blue curve will slide over green algebraically once, so it should pierce the disc Γ algebraically once.

Finally, to establish that $Z_n \underset{\text{sm}}{\not\cong} Z'_n$, we have the following:

Lemma 2.1.10

For all n, there exists $T^2 \hookrightarrow Z'_n$ generating H_2.

This is just recognition of the fact that the 4-ball genus of the trefoil is 1 (and the trefoil generates $H_2(Z'_n)$). To finish the proof, we will use adjunction and trichotomy for the ν-invariant:

This chain of reasoning shows that Z_n and Z'_n are *absolutely* exotic, since any diffeomorphism between them can only carry a spin^c structure $\mathfrak{s}_s$ to $\mathfrak{s}_s$ or $\mathfrak{s}_{-s}$, and our condition on $\mathfrak{s}$ related to the ν invariant only depends on $|s|$.

- Adjunction then tells us that if $F^\circ_{X^\circ,\mathfrak{s}} \neq 0$ then $|s| + n \leq 2g(\Sigma)$. Thus, for our use case, if $|s| + n > 2$ then $F^\circ_{(Z'_n)^\circ,\mathfrak{s}_s} = 0$.
- Meanwhile, for Z_n, a (very fast) computer calculation tells us that $\nu(J_0) = 3$, so, by Lemma 1.8.8, $\nu(J_n) \geq 3$ for all $n \leq 0$ (since we can only insert negative twists).
- The Trichotomy Lemma 1.8.5 then tells us that if $|s|+n < 6$, $\hat{F}_{Z_n^\circ,\mathfrak{s}} \neq 0$.
- Thus, if we can find compatible spin^c structures $\mathfrak{s}$ on Z_n and Z'_n such that $2 < |s| + n < 6$, $F^\circ_{(Z'_n)^\circ,\mathfrak{s}_s} = 0$ and $\hat{F}_{Z_n^\circ,\mathfrak{s}} \neq 0$.
- Suppose $Z_n \underset{\text{sm}}{\cong} Z'_n$ via a diffeomorphism G. Then the spin^c structures on Z_n are in bijection with those on Z'_n, so $G_*(\mathfrak{s}_s) = \mathfrak{s}_{\pm s}$ (since $H_2(Z_n, \partial) \cong H_2(Z'_n, \partial) = \mathbb{Z}$). Choosing s so that $|s|+n = 4$, we obtain a contradiction from the above line, so $Z_n \underset{\text{sm}}{\ncong} Z'_n$. ■

Lecture 11: February 20th

I asked whether an alternate proof of $Z_n \underset{\text{sm}}{\ncong} Z'_n$ might be given by just calculating the 4-ball genus (denoted g_4) of J_n and showing that it is greater than $1 = g_4(\text{LHT})$ — this almost works, and led to the discussion of shake genus.

Definition 2.1.11: Shake Genus

The *n-shake genus* $g^n_{\text{sh}}(K)$ of K is the minimum genus of a closed surface $\Sigma^2 \hookrightarrow X_n(K)$ generating the second homology.

Observe that $g^n_{\text{sh}}(K) \leq g_4(K)$ since any slice surface for K can be capped off by the core disc to give a surface generating the second homology, but it's possible that we could do better by taking a surface that passes over the core disc geometrically several times but algebraically once.

Exercise 2.1.12

Come up with a diagrammatic definition of $g^n_{\text{sh}}(K)$.

Exotic Contractibles

Theorem 2.1.13: Akbulut '91, Akbulut-Ruberman '14[4]

There exist exotic (both relative and absolute) contractible 4-manifolds.

[4] Akbulut and Ruberman, *Absolutely exotic compact 4-manifolds*; Akbulut, *An exotic 4-manifold*; Yasui, *Corks, exotic 4-manifolds and knot concordance*

Remark 2.1.14

This is almost a disproof of the smooth Poincaré conjecture, which is equivalent to the nonexistence of (absolutely) exotic B^4s. Unfortunately, the homotopy balls we will build will not have S^3 boundary.

Definition 2.1.15: Corks

A *cork* is a pair (C, C') of contractible 4-manifolds equipped with a boundary diffeomorphism $f : \partial C \to \partial C'$ which does not extend smoothly to a diffeomorphism $F : C \to C'$.

Theorem 2.1.16: Freedman '82

f as above extends to $F : C \to C'$ as a *homeomorphism.*

Freedman's proof that f extends (topologically) follows by gluing C to C' along f to form a closed 4-manifold W, which is a homotopy 4-sphere by Mayer-Vietoris and Whitehead. Thus, $W \underset{\text{top}}{\cong} S^4$ by the topological Poincaré conjecture (also due to Freedman) and therefore bounds D^5. Decomposing W as

$$W = C \cup_{\partial C \times \{0\}} \partial C \times [0,1] \cup_f C'$$

we may view D^5 as a relative h-cobordism between $(C, \partial C)$ and $(C', \partial C')$, and it follows from the topological h-cobordism theorem (due to Freedman) that this h-cobordism is (topologically) a cylinder, from which we may construct the desired extension F. h-cobordisms are defined formally below.

Theorem 2.1.17: Cork Theorem

Given any pair X, X' of smooth simply-connected 4-manifolds which are homotopy equivalent (not necessarily closed), there exists a cork (C, C') with $C \hookrightarrow X$ such that $(X\ C) \cup_f C' \underset{\text{sm}}{\cong} X'$.

Note that, despite being contractible, corks are not necessarily local (i.e., they don't necessarily embed as $C \hookrightarrow B^4 \hookrightarrow X$). (C, C') being a cork doesn't tell us whether any given embedding of C produces an exotic pair (X, X') — in particular, that f does not extend smoothly over the interior of C does not suffice to show that $X \underset{\text{sm}}{\ncong} X'$.

The cork theorem is due to Curtis, Freedman, Hsiang, Stong, and, separately, Matveyev, in 1995. Bižaca's name is sometimes included in this list, but I've never been able to find a relevant citation.

Definition 2.1.18: h-cobordisms

An *h-cobordism* $W : X_0 \to X_1$ is a cobordism such that the inclusion maps $\iota_i : X_i \hookrightarrow W$ are homotopy equivalences i.e. W deformation retracts onto either boundary component. If such a W exists, we say that X_0 and X_1 are *h-cobordant.*

Note that, by Whitehead, if $\pi_1(X_i) = 1$, then a homology cobordism (denoted an H-cobordism) is an h-cobordism.

Theorem 2.1.19: Smale '62, Freedman '82

For $n \geq 5$ in any category (smooth, PL, topological), or $n = 4$ in the topological category, any simply-connected h-cobordism $W : X_0 \to X_1$ is isomorphic (in its respective category) to a product $X_0 \times I$.

The $n \geq 5$ smooth case is due to Smale, extended to the other categories by various others, and the $n = 4$ case is due to Freedman. The h-cobordism theorem is *false* for smooth 4-manifolds, which follows (e.g.) by the existence of nontrivial corks.

Exercise 2.1.20

Suppose $W : X_0 \to X_1$ is a smooth h-cobordism between simply-connected 4-manifolds, and assume that W is built (relatively) from a unique 2-handle and a unique 3-handle. Describe $X_{\frac{1}{2}}$, the level after 2-handle attachment but before 3-handle attachment. If you were given X_0, X_1, and wanted to build an h-cobordism between them, what are some necessary conditions on the X_i for this to be possible? Are these conditions sufficient?

Bonus

Build an explicit h-cobordism from handles.

Proof : In the general case, where we have an equal number of 2 and 3-handles, the $5D$ 2-handles are attached along circles in the outgoing boundary of $X_0 \times I$, i.e., X_0, which is a simply-connected 4-manifold. Since X_0 is simply-connected, the attaching circles are all nullhomotopic, and since X_0 is 4-dimensional, homotopy implies isotopy, so the 2-handles are attached along disjoint unknots. Thus, $X_{\frac{1}{2}} = X_0 \#_n (S^2 \times S^2) \#_m (S^2 \tilde{\times}) S^2$ since there are two framings for 2-handles in this dimension, corresponding to $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$, and where $m + n$ is the number of 2-handles.

A result of Wall says that we may avoid $S^2 \tilde{\times} S^2$ summands so

$$X_{\frac{1}{2}} = X \#_{m+n} S^2 \times S^2$$

but I have no idea how the proof goes.

For X_0 and X_1 to have a shot at being smoothly h-cobordant, the X_i must at least be homotopy equivalent. ■

In fact, homotopy equivalence is all we need:

Theorem 2.1.21: Wall '64

Smooth, simply-connected, homotopy equivalent 4-manifolds X, X' are smoothly h-cobordant.

Morally, this result tells us that exoticness in dimension 4 comes from a "different place" than it does in higher dimensions. For example, Milnor's exotic 7-spheres are not h-cobordant, since this would make them diffeomorphic, whereas exotic pairs in dimension 4 *are* h-cobordant, we just can't straighten this h-cobordism out to a cylinder.

Lecture 12: February 25th

If we try to run the high-dimensional proof of the h-cobordism theorem in dimension 4, the standard tricks work to cancel the (5-dimensional) 1 and 4-handles, but we are unable to cancel the 2 and 3-handles. Because $W : X \to X'$ is an h-cobordism, we know that the 2 and 3-handles are equal in number, and *homologically* cancelling, but, in general, we won't be able to cancel excess intersections to upgrade this to geometric cancellation. However, we may use the fact that the nontriviality of the h-cobordism is isolated to the 2 and 3-handles to try to split W into two sub h-cobordisms, one of which contains all of the handles, and the other of which is trivial (i.e. a cylinder).

Proposition 2.1.22

There exists a sub h-cobordism $N \hookrightarrow W$ where N contains all the handles of W, such that $W \setminus N$ is a product, and $N \cap X$ and $N \cap X'$ are contractible.

$N \cap X$ and $N \cap X'$ will give us our cork, with the boundary identification coming from regarding the product h-cobordism $W \setminus N$ as a mapping cylinder.

Note that these nontrivial h-cobordisms also give us two natural notions of "complexity" for exotica:

2/3 pairs If there are no algebraically cancelling pairs of 2 and 3-handles, then the h-cobordism W is a cylinder; thus, the minimal number of such pairs between an exotic pair X, X' gives a natural notion of complexity. Looking at the middle level of W (and applying an aforementioned result of Wall), we obtain a diffeomorphism $X\#_k(S^2 \times S^2) \underset{\text{sm}}{\cong} X'\#_k(S^2 \times S^2)$ where k is the number of 2/3 pairs, so this notion of complexity is equivalent to the *stabilization distance* k between an exotic pair X, X'. A major open problem in this area for closed manifolds (referred to as "one is not enough") is whether we may take $k = 1$.

Excess Intersection Another feature of our nontrivial h-cobordisms is that the 2/3 pairs intersect algebraically once (which is necessary for W to be an h-cobordism) but geometrically some odd number of times; another natural notion of complexity is the minimum value of this excess geometric intersection. Morgan-Szabó[5] show that there exist h-cobordisms of arbitrarily large excess intersection; however, it is still open (as above) whether there exists an exotic pair X, X' such that *any* h-cobordism between them has larger than minimal complexity.

[5] Morgan and Szabo, *Complexity of 4-dimensional h-cobordisms*.

Explicitly, if A_i, B_j are the attaching and belt spheres of the 3 and 2-handles of the h-cobordism, the *complexity* is defined as the minimum over all h-cobordisms between a given exotic pair of the sum

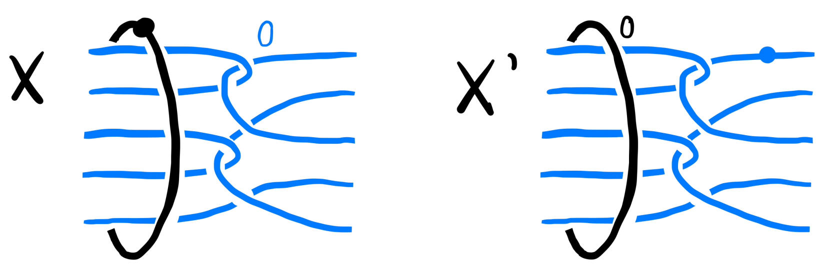
$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|A_i \cap B_j|\right) - n$$

where there are n 2/3 pairs, so the minimum complexity of a nontrivial h-cobordism is 2, achieved by $n = 1$ and $|A_1 \cap B_1| = 3$.

Theorem 2.1.23: Akbulut '92

There exist relatively exotic contractible manifolds.

PROOF : As usual, we begin with our candidates:



We are working relative to the obvious boundary identification $f : X \xrightarrow{\sim} X'$ that comes from the fact that both boundaries have identical Dehn surgery descriptions.

There are less complex exotic contractibles in the sense that the geometric linking number of these unknots is 5 and there are exotic contractibles with geometric linking 3 (the minimum). We're reviewing this particular example because the Thurston-Bennequin number turns out to give us bounds on ν that suffice to prove that $X \underset{\text{sm}}{\ncong} X'$ (really, this is why we chose this particular link of unknots when defining Z_n and Z'_n above).

By Theorem 2.1.16, f extends to a homeomorphism $F : X \to X'$, so all that's left is to show that f does not extend as a diffeomorphism G. Suppose such a G exists; we will build W, W' from X, X' by adding a 2-handle to each side along a 0-framed trefoil to obtain Z_0 and Z_0' respectively, since the boundary identification takes an attaching curve γ in ∂X to the same curve $f(\gamma)$ in $\partial X'$. We may extend G over the 2-handles by the identity and obtain a contradiction since we know that $Z_0 \underset{\text{sm}}{\ncong} Z_0'$ from arguments above.

This only gives us relatively exotic contractibles since our argument relies on $f(\gamma)$ being a particular curve in $\partial X'$, and that the trefoil is 0-framed is not actually important, we just need the induced framing on $f(\gamma)$ to agree with the framing on γ. ■

Note that relatively exotic is the best we can do for the given X and X' since the link in the Kirby diagram turns out to be symmetric in its two components, so in fact $X \underset{\text{sm}}{\cong} X'$.

Morally, our invariants some ambient topology (e.g., H_2) to exist, so we need to enlarge our contractibles by a 2-handle to obtain a contradiction. Note that this is not Akbulut's original proof, which used Donaldson invariants.

Surface Bundles

Definition 2.2.1: Surface Bundles

A *surface bundle* M^n is a fiber bundle $\Sigma^2 \hookrightarrow M^n \twoheadrightarrow B^{n-2}$ where π is a locally trivial surjection and B and M are smooth manifolds. Note that both B and Σ can have boundary.

For our purposes, we will be interested in $n \in \{3, 4\}$. In the $n = 3$ case, there are only two choices of base manifold, $B = I$ and $B = S^1$. $B = I$ is uninteresting since fiber bundles over a contractible base are trivial, so all such M are of the form $\Sigma \times I$. Fiber bundles over S^1 are just geometric realizations of a self-diffeomorphism of the fiber, i.e., mapping tori (this follows by the clutching construction), so the data of such bundles is just an element $\varphi \in \text{MCG}(\Sigma)$ (called the *monodromy* of the bundle).

The *clutching construction* is a method for constructing and/or prescribing the data of a fiber bundle over S^n; in particular, S^n has two standard charts given by the complements of (say) antipodal points. Any fiber bundle is trivial over each of these charts since they are contractible, so the data of the fiber bundle is given entirely by the transition map on the intersection which deformation retracts to the equatorial S^{n-1}, i.e., a map $\varphi : S^{n-1} \to \text{Aut}(F)$ where F is the fiber. It turns out that the fiber bundle only depends on the homotopy type of φ so the F-fiber bundles over S^n are in bijection with $\pi_{n-1}(\text{Aut}(F))$.

Let $\Sigma_{g,b}$ denote the genus g surface with b boundary components, then we have the following:

Theorem 2.2.2: Dehn

$\text{MCG}(\Sigma_{g,b})$ is generated by Dehn twists.

Note that this is a *monoid* (or at least pointed set) isomorphism (where the monoid structure on the set of fiber bundles is given by connect sum of base spheres), unlike the superficially similar *torsor* isomorphism we obtain for framings. In particular, the trivial map $S^n \mapsto \text{id}_F$ gives the trivial bundle $S^n \times F$, whereas there is not generally a natural choice of trivial framing.

Therefore, it is easy to build a fibered M^3 — pick a fiber $\Sigma_{g,b}$, and $\varphi \in \text{MCG}(\Sigma_{g,b})$ prescribed by a sequence of powers of curves on Σ. It is somewhat harder to determine the fiber and automorphism of a given fibered

3-manifold, and it is quite hard to decide whether an arbitrary 3-manifold is fibered (over a circle) at all.

There are a few ways to obstruct fiberedness of a 3-manifold — there are π_1 criteria due to Stallings and Neuwirth, and $\widehat{\text{HF}}$ obstructions. We will touch on the latter below.

Proposition 2.2.3: Murasugi

Let Y be the $\Sigma_{1,1}$ bundle over S^1 where $\varphi = \tau_b \circ \tau_a$ as in Figure 2.1. Then Y is diffeomorphic to $S^3 \setminus \nu(\text{LHT})$.

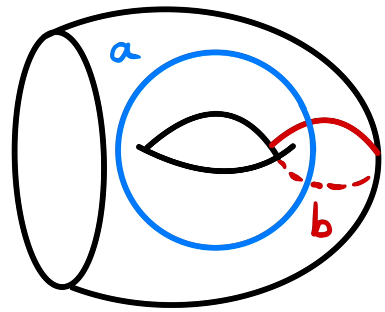
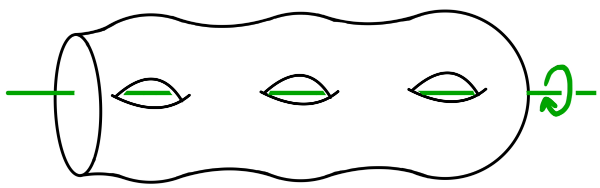


Figure 2.1: By convention, a positive Dehn twist is given by approaching the curve and then twisting along it to the left.

Braid Groups

Even though the mapping class groups of surfaces are well-understood, in some cases, we may simplify our life further by passing to the *braid group*; to motivate this, consider the *hyperelliptic involution* on $\Sigma_{g,\leq 2}$ is the following map, given by rotation through angle π about a central axis:



Exercise 2.2.4

Using the hyperelliptic involution, one can see $\Sigma_{g,1}$ as the double-branched cover of Γ: describe Γ and the branching set.

Bonus

In the case $g = 1$, let $\pi : \Sigma_{1,1} \to \Gamma$ be the quotient map by the hyperelliptic involution. Can you find a diffeomorphism f of Γ such that $\pi \circ \tau_b = f \circ \pi$ where τ_b is defined as above?

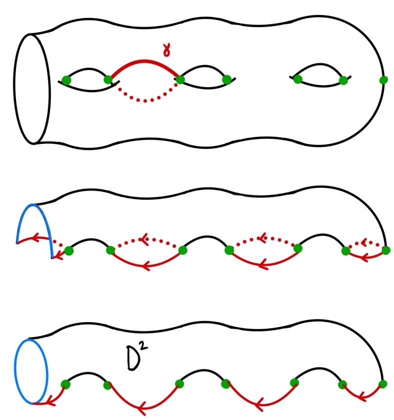
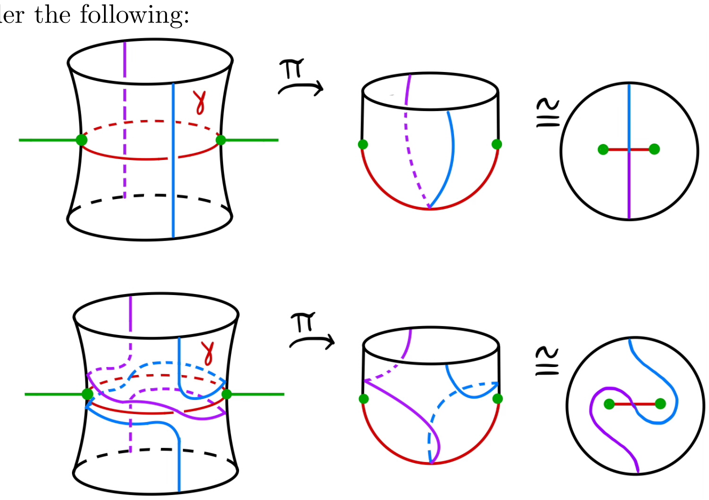


Figure 2.2: Finding the quotient $\Gamma = \Sigma_{g,1}/\sim$ by approximating a fundamental domain, then sewing up what's left.

PROOF : Γ is D^2, branching along $2g + 1$ points (see Figure 2.2). For the bonus, consider the following:



The effect of a Dehn twist upstairs in $\Sigma_{g,1}$ is to interchange the branch points in D^2; thus, any such f must swap these points. It turns out that this is a sufficient condition as well. ■

Lecture 13: February 27th

Let $\Sigma_{g,b,m}$ denote the genus g surface with b boundary components and m (labeled) marked points.

Definition 2.2.5: Braid Groups

The *m-strand braid group* B_m is defined to be the set of orientation preserving homeomorphisms of $\Sigma_{0,1,m}$ fixing the boundary pointwise and marked points setwise, up to isotopy fixing the boundary and marked points *pointwise*.

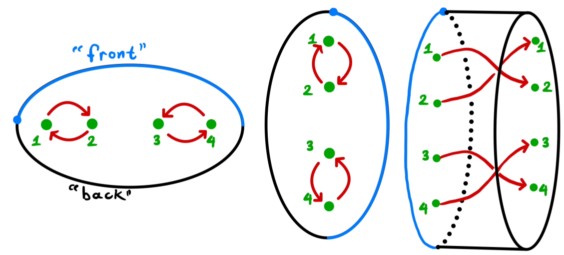
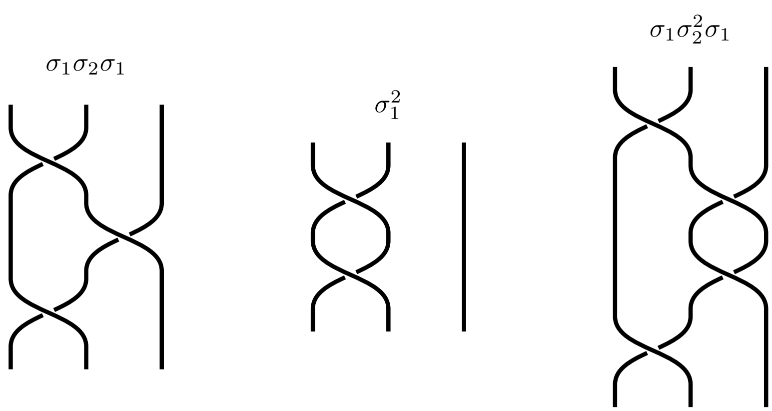


Figure 2.3: How to obtain an abstract braid from an automorphism of D^2 with some marked points.

Example 2.2.6: Some 3-Stranded Braids



Theorem 2.2.7: Artin ’23

B_m has the following presentation:

$$B_m = \langle \sigma_1, \cdots, \sigma_{m-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

The relation

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

written in R-matrix form shows up all over the place in physics as the *Yang-Baxter equation.* This relation is essentially a Reidemeister III move.

Theorem 2.2.8: Birman-Hilden ’73

Given $f \in \mathrm{MCG}(\Sigma_{g,1})$ such that $f \circ h \cong h \circ f$ where h is the hyperelliptic involution, f descends to a well-defined element of B_{2g+1}, giving rise to a map $\pi : \mathrm{MCG}^h(\Sigma_{g,1}) \to B_{2g+1}$ where MCG^h denotes the group of automorphisms commuting with the hyperelliptic involution h. Moreover, all $\varphi \in \mathrm{MCG}(\Sigma_{1,1})$ and $\mathrm{MCG}(\Sigma_{2,0})$ commute with h.

Therefore, we may specify, in some cases, punctured surface bundles over S^1 via braid group elements. For example, as in Proposition 2.2.3, $S^3 \backslash \nu(\mathrm{RHT})$ is the double branched cover of the mapping torus of $\sigma_2\sigma_1 \in B_3$ acting on D^2 with three marked points. The branching is over the knot K obtained by closing up the braid in the solid torus (see Figure 2.4).

The map we’ve written down here is only one part of a much deeper story; in particular, π fits into an explicit exact sequence containing more information about the mapping class group. The image of π in the case of an unpunctured surface such as $\Sigma_{2,0}$ is the *spherical braid group*, which has a presentation as above with the extra relation

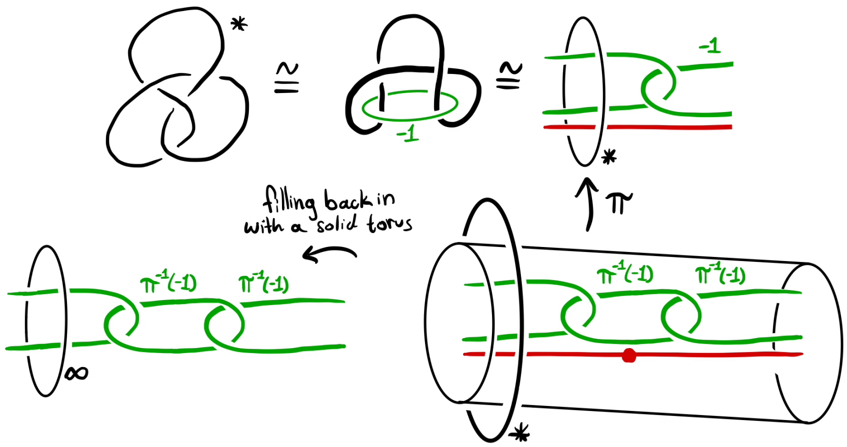
$$\sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_{n-1}\sigma_{n-2} \cdots \sigma_1$$

which corresponds to the ability to change crossings by pulling a strand along the back of the sphere (i.e. the light bulb trick).

Exercise 2.2.9

Describe the double branched cover of S^3 along 4_1.

PROOF : We may obtain a surgery diagram for this branched cover by first making our knot appear unknotted in the diagram, so that we are more easily able to see the branched double cover in the black curve's complement which is $S^1 \times D^2$ (this was referred to in class as the Rolfsen method):



■

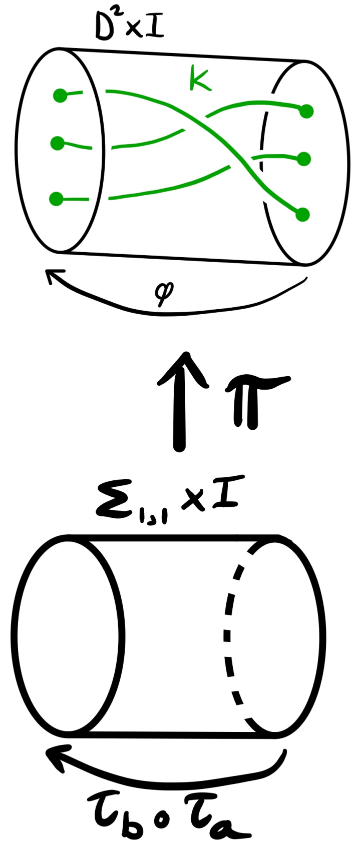


Figure 2.4: The branched cover of a mapping torus is the mapping torus of the branched cover.

Interlude on Floer Theoretic Calculations

Genus and Thurston Norm

Lecture 14: March 4$^{\text{th}}$

This lecture and the next were given by Tye Lidman.

Through examples, we want to explore how much HF$^\circ$ and HFK$^\circ$ know about knots and 3-manifolds.

Theorem 2.3.1: Ozsváth-Szabó, Ghiggini, Ni

Let $K \subseteq S^3$ and let s denote the Alexander grading:

1. $\widehat{\mathrm{HFK}}(K, s) = 0$ if $|s| > g(K)$
2. $\widehat{\mathrm{HFK}}(K, \pm g(K)) \neq 0$
3. $\widehat{\mathrm{HFK}}(K, \pm g(K)) = \mathbb{F} \iff K$ is fibered

The first result is not too hard to show, the second and third are deeper and factor through some hard results on taut foliations by Gabai.

So $\widehat{\mathrm{HFK}}$ detects both the genus and fiberedness.

Example 2.3.2

$\widehat{\mathrm{HFK}}(\mathrm{RHT}) = \mathbb{F}^3$ supported in (Maslov, Alexander) gradings $(m, s) = (0,1), (1,0), (2,1)$. Thus we can see that the genus of the right handed trefoil is 1, and that it is fibered. In fact, $\widehat{\mathrm{HFK}}$ is a RHT detector since there are only three genus 1 fibered knots: the two trefoils and 4_1 (this is classical).

HF° gives us analogous information about 3-manifolds, once we have developed the correct analogue for genus:

Definition 2.3.3: Thurston Norm

Let $\alpha \in H_2(Y^3)$, with $[\Sigma^2] = \alpha$ an embedded surface representative, possibly with several components Σ_i. Then define

$$\Theta(\Sigma) = \sum_{\Sigma_i \neq S^2} 2g(\Sigma_i) - 2$$

The *Thurston norm* of α is defined as $\Theta(\alpha) := \min_{[\Sigma]=\alpha} \Theta(\Sigma)$.

For formal reasons, we have to exclude spheres from the Thurston norm, as we can make $\Theta = -\infty$ by adding on arbitrarily many nullhomologous spheres with $\chi = -2$. However, this just begs the question why we use χ instead of g in the definition of the Thurston norm (we could also exclude nullhomologous surfaces from the sum), so probably something deeper is going on with the exclusion of spheres.

Theorem 2.3.4: Thurston

Θ is a semi-norm, i.e., $\Theta(k\alpha) = k\Theta(\alpha)$ for $k \in \mathbb{N}$ and $\Theta(\alpha+\beta) \leq \Theta(\alpha) + \Theta(\beta)$.

Θ fails to be a norm since homology classes represented by tori or spheres have 0 Thurston norm (but do not necessarily represent 0 in homology).

Example 2.3.5

Let $Y = T^3$. Then $\Theta([S^1 \times S^1 \times \bullet]) = 0$ (and similarly for the other two coordinate tori), so $\Theta : H_2(T^3) \to \mathbb{Z}_{\geq 0}$ vanishes identically by the triangle inequality, since the $\binom{3}{2}$ coordinate tori give a basis for $H_2(Y)$.

Example 2.3.6

Let $Y = \Sigma_g \times S^1$, then $\Theta([\gamma \times S^1]) = 0$ and $\Theta([\Sigma_g \times \bullet]) = 2g - 2$.

Note that the Thurston norm is not even a seminorm for 4-manifolds; one can see this from e.g. the Thom conjecture (now a theorem) which states that the genus function on $\mathbb{CP}^2$ is given by

$$g(d[\mathbb{CP}^1]) = \frac{(|d|-1)(|d|-2)}{2}$$

which violates the linearity condition.

Example 2.3.7

Let $Y = S^3_0(K)$ for some knot K, with $H_2(Y) = \mathbb{Z}$ generated by a capped off Seifert surface $\hat{\Sigma}$ (the 0-surgery gives us a natural disc to cap off the Seifert surface). Then $\Theta([\hat{\Sigma}]) = 2g(K) - 2$.

This example is another hard theorem of Gabai whose proof utilizes foliations.

Theorem 2.3.8: Ozsváth-Szabó, Ni

Let $\mathfrak{s} \in \mathrm{Spin}^c(Y) = 2H_1(Y)$, and fix $\alpha \in H_2(Y)$.

1. If $|\alpha \cdot \mathfrak{s}| > \Theta(\alpha)$ then $\mathrm{HF}^\circ(Y, \mathfrak{s}) = 0$
2. $\bigoplus_{\mathfrak{s}\cdot\alpha = \Theta(\alpha)} \mathrm{HF}^\circ(Y, \mathfrak{s}) \neq 0$
3. If Y is prime and α is represented by a connected surface Σ realizing $\Theta(\alpha) > 0$, then $\bigoplus_{\mathfrak{s}\cdot\alpha = \Theta(\alpha)} \mathrm{HF}^-(Y, \mathfrak{s}) = \mathbb{F}$ if and only if Y fibers with Σ as the fiber surface.

Comparing with the above theorem about HFK, we can see that the Thurston norm plays the same role here as the genus did for HFK.

Example 2.3.9

Consider $Y = S^3_0(\mathrm{RHT}\#\mathrm{LHT})$. $H_1(Y) = \mathbb{Z}$ so our spin^c structures are in bijection with $2\mathbb{Z}$. We have that

$$\mathrm{HF}^-(Y, \mathfrak{s}_{2i}) = \begin{cases} \mathbb{F}[U]/U & i = -1 \\ \mathbb{F}[U] \oplus \mathbb{F}[U] & i = 0 \\ \mathbb{F}[U]/U & i = 1 \\ 0 & \text{else} \end{cases}$$

Since genus is additive, $\Theta([\hat{\Sigma}]) = 2$ by the above example, where $\hat{\Sigma}$ is the capped off minimal genus Seifert surface. The above theorem then implies that Y is a fibered knot.

If we set $Y' = S^3_0(6_1)$ to be 0-surgery on the stevedore knot 6_1,

$$\mathrm{HF}^-(Y', \mathfrak{s}_{2i}) = \begin{cases} \mathbb{F}[U]^2 \oplus (\mathbb{F}[U]/U)^2 & i = 0 \\ 0 & \text{else} \end{cases}$$

$g(6_1) = 1$ so $\Theta = 0$, and it turns out that Y' is not fibered, though the above theorem is not sufficient to deduce this.

Tye remarks that Heegaard-Floer homology with twisted coefficients can detect fiberedness with fiber T^2 (i.e. $\Theta = 0$), and can in fact tell the difference between S^2 and T^2.

Example 2.3.10

Consider $Y = \Sigma_g \times S^1$ for $g \geq 1$, and let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. A generic spin^c structure $\mathfrak{s}$ is given by $\mathfrak{s} = [\gamma] + 2q[S^1]$ where $[\gamma] \in H_1(\Sigma_g)$. By the previous theorem, if $\mathrm{HF}^\circ(Y, \mathfrak{s}) \neq 0$ then $|\mathfrak{s} \cdot (\Sigma_g \times \bullet)| \leq 2g - 2$ and $\mathfrak{s} \times (\eta \times S^1) = 0$ for all $[\eta] \in H_1(\Sigma_g)$. The former condition implies that $2|q| \leq 2g - 2$ and the latter that γ is nullhomologous in Σ_g so $[\gamma] = 0$. Thus, we may calculate $\mathrm{HF}^-(Y, (2g-2)[S^1]) = \mathbb{F}$ since Y is evidently fibered with fiber Σ_g, and the only spin^c structure $\mathfrak{s}$ satisfying $\mathfrak{s} \cdot [\Sigma_g] = 2g - 2$ is $(2g-2)[S^1]$.

The criterion on fiberedness also allows us to extend our previous proof in Exercise 1.7.4 of adjunction for spheres with self-intersection 0 to all $g \geq 1$ surfaces with self-intersection 0 — i.e., that if $F^\circ_{W,\mathfrak{s}} \neq 0$ where W is a cobordism, then any closed surface Σ of self-intersection 0 and $g \geq 1$ in the interior of W satisfies $|s| \leq 2g(\Sigma) - 2$.

PROOF : If we have Σ as above, self-intersection 0 implies that the tubular neighborhood of Σ in W is $\Sigma \times D^2$, with boundary $\Sigma \times S^1$. Factoring our cobordism as in Exercise 1.7.4, $F^-_{W,\mathfrak{s}}$ factors through $\mathrm{HF}^-(\Sigma \times S^1, \mathfrak{s}|_{\Sigma \times S^1})$, so if $F^-_{W,\mathfrak{s}}$ is nonzero, this group must be nonzero. However, if $|\Sigma \cdot \mathfrak{s}| = |s| > 2g - 2$, HF° vanishes (by the first statement in the above theorem), so we must have that $|s| \leq 2g - 2$. ■

The obstruction to us immediately extending this proof to a general proof of adjunction is a lack of control of the Thurston norm on arbitrary surface bundles over S^1.

The Surgery Exact Triangle

Lecture 15: March 6th

Theorem 2.3.11: Ozsváth-Szabó

Let $K \subseteq Y^3$ be a knot, then there exists an exact triangle

$$\begin{array}{ccc} \mathrm{HF}^\circ(Y) & \xrightarrow{F^\circ_{W_1}} & \mathrm{HF}^\circ(Y_\lambda(K)) \\ {\scriptstyle F^\circ_{W_3}}\nwarrow & & \swarrow{\scriptstyle F^\circ_{W_2}} \\ & \mathrm{HF}^\circ(Y_{\lambda+\mu}(K)) & \end{array}$$

where λ is a choice of framing for K, μ is a meridian of K, and the W_i are given by 2-handle attachments as specified in Figure 2.5. For HF^-, the triangle only holds after we tensor with $\mathbb{F}[[U]]$.

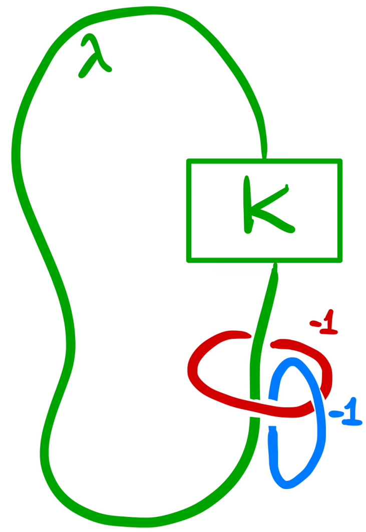


Figure 2.5: The surgery curves for the surgery exact triangle. W_1 is given by 2-handle attachment along the green curve, K, W_2 by -1-framed 2-handle attachment along the red meridian, and W_3 by -1-framed 2-handle attachment along the blue curve. The result of the final surgery is Y again since sliding red over blue makes the red curve's framing 0 and separates it from the blue curve (which is now -1 surgery on a disjoint unknot, which does nothing), and the (now) 0-framed meridian cancels the integral surgery on K.

Note that, as integers (if this makes sense in context), $\lambda + \mu = \lambda + 1$. Attaching a -1-framed meridian to K has the same effect as increasing the framing of K by one since we can slide K over the meridian to make them disjoint, which increases λ by one (and we may blow down the disjoint -1-framed unknot).

Example 2.3.12

Recall from Example 1.6.1 that $\mathrm{HF}^-(S^3) = \mathbb{F}[U]$ and $\widehat{\mathrm{HF}}(S^3) = \mathbb{F}$. We want to calculate $\widehat{\mathrm{HF}}(\mathbb{RP}^3)$, using the fact that $\mathbb{RP}^3 = S^3_2(U)$. Set $Y = S^3$, $K = U$, $\lambda = 1$ so that $\lambda + \mu = 2$.

The first cobordism W_1 is from S^3 to 1-surgery on the unknot, which is $X_{-1}(U)^\circ = (\mathbb{CP}^2)^{\circ\circ}$. Since $[\mathbb{CP}^1]^2 = 1$, adjunction implies that $\hat{F}_{X_1(U)^\circ,\mathfrak{s}} = 0$ for all spin^c structures $\mathfrak{s}$, so our surgery exact triangle is as follows:

$$\begin{array}{ccc} \widehat{\mathrm{HF}}(S^3) & \xrightarrow{0} & \widehat{\mathrm{HF}}(S^3) \\ {\scriptstyle F^\circ_{W_3}}\nwarrow & & \swarrow{\scriptstyle F^\circ_{W_2}} \\ & \widehat{\mathrm{HF}}(\mathbb{RP}^3) & \end{array}$$

Thus, $\widehat{\mathrm{HF}}(\mathbb{RP}^3) = \mathbb{F}^2$.

Exercise 2.3.13

Compute $\widehat{\mathrm{HF}}(L(n,1))$ for all $n \geq 3$.

PROOF : Let $Y = S^3$ as above, and K the $n-1$-framed unknot. The map induced by $W_1 : S^3 \to S^3_{n-1}$ for any spin^c structure is 0 as above by adjunction since there is an evident sphere with self-intersection $n-1$. Thus, $\widehat{\mathrm{HF}}(L(n,1)) = \widehat{\mathrm{HF}}(L(n-1,1)) \oplus \widehat{\mathrm{HF}}(S^3) = \mathbb{F}^n$ by induction. ■

Example 2.3.14

Consider the surgery exact triangle arising from $Y = S^3$ and K the -1-framed unknot:

$$\cdots \to \widehat{\mathrm{HF}}(S^3) \to \widehat{\mathrm{HF}}(S^3_{-1}(U)) \to \widehat{\mathrm{HF}}(S^3_0(U)) \to \cdots$$

Note that $S^3_0(U) = S^1 \times S^2$ and $S^3_{-1}(U) = S^3$; we want to calculate $\widehat{\mathrm{HF}}(S^1 \times S^2)$. We cannot immediately win by adjunction as above since the -1-framing means that we don't have any positive surfaces. However, $\nu(U) = 0$ so by Lemma 1.8.5, $\widehat{F}_{X_{-1}(U)^\circ,\mathfrak{s}_i} = 0$ if $|i| > 1$ and $\widehat{F}_{X_{-1}(U)^\circ,\mathfrak{s}_i} \neq 0$ if $|i| = 1$ where i must be odd. Thus,

$$\sum_{i=2k+1} \widehat{F}_{X_{-1}(U)^\circ,\mathfrak{s}_i} = \widehat{F}_{X_{-1}(U)^\circ,\mathfrak{s}_1} + \widehat{F}_{X_{-1}(U)^\circ,\mathfrak{s}_{-1}} = 0 \in \mathbb{F} = \mathbb{Z}/2$$

i.e. two *explicitly* (not just potentially) nonzero maps sum to 0 in $\mathbb{F}$. Therefore, as above, $\widehat{\mathrm{HF}}(S^1 \times S^2) = \mathbb{F}^2$.

Since the ν invariant does not (as far as we know) tell us anything about $F^-_{W,\mathfrak{t}}$, we don't yet have enough information to extend the above calculation to $\mathrm{HF}^-(S^1 \times S^2)$; however, there is an additional exact triangle (i.e., the rows repeat) which we can use to sometimes bootstrap HF^- information from knowledge about $\widehat{\mathrm{HF}}$:

As above, and as with any exact sequence we might encounter involving HF^-, we need to tensor by $\mathbb{F}[[U]]$ in order for this to actually work. As above, we will ignore this.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{HF}^-(Y,\mathfrak{s}) & \xrightarrow{\times U} & \mathrm{HF}^-(Y,\mathfrak{s}) & \longrightarrow & \widehat{\mathrm{HF}}(Y,\mathfrak{s}) & \longrightarrow & \cdots \\
 & & \downarrow F^-_{W,\mathfrak{t}} & & \downarrow F^-_{W,\mathfrak{t}} & & \downarrow \widehat{F}_{W,\mathfrak{t}} & & \\
\cdots & \longrightarrow & \mathrm{HF}^-(Y',\mathfrak{s}') & \xrightarrow{\times U} & \mathrm{HF}^-(Y',\mathfrak{s}') & \longrightarrow & \widehat{\mathrm{HF}}(Y',\mathfrak{s}') & \longrightarrow & \cdots
\end{array}$$

Note that, forgetting the $\mathbb{F}[U]$ structure, this exact sequence implies that $\widehat{\mathrm{HF}} = \ker(U) \oplus \operatorname{coker}(U)$, i.e., HF^- determines $\widehat{\mathrm{HF}}$.

In the above, $W : Y \to Y'$ is a cobordism, $\mathfrak{s} = \mathfrak{t}|_Y$, and $\mathfrak{s}' = \mathfrak{t}|_{Y'}$. For our purposes, $W = X_{-1}(U) : S^3 \to S^3$. The above diagram is commutative, so when $\widehat{F}_{X_{-1}(U),\mathfrak{t}} = \mathrm{id}$, $F^-_{X_{-1}(U),\mathfrak{t}} : \mathrm{HF}^-(S^3) \to \mathrm{HF}^-(S^3)$ is forced to be the identity as well, since $\mathrm{HF}^-(S^3) = \mathbb{F}[U]$. When $\widehat{F}_{X_{-1}(U),\mathfrak{t}} = 0$, one can deduce from grading shift formulae (which we have not thus far discussed) that $F^-X_{-1}(U),\mathfrak{t}$ is multiplication by a power of U. Putting these two facts together (and generalizing slightly), we have the following blowup formula:

Proposition 2.3.15: Blowup Formula

Suppose W is a cobordism as above, and $W' = W\#\overline{\mathbb{CP}^2}$. Then

$$F^\circ_{W',\mathfrak{t}\#\mathfrak{s}} = \begin{cases} F^\circ_{W,\mathfrak{t}} & s = \pm 1 \\ F^\circ_{W,\mathfrak{t}} \times U^{\binom{|k|}{2}} & \text{else} \end{cases}$$

where $s = 2k + 1$.

Without calculating the precise power of U by which F^- acts, we can see that the sum of the F^- maps is zero by noting that the degree shift formula depends only on $c_1(\mathfrak{t})^2$, so conjugate spin^c structures have the same degree shift, and their sum cancels out over $\mathbb{F}[U]$.

Thus, returning to our example, $F^-_{X_{-1}(U)^\circ,\mathfrak{s}_i} = \times U^{\binom{|k|}{2}}$ where $i = 2k + 1$. The sum of such maps over i cancels out by symmetry, so by the surgery exact triangle, $\mathrm{HF}^-(S^1 \times S^2) = \mathbb{F}[U] \oplus \mathbb{F}[U]$ as above.

Technically, all we know is that

$$\mathrm{HF}^-(S^1 \times S^2) \otimes_{\mathbb{F}[U]} \mathbb{F}[[U]] \cong \mathbb{F}[[U]] \otimes_{\mathbb{F}[U]} \mathbb{F}[[U]]$$

and it's not obvious to me that algebra alone then implies that $\mathrm{HF}^-(S^1 \times S^2) = \mathbb{F}[U] \oplus \mathbb{F}[U]$.

Invariants for Closed Exotica

Lecture 16: March 11th

Tye and Lisa tag-teamed today's lecture. Tye took the Floer theory, Lisa took the rest.

Recall that if X is a closed 4-manifold, we cannot obtain any interesting information from regarding $X^{\circ\circ}$ as a cobordism by Theorem 1.6.5. Moreover, it turns out that if $W : Y \to Y'$ has $b_2^+(W) > 0$, $\mathrm{Im}(F^-_{W,\mathfrak{t}}) \subseteq \mathrm{HF}_{\mathrm{red}}(Y')$ i.e. the image of the cobordism map is torsion, so for closed 4-manifolds with $b_2^+ > 0$, $F^-_{X^{\circ\circ},\mathfrak{t}}$ lands in $\mathrm{HF}_{\mathrm{red}}(S^3) = 0$ so is in fact 0. With these nonexistence results in mind, we are finally ready to define nontrivial Floer invariants of closed 4-manifolds.

Mixed Invariants

For $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, there exists a non-degenerate pairing

$$\langle -, - \rangle : \mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{s}) \otimes_{\mathbb{F}[U]} \mathrm{HF}_{\mathrm{red}}(-Y, \mathfrak{s}) \to \mathbb{F}$$

(which we will not define here). In the simplest possible case, if $\mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{s}) = \mathbb{F}$ (e.g., if Y is fibered and $\mathfrak{s}$ is as in Theorem 2.3.8) then $\langle -, - \rangle : \mathbb{F} \otimes \mathbb{F} \to \mathbb{F}$ is very simple, given by $\langle 1, 1 \rangle = 1$ (and all other pairings are automatically trivial).

This mixed invariant is a special case of the *Ozsváth-Szabó invariant* of 4-manifolds, that is conjecturally equal to the Seiberg-Witten invariant (perhaps after reduction (mod 2)). The case of the invariant that we have defined corresponds to the case where the virtual dimension of the Seiberg-Witten moduli space is 0 (perhaps assuming the Seiberg-Witten simple type conjecture). In the Floer setting, the virtual dimension appears (conjecturally?) as grading shifts in the maps.

One can recover/define the Seiberg-Witten invariant via a pairing on *monopole* Floer homology, so Ozsváth-Szabó = Seiberg-Witten probably factors through monopole Floer = Heegaard Floer.

Definition 2.4.1: Admissible Cuts and Mixed Invariants

Let X be a closed 4-manifold with $b_2^+ \geq 3$. An *admissible cut* is a separating 3-manifold Y such that $X = X_1 \cup_Y X_2$ where $b_2^+(X_i) > 0$ and the induced map on spin^c structures $H^2(X) \to H^2(X_1) \oplus H^2(X_2)$ is injective (e.g., if Y is a rational homology sphere), i.e., spin^c structures on X are determined by their restrictions to the X_i.

Given an admissible cut and a spin^c structure $\mathfrak{t}$ on X, consider

$$F^-_{X_1^\circ,\mathfrak{t}|_{X_1}}(1) \in \mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{t}|_Y) \qquad F^-_{X_2^\circ,\mathfrak{t}|_{X_2}}(1) \in \mathrm{HF}_{\mathrm{red}}(-Y, \mathfrak{t}|_Y)$$

where we are evaluating F^- on $1 \in \mathrm{HF}^-(S^3) = \mathbb{F}[U]$. These elements pair to

$$\langle F^-_{X_1^\circ,\mathfrak{t}|_{X_1}}(1), F^-_{X_2^\circ,\mathfrak{t}|_{X_2}}(1) \rangle_{\mathrm{HF}_{\mathrm{red}}(Y,\mathfrak{t}_Y)} =: \Phi_{X,\mathfrak{t}}$$

where $\Phi_{X,\mathfrak{t}}$ is called the *mixed invariant*.

Theorem 2.4.2: Ozsváth-Szabó

Admissible cuts always exist, and $\Phi_{X,\mathfrak{t}}$ is independent of the choice of cut.

SKETCH : Let $\Sigma \subseteq X$ have $[\Sigma]^2 > 0$, then $X_1 = \nu(\Sigma)$, $X_2 = X \setminus X_1$ is an admissible cut. The second claim is much more difficult to show and is omitted. ■

Exercise 2.4.3

Suppose X_1, X_2 are closed with $b_2^+(X_i) > 0$ and $b_2^+(X_1) + b_2^+(X_2) \geq 3$. Then $\Phi_{X_1 \# X_2, \mathfrak{t}} = 0$ for all spin^c structures $\mathfrak{t}$.

PROOF : The ball we connect sum along gives us an admissible S^3 cut, which has no $\mathrm{HF}_{\mathrm{red}}$ so the mixed invariant must vanish. ■

This result (that Seiberg-Witten invariants of connect sums vanish) is more difficult to prove in the gauge theoretic setting. Briefly, if $X = X_1 \# X_2$, we look at the bicollar or neck $S^3 \times I$ along which the connect sum is performed, and perturb the metric to stretch this neck. The Weitzenböck formula then implies than any solutions of the Seiberg-Witten equations along the neck vanish, so a solution on X decomposes into separate solutions on the X_i. However, the dimension of the moduli space of solutions on X is one more than the sum of the dimensions of the moduli spaces of solutions on the X_i, which leads to a contradiction.

Exercise 2.4.4

If X is closed with $b_2^+(X) \geq 3$, then $\Phi_{X,\mathfrak{t}} \neq 0$ implies that there exists $\mathfrak{t}' \in \mathrm{Spin}^c(X \# \overline{\mathbb{CP}^2})$ such that $\Phi_{X \# \overline{\mathbb{CP}^2}, \mathfrak{t}'} \neq 0$.

PROOF : This follows from the first case of the blowup formula (Proposition 2.3.15). ■

Remark 2.4.5: Adjunction Redux

Note that we can reformulate adjunction in terms of mixed invariants: if $\Phi_{X,\mathfrak{t}} \neq 0$ and $\Sigma \hookrightarrow X$ with $\Sigma^2 \geq 0$, then

$$|\langle \mathfrak{t}, \Sigma \rangle| + \Sigma \cdot \Sigma \leq 2g(\Sigma) - 2$$

$|\langle \mathfrak{t}, \Sigma \rangle|$ is just another way of writing the integer $|t|$ associated to a spin^c structure as we defined it above Theorem 1.7.1.

The Lickorish Trick

Suppose we have $K \hookrightarrow \Sigma^2 \hookrightarrow Y^3$. The surface Σ gives rise to a preferred framing:

Definition 2.4.6: Surface Framings

The *surface framing* λ_Σ on K is the longitude of K such that $\lambda_\Sigma \hookrightarrow \Sigma$.

From this data, there are two 3-manifolds we are interested in: the first is given by Dehn surgery on K: $Y_1 = Y_{\lambda_\Sigma \pm 1}(K)$; the second is given by replacing a neighborhood of Σ by a mapping cylinder:

$$Y_2 = Y \setminus \nu(\Sigma) \cup_{\Sigma \times S^0} \Sigma \times I / (\Sigma \times \{0\} \sim_{\tau_K^\mp} \Sigma \times \{1\})$$

where τ_K is the Dehn twist along K.

Theorem 2.4.7: Lickorish, '64

$$Y_1 \cong Y_2$$

Exercise 2.4.8

Suppose that Y is fibered with fiber Σ and $K \hookrightarrow \Sigma \hookrightarrow Y$. Prove that $Y_{\lambda_\Sigma \pm 1}(K)$ is still fibered. What is the monodromy?

Bonus

If $[K] \neq 0 \in H_1(\Sigma)$, what can you say about the Thurston norm of $[\Sigma]$ in $Y_{\lambda_\Sigma}(K)$?

PROOF : Using the Lickorish trick, the first part is immediate. The new monodromy is the old monodromy together with an extra Dehn twist along K. For the bonus, note that surgering, for example, a standard H_1-generator in Σ_g reduces the genus by 1. In fact, as long as $[K]$ is primitive in $H_1(\Sigma)$ (which it must be since it is embedded and connected), there is a homeomorphism of Σ that takes K to some standard generator, so passing from Y to $Y_{\lambda_\Sigma}(K)$ decreases the genus of Σ by one, and therefore the Thurston norm of $[\Sigma]$ by two. ■

Assuming as in the above exercise that $[K] \neq 0 \in H_1(\Sigma)$, there exists a spin^c refinement of the surgery exact triangle:

This is a "spin^c refinement" in the sense that we are summing over a subset of the spin^c structures rather than all of them as in the original surgery exact triangle. I assume from the way this information has been presented that we can't, in general, produce a surgery exact triangle for single spin^c structure on each 3-manifold (assuming the appropriate notion of compatibility).

Theorem 2.4.9: spin^c-Refined Surgery Exact Triangle

Fix a connected surface $\Sigma \hookrightarrow Y^3$ with $K \subseteq \Sigma$ and $g(\Sigma) \geq 2$, and set $\sigma := \Theta([\Sigma])$ to be the Thurston norm of $[\Sigma]$ in Y, then there exists the following exact triangle:

$$\begin{array}{ccc} \mathrm{HF}^\circ(Y|\Sigma) & \xrightarrow{F^\circ_{W_1|\Sigma}} & \mathrm{HF}^\circ(Y_\lambda(K)|\Sigma) \\ {\scriptstyle F^\circ_{W_3|\Sigma}} \nwarrow & & \swarrow {\scriptstyle F^\circ_{W_2|\Sigma}} \\ & \mathrm{HF}^\circ(Y_{\lambda+\mu}(K)|\Sigma) & \end{array}$$

where

$$\mathrm{HF}^\circ(Z|\Sigma) := \bigoplus_{\mathfrak{s}\cdot\Sigma=\sigma} \mathrm{HF}^\circ(Z,\mathfrak{s})$$

and the cobordism maps are defined as

$$F^\circ_{W_i|\Sigma} := \sum_{\mathfrak{s}\cdot\Sigma=\sigma} F^\circ_{W_i,\mathfrak{s}}$$

Note that we are abusively referring to one surface Σ in three different manifolds, even though surgery may modify this surface (such as by decreasing its genus as we saw above).

The usual caveats of needing to tensor by power series to achieve exactness for the minus flavor apply.

Lemma 2.4.10

Let $K \hookrightarrow \Sigma \hookrightarrow Y$ with $g(\Sigma) \geq 2$, and Σ is Thurston norm minimizing in its homology class. Set $\lambda = \lambda_\Sigma - 1$, then $\mathrm{HF}^\circ(Y|\Sigma) \to \mathrm{HF}^\circ(Y_\lambda|\Sigma)$ is an isomorphism.

PROOF : It will suffice to prove that $\mathrm{HF}^\circ(Y_{\lambda+\mu}(K)|\Sigma) = 0$. Note that $\lambda+\mu = \lambda_\Sigma$, so by the bonus above, the Thurston norm of $[\Sigma]$ is decreased by 2 in $Y_{\lambda+\mu}(K)$, so the direct sum in the definition of $\mathrm{HF}^\circ(Y_{\lambda+\mu}(K)|\Sigma)$ is taken over spin^c structures $\mathfrak{s}$ satisfying $\mathfrak{s}\cdot\Sigma = \sigma = \Theta_{Y_{\lambda+\mu}(K)}([\Sigma]) + 2$. By Theorem 2.3.8, the HF^- groups vanish over such spin^c structures, hence the $\widehat{\mathrm{HF}}$ groups vanish as well, and we are done. ■

Surface Bundles and Lefschetz Fibrations

Lecture 17: March 13th

Let $F^2 \hookrightarrow M^4 \twoheadrightarrow B$ be a surface bundle over a surface B, with F, B closed and $\pi_1(M) = 1$. The long exact sequence of a fibration implies that $\pi_1(B) = 1$, i.e., $B = S^2$, so $\pi_2(S^2) \twoheadrightarrow \pi_1(F)$ (again by the long exact sequence), and the only way to obtain such a surjection for closed F is if $F = S^2$ so M is an S^2-bundle over S^2. There are two such bundles, $S^2\times S^2$, and $S^2\tilde{\times}S^2 = \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. In order to obtained more complicated simply-connected 4-manifolds via surface bundles, we need to generalize our approach.

Lefschetz fibrations are a particular type of singular surface bundles over surfaces. There are interesting simply-connected Lefschetz fibrations with nonvanishing invariants, and whose total spaces we can sew up into a closed manifold using some information from $\mathrm{MCG}(F)$. This is an improvement over trying to close up a generic handlebody since the mapping class groups of 3-manifolds are more complicated than the mapping class groups of surfaces (which are completely understood). The boundary of a Lefschetz fibration is still a 3-manifold, of course, but it inherits a fibered structure which will reduce the complexity of closing it up.

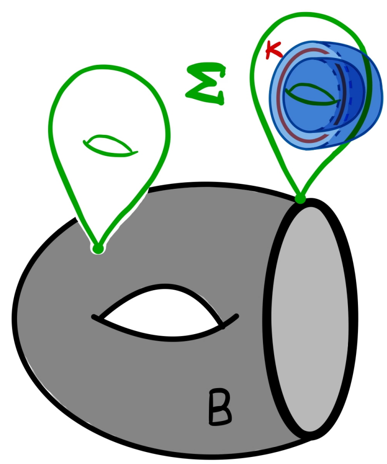


Figure 2.6: The first half of the construction of a Lefschetz fibration: attaching a 2-handle to a fiber above the boundary.

The construction of a Lefschetz fibration is as follows: let X be a surface bundle over a base B with nonempty (say, connected) boundary. Then $Y = \partial X$ is fibered over $S^1 = \partial B$. Thus, we may attach a $\lambda_\Sigma \pm 1$-framed 2-handle to Y along some knot K in the fiber Σ (see Figure 2.6); since the boundary is still fibered (by the Lickorish trick), this preserves the structure of a fiber bundle on $X' = X \cup$ 2-handle. The new boundary $Y' = Y_{\lambda_\Sigma\pm 1}(K)$ can be realized as the boundary of some other surface bundle over a base B', and we may obtain a closed 4-manifold by gluing these bundles together.

One may also think of Lefschetz fibrations as ordinary bundles over surfaces of genus g_1, g_2, glued together along a singular bundle over an annulus isolating the singular fiber(s).

Definition 2.5.1: Lefschetz Fibrations

A *Lefschetz fibration* is a singular surface bundle over a surface where all singularities are of "vanishing cycle type," (i.e., as above) and the 2-handle framings are all $\lambda_\Sigma - 1$. An *achiral Lefschetz fibration* allows both framings $\lambda_\Sigma \pm 1$.

Only allowing minus signs has to do with the fact that Lefschetz fibrations come from complex geometry; think about how connect summing an $\overline{\mathbb{CP}^2}$ is a valid operation on complex manifolds but connect summing with $\mathbb{CP}^2$ is not.

The generic fiber of a Lefschetz fibration is called *regular*, and the other fibers *singular*. The 2-handle attaching curves are called *vanishing cycles*.

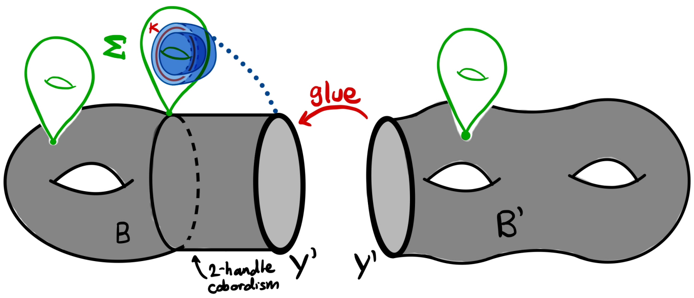


Figure 2.7: A completed (achiral) Lefschetz fibration with one singular fiber. The curve K is a vanishing cycle.

Theorem 2.5.2: Kas '80[6]

This definition agrees with the ordinary definition for Lefschetz fibrations.

[6] Kas, *On the handlebody decomposition associated to a Lefschetz fibration*

One ordinary definition for Lefschetz fibrations is a compact, connected, oriented, smooth 4-manifold X together with a map $f : X \to \Sigma$ where Σ is a surface (possibly with boundary), and such that each critical point of f has an orientation-preserving chart on which $f : \mathbb{C}^2 \to \mathbb{C}$ is given by

$$f(z, w) = zw$$

It turns out that for Lefschetz fibrations, $\pi_1(B) \hookrightarrow \pi_1(X)$ so in order for X to be simply-connected and closed, we are forced to take $B = S^2$. Moreover, for such X, $b_2^+(X) \geq 1$; this follows from the fact that the regular fibers of the Lefschetz fibration have self-intersection 0, together with non-degeneracy of the intersection form. Explicitly, if $0 \neq a \in H_2(X)$ and $a^2 = 0$, since the intersection form is non-degenerate, there exists $b \in H_2(X)$ such that $a \cdot b \neq 0$. Assume $a \cdot b = c > 0$ (since we may replace b by $-b$), and $b_2 = d$. Then

$$(ka + b)^2 = 2kc + d$$

which we can make positive by choosing k sufficiently large, so $b_2^+(X) > 0$.

Thus, if a non-trivial homology class has self-intersection 0, then $b_2^{\pm} \geq 1$. The prototype for this situation is $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ i.e. the intersection form of $S^2 \times S^2$.

Note that $B = S^2$ is not enough to guarantee that X is simply-connected (e.g. $X = S^2 \times T^2$); we obtained Lefschetz fibrations above by attaching 2-handles to surface bundles, so for X to be simply-connected, we need enough 2-handles (i.e. singular fibers) to kill π_1 of the generic fiber.

Another benefit of working with Lefschetz fibrations is that their mixed invariants are nonvanishing:

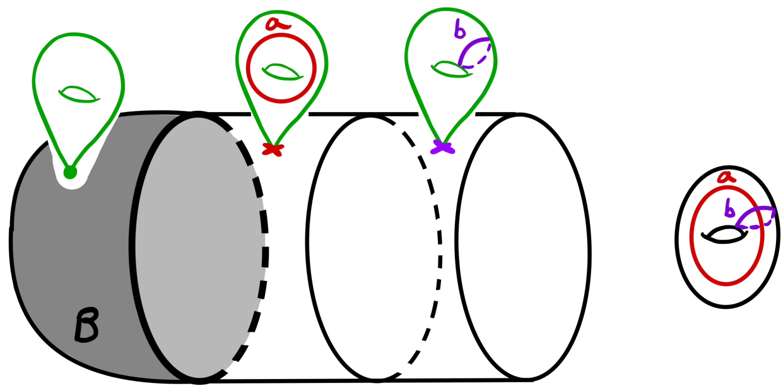
Theorem 2.5.3: Ozsváth-Szabó

If X is a closed Lefschetz fibration with $b_2^+ \geq 3$ then there exists $\mathfrak{t} \in \mathrm{Spin}^c(X)$ with $\Phi_{X,\mathfrak{t}} \neq 0$.

An analogous result holds with a slightly relaxed restriction on b_2^+ but we haven't defined mixed invariants for $b_2^+ < 3$.

Example 2.5.4

Let $X = D^2 \times T^2$, with two vanishing cycles a, b as shown:



Since the starting bundle X is trivial, we can think of the two vanishing cycles as living on the same T^2 fiber, as shown above right.

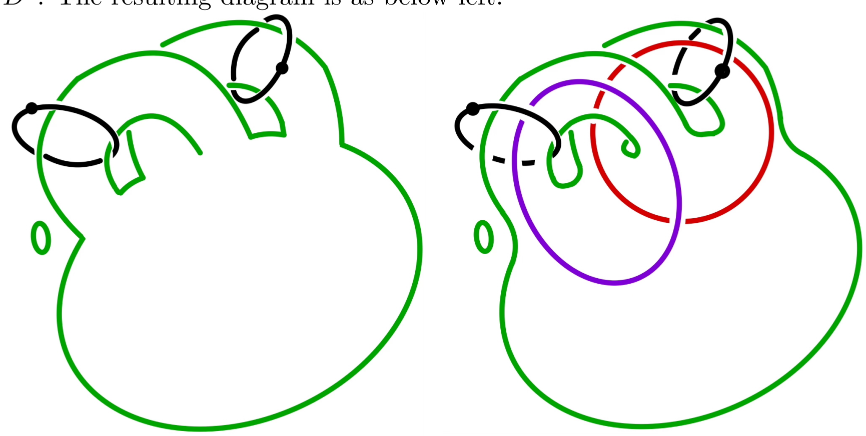
Exercise 2.5.5

Draw a handle diagram of $T^2 \times D^2$. Pick a curve a on T^2 and draw $a \times \{1\}$ in the diagram.

Bonus

Choose $b \neq a$ on T^2, and draw $b \times \{-1\}$ on your diagram.

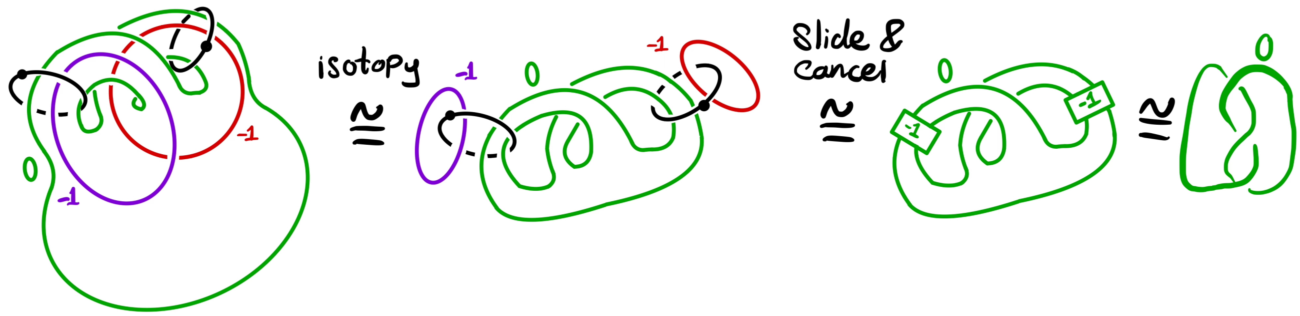
PROOF : Taking the standard handle decomposition of a torus, with a single 0-handle, two 1-handles (bands) and a single 2-handle, we automatically obtain a handle decomposition for $T^2 \times D^2$ by thickening all the handles by D^2. The resulting diagram is as below left:



For the bonus, see above right. Our curves a, b are as in the previous example, and we vary the second coordinate (i.e. $a \times \{1\}$ and $b \times \{-1\}$) so that the curves do not intersect in the diagram. ∎

Not clear to me how the second coordinate in e.g. $a \times \{1\}$ pins down the crossing information with respect to $b \times \{-1\}$.

Continuing our example from above, we may now produce a handle diagram for our Lefschetz fibration with two (-1-framed) singular fibers:



In the language of Lefschetz fibrations, a torus $S^1 \times S^1$ where both S^1 generators are vanishing cycles is called a *cusp singularity*. The singular fiber in this case is topologically a sphere. When only one generator is a vanishing cycle, the singular fiber is a *fishtail singularity*; these names correspond to the appearances of the corresponding singular elliptic curves.

Corollary 2.5.6

$S^3_0(\text{RHT})$ is fibered.

This follows immediately from the above since the boundary of a Lefschetz fibration is itself fibered.

Exercise 2.5.7

Show that RHT is fibered implies that $S^3_0(\text{RHT})$ is fibered.

This boils down to showing that the fibration extends over the surgery solid torus.

Once we have attached 2-handles to our vanishing cycles, we will have some boundary 3-manifold Y'. In order to obtain a closed 4-manifold, we need to build some other surface bundle with boundary Y'. Since we are interested in $B = S^2$ for fundamental group reasons, both B and B' (as in Figure 2.5) are D^2, so a surface bundle over B' is forced to be trivial. Thus, we want Y' to be a trivial bundle (with fiber F) over S^1. Since $B = D^2$, the bundle we started with (before attaching 2-handles) was trivial, so by the Lickorish trick, Y' is the result of some sequence of positive Dehn twists on $F \times S^1$.

Explicitly, Y' is the mapping torus of F by the automorphism $\circ_\alpha \tau_\alpha$ where α indexes our vanishing cycles. All our Dehn twists are positive since we will restrict ourselves to -1 framings. For Y' to be a trivial bundle, $\circ_\alpha \tau_\alpha$ must be isotopic to the identity on F, so our ability to close up a Lefschetz fibration is reduced to the following:

Lecture 18: March 25th

Question 2.5.8

When is a product of positive Dehn twists isotopic to the identity? Such products are called *positive factorizations of the identity*.

Example 2.5.9

Let $B = S^2$, $F = S^2$ with one singularity, attached along an unknot in S^2 (since no other knots lie on S^2). We may decompose the base into $B = B_L \cup_{S^1} A \cup_{S^1} B_R$ where the annulus A isolates the singular fiber, since $X|_{B_L}$ and $X|_{B_R}$ are both the trivial bundle $D^2 \times S^2$,

$Y := X|_A$ is necessarily a cobordism from $S^1 \times S^2$ to itself.

We may regard Y as the mapping torus of S^2 along some automorphism τ_γ specified by the singular fiber. However, the mapping class group of S^2 is trivial, so $Y = S^2 \times S^1$. We may draw a handle diagram for this Lefschetz fibration: before the attachment of the singular fiber, our total space is $S^2 \times S^2$, given by a 0-framed Hopf link. The -1-framed 2-handle is attached (along its vanishing cycle) on the surface of one of the spherical fibers, which we may regard as being the core disc of one of the 2-handles, hence, the vanishing cycle sits inside the disc bounded by one of the unknots, unlinked from both of them, which gives a handle diagram for $S^2 \times S^2 \# \overline{\mathbb{CP}^2}$.

In fact, thinking along these lines, we can see that any closed Lefschetz fibration with base and (generic) fiber S^2 will be $S^2 \times S^2$ blown up some number of times.

Increasing the complexity slightly, let a, b be curves on T^2 as in Example 2.5.4. It is a fact (called the 2-*chain relation*) that $(\tau_b \circ \tau_a)^6 = \mathrm{id}$. Let $E(n)$ be the *elliptic surface* Lefschetz fibration with $B = S^2$, $F = T^2$, and $6n$ vanishing cycles along each of a, b (i.e. $6n$ cusp singularities). For example, K3 $= E(2)$.

The 2-chain relation generalizes to the *k-chain relation* in the mapping class groups of surfaces: in general, let $\alpha_1, \cdots, \alpha_k$ be circles in a surface Σ, and N a closed normal neighborhood of $\rho_1 \cup \cdots \cup \rho_k$. Then, for k even

$$(\tau_{\alpha_1} \cdots \tau_{\alpha_k})^{2k+2} = \tau_\delta$$

and for k odd

$$(\tau_{\alpha_1} \cdots \tau_{\alpha_k})^{k+1} = \tau_{\delta_1}\tau_{\delta_2}$$

where $\delta = \partial N$ in the former case and $\delta_1 \cup \delta_2 = \partial N$ in the latter. In our application above, we took the α_i to be the standard generators for the homology of the torus, so the boundary of the resulting neighborhood was nullhomotopic.

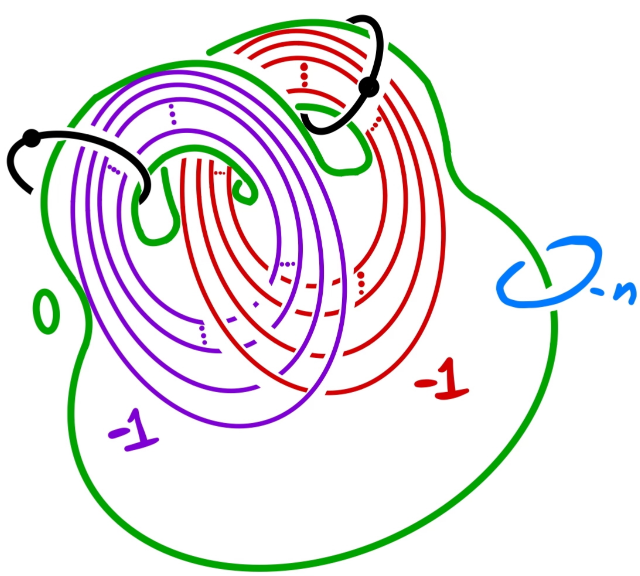
Exercise 2.5.10

Draw a handle diagram of $E(1)$.

Bonus

Compute $\pi_1(E(n))$ and the parity of $Q_{E(n)}$.

PROOF : As in Example 2.5.4, we start with vanishing cycles along the two 1-handles of T^2, and then continue placing them alternatively in essentially the same spot, resulting in following handle diagram:



The $-n$-framed 2-handle is required to close up the diagram (and two 3-handles and the 4-handle are omitted). $E(n)$ is simply-connected since Example 2.5.4 shows that two vanishing cycles suffice to kill the fundamental group. $E(n)$ is even iff n is even, though this is not obvious from our diagram. ■

This is framed as an exercise but there's a couple important (but standard) facts in the above that we don't show (i.e. that $E(n)$ is even and the presence of the $-n$-framed 2-handle).

Theorem 2.5.11: Matsumoto, Endo

If X is a Lefschetz fibration over S^2 with $F = T^2$, and with n non-separating and s separating vanishing cycles, then

$$\chi(X) = n + s \qquad \sigma(X) = -\frac{2}{3}n - s$$

Corollary 2.5.12

$$\chi(E(n)) = 12n \qquad \sigma(E(n)) = -8n$$

In particular, $\chi(E(1)) = 12$ and $\sigma(E(1)) = -8$ so $b_2^+(E(1)) = 1$, and our mixed invariants are not (yet) defined, so we have no hope of distinguishing $E(1)$ from $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$; worse still, there is a more fundamental reason why we cannot distinguish $E(1)$ from $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$: they are diffeomorphic.

We now have an explicit family of (eponymous) closed smooth 4-manifolds, and we know something about their algebraic topology. Our great fear is that these, too, will ultimately be diffeomorphic to a connect sum of boring manifolds (as $E(1)$ was). They are not:

Example 2.5.13

K3 $= E(2)$ is even. $b_2^+(E(2)) = 3$ and $b_2^-(E(2)) = 19$ but $E(2) \underset{\text{sm}}{\ncong} \#_3\mathbb{CP}^2\#_{19}\overline{\mathbb{CP}^2}$ since the latter is odd. Since indefinite quadratic forms are classified by their rank, signature, and type, we can determine that $Q_{E(2)} = -2E_8 \oplus 3H$. For $E(2)$ to split as a direct sum $X\#Y$, we would have $Q_{E(2)} = Q_X\#Q_Y$. Using a combination of Donaldson's theorem and Furuta's $\frac{10}{8}$-theorem, one can show that this is not possible, so $E(2)$ truly is irreducible.

Donaldson's theorem states that the definite intersection form of a compact, oriented, smooth 4-manifold is diagonalizable (with no restrictions on fundamental group). Furuta's $\frac{10}{8}$-theorem states that, if M is a closed, oriented, smooth, *spin* 4-manifold, then

$$b_2(M) \geq \frac{10}{8}|\sigma(M)| + 2$$

The clunky inclusion of $\frac{10}{8}$ rather than $\frac{5}{4}$ is in reference to the $\frac{11}{8}$-*conjecture*, that for M as above,

$$b_2(M) \geq \frac{11}{8}|\sigma(M)|$$

The conjectural bound is sharp if it is true, since equality is achieved by $E(2)$.

If we instead consider $E(2)\#\overline{\mathbb{CP}^2}$, whose intersection form is odd, by the classification of indefinite forms as above, $E(2)\#\overline{\mathbb{CP}^2} \underset{\text{top}}{\cong} \#_3\mathbb{CP}^2\#_{20}\overline{\mathbb{CP}^2}$. However, by Theorem 2.5.3 together Exercises 2.4.3 and 2.4.4, we may conclude that $E(2)\#\overline{\mathbb{CP}^2}$ has non-vanishing mixed invariants, while $\#_3\mathbb{CP}^2\#_{20}\overline{\mathbb{CP}^2}$ does not. Hence, we may conclude the following:

Theorem 2.5.14: Z is exotic

$X = E(2)\#\overline{\mathbb{CP}^2}$ is an exotic copy of $\#_3\mathbb{CP}^2\#_{20}\overline{\mathbb{CP}^2}$.

Lecture 19: April 1st

Having tasted the payoff, we now want to return to an unproven fact upon which the entire construction of $E(n)$ rests: the 2-chain relation $(\tau_b \circ \tau_a)^6 = \text{id}$. This relation is not derivable from the braid relations, Markov relations, or the spherical braid relation; rather, it arises from non-trivial automorphisms of the 3-manifold in which the braid lives. Developing the correct framework for understanding this identity will allow us to produce further identities, hence, further potentially interesting Lefschetz fibrations (in particular, with higher genus generic fiber).

Key to our discussion will be other instances of the Birman-Hilden correspondence: Theorem 2.2.8 covered the case for surfaces with a single boundary component, where we will now be interested in closed surfaces. Our setup is a surface Σ_g, with some automorphism $\prod_i \tau_{\alpha_i}$, such that $\sigma_{\alpha_i} \circ \pi = \pi \circ \tau_{\alpha_i}$ where $\pi : \Sigma_g \to S^2$ is the (branched) double covering map, and σ_{α_i} is the corresponding automorphism of the branch points of S^2 (see e.g. the figure in Exercise 2.2.4).

Let Y be a fibered 3-manifold, given by the mapping torus of $\varphi \in \text{MCG}(\Sigma)$. Unwrapping this to $\Sigma \times I$, we may use π to project down to $S^2 \times I$ where the branch points form a braid as in Figure 2.8.

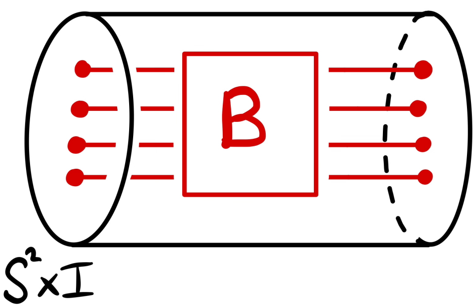


Figure 2.8: Braiding the branch points can capture the data of the progenitor automorphism φ.

Theorem 2.5.15: Birman-Hilden '73

If Σ_g is the closed surface of genus g with $g \leq 2$, then $Y \cong \Sigma \times S^1$ if and only if $(S^1 \times S^2, B) \sim (S^1 \times S^2, \text{id})$.

At present, this theorem has no content, as we have not yet defined the appropriate notion of equivalence for such braids; developing this notion is precisely our goal. Recall that the branched double cover of $S^1 \times S^2$ over B is defined as the standard branched cover on $\nu(B)$ and a double cover on $S^1 \times S^2 \setminus \nu(B)$ defined by a map $f : H_1(S^1 \times S^2 \setminus \nu(B)) \to \mathbb{Z}/2$.

By the letter of the law, f should be a map from π_1, but the target group $\mathbb{Z}/2$ is abelian, so any such map factors through the abelianization.

$H_1(S^1 \times S^2 \setminus \nu(B))$ is generated by the s_i, the meridian to the i^{th} strand (of which there are $2g+2$), together with t, which corresponds to the S^1 direction as in Figure 2.9. In order for the double cover produced by f to cohere with the branching action, we want $f(s_i) = 1$ and $f(t) = 0$ in $\mathbb{Z}/2$. Note that, depending on the braid in question, some of the s_i may in fact be equal, so our generating set may be redundant.

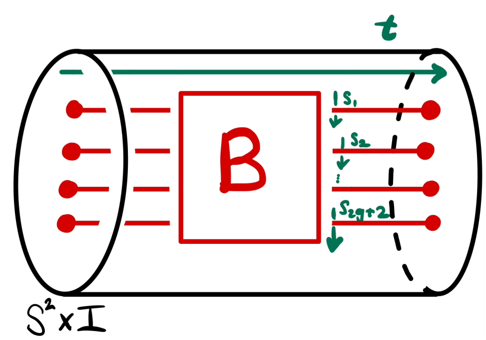


Figure 2.9: Generators for $H_1(S^1 \times S^2 \setminus \nu(B))$.

Proposition 2.5.16

Let M, M' be manifolds, with covering homomorphisms $\pi_1(M) \xrightarrow{c} G$, $\pi_1(M') \xrightarrow{c'} G$, and $\varphi : M \to M'$. Then φ lifts to the covers induced by c, c' if and only if $c' \circ \varphi_* \cong c$.

Corollary 2.5.17

If $\varphi : S^1 \times S^2 \setminus \nu(B) \to S^1 \times S^2 \setminus \nu(B')$ has $f \circ \varphi_* = f$ then the branched double covers along B and B' are diffeomorphic.

Thus, the appropriate notion of equivalence (which we will refer to as *Birman-Hilden equivalence*) for braids is such automorphisms φ. As we shall see, some trivial spherical braids are *not* BR-equivalent to the identity braid, and some nontrivial spherical braids are.

When a trivial spherical braid is not BR-trivial, the obstruction is that its action on H_1 is not as prescribed above. Conversely, when a nontrivial spherical braid is BR-trivial, this usually corresponds to some global diffeomorphism of the ambient 3-manifold, i.e., a Glück twist.

Exercise 2.5.18

What does $\tau_b \circ \tau_a$ as in Example 2.5.4 descend to as a spherical braid? What is the resulting mapping torus?

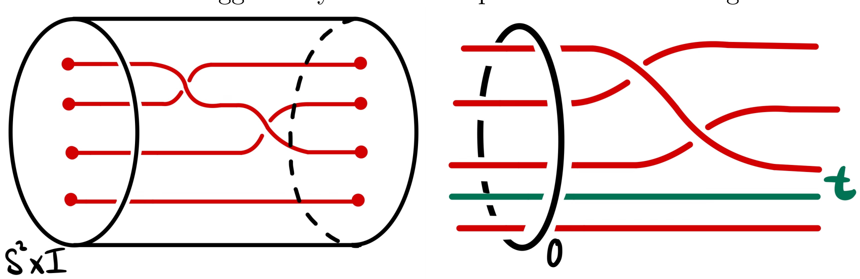
Bonus

$S^2 \times S^1$ has essentially a single nontrivial automorphism (which is of order 2), the *Glück twist*, which is given in coordinates by

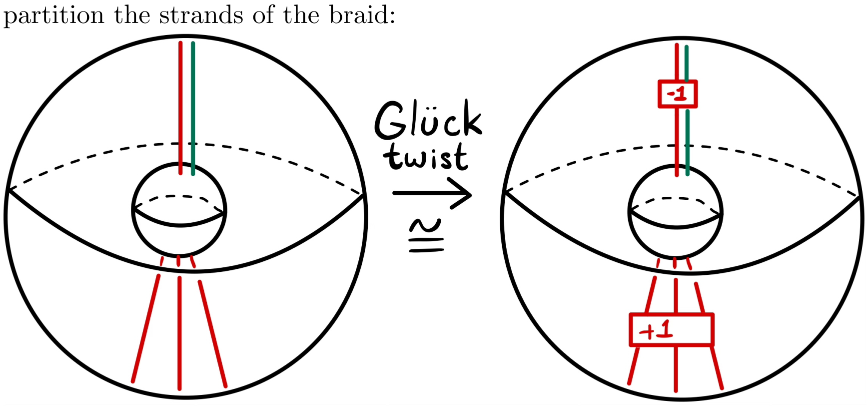
$$S^2 \times S^1 \ni (p, \theta) \mapsto (\mathrm{rot}_\theta(p), \theta)$$

where rot_θ refers to rotation through angle θ about some fixed axis. What is the effect of a Glück twist on a spherical braid?

Proof : Drawing the braid itself is straightforward (below left); using the standard surgery description of $S^1 \times S^2$, we obtain the drawing below right, where we have suggestively drawn a representative for the generator t.



For the bonus, we pass to yet another perspective on $S^2 \times I$, thought of as nested spheres. Then, the effect of a Glück twist is dependent on how we partition the strands of the braid:



We may abusively refer to the automorphism $\tau_{a_2} \circ \tau_{a_1}$ as $a_2 \circ a_1$ (and eventually $a_2 a_1$), as is common in the literature.

Twisting as shown adds a full twist to three of the strands and leaves the other strand alone. Now, note that $b \circ a$ is one third of a full twist on three strands, so we may draw $(ba)^6$ as in Figure 2.10, where we have chosen t' homologous to t, again suggestively.

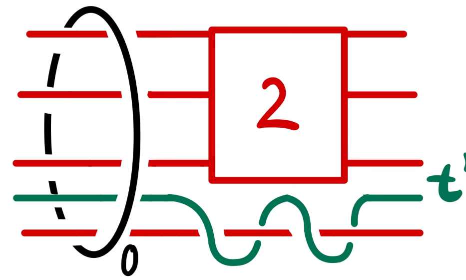


Figure 2.10: $(ba)^6$ is two full twists on three strands.

Then, partitioning the strands of the braid as above, and doing two Glück twists (the result of which is isotopic to the identity), the resulting braid is the identity braid, and t' has been untwisted to t (as drawn above).

To check via Corollary 2.5.17 that *the square of* a Glück twist φ has the correct lifting properties (without using the nonobvious fact that the Glück twist is an order 2 automorphism), note that $\varphi_*(s_i) = s_i$ and $\varphi_*(t) = t + 2s_4 = t \in \mathbb{Z}/2$. Thus, the lifting criterion is satisfied.

However, a single Glück twist with the strands partitioned as above is *not* an allowable automorphism in the sense of Corollary 2.5.17, since $t \mapsto t + s_4 \neq t$. In particular, this shows that Glück twists are a nontrivial automorphism of $S^2 \times S^1$.

Exercise 2.5.19

Consider the curves a_i on T^2 as drawn in Figure 2.11. Prove that

$$(a_3a_2a_1)^4 = \text{id} \text{ and } (a_1a_2a_3^2a_2a_1)^2 = \text{id}$$

Bonus

What are the fundamental group, Euler characteristic, and signature of the associated Lefschetz fibration?

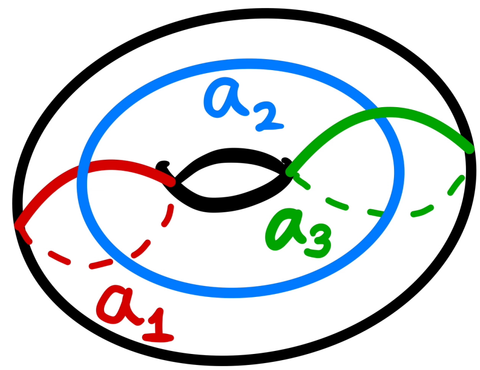


Figure 2.11: A redundant system of curves on T^2 whose Dehn twists correspond to generators of the braid group.

PROOF : The associated braid word for the first putative equality is

$$(\sigma_3\sigma_2\sigma_1)^4$$

which is a single full twist on 4 strands. A single Glück twist can undo this full twist, and maps the s_i to themselves, and t to $t + s_1 + s_2 + s_3 + s_4$. But the s_i are all homologous since the closure of this braid is a single circle, so $t \mapsto t + 4s_1 = t \in \mathbb{Z}/2$.

The second associated braid is

$$(\sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_1)^2$$

The parenthesized expression is the spherical braid relation: the given braid consists of the first strand sweeping over all the other strands, then returning to its original position behind all the other strands, and we can undo the given braid word by two Glück twists, and one can check that this induces $t \mapsto t$ as above.

That the second parenthesized expression is the spherical braid relation (i.e, is already trivial in the spherical braid group) but needs to be *squared* in order to be Birman-Hilden equivalent to the identity braid highlights an important point, which is that the set of allowed operations for BR-equivalence is *not* simply (braid group moves) + (diffeomorphisms of the 3-manifold).

Let Z_1, Z_2 respectively be the closed Lefschetz fibrations associated to the above braids. Using Theorem 2.5.11, we can calculate that $\chi(Z_i) = 12$ and

$\sigma(Z_i) = -8$ (there are no separating curves on a torus). $\pi_1(Z_i) = 1$ since we only need a_1 and a_2 vanishing cycles to kill the fundamental group of T^2. From this, we can calculate that $b_2^+(Z_i) = 1$ and $b_2^-(Z_i) = 9$. Thus, by the classification of indefinite odd lattices, $Z_i \underset{\text{top}}{\cong} \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}$. ■

We will later see that the above homeomorphism may in fact be upgraded to a diffeomorphism. It turns out that we can't get an exotic copy of $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}$ via Lefschetz fibrations alone; to do this, we will need subtler methods of modifying 4-manifolds. The manifold identified by either of the braid words in the above example is the elliptic surface $E(1)$.

Lecture 20: April 3rd

To recap, a recipe for producing potentially interesting (eponymous) closed smooth 4-manifolds is to write down an interesting braid in $S^1 \times S^2$ that is BR-equivalent to the identity. In ideal circumstances, this lifts to a product of Dehn twists on Σ_g whose composition is the identity. These data produces a Lefschetz fibration (by attaching -1-framed 2-handles along the Dehn twist curves) whose algebraic topology is readily calculable. Another useful feature of Lefschetz fibrations is that, by construction, they come with many explicit surfaces of known self-intersection.

In the above example, we built $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}$, which is large for our tastes. We want to develop a technique for modifying Lefschetz fibrations that preserves $\pi_1 = 1$ and $\Phi_{X,\mathfrak{t}} \neq 0$ for some $\mathfrak{t}$ (ignoring, for the moment, that mixed invariants as we have defined them are defined on $b_2^+ = 1$ manifolds).

Lisa mentions that Lefschetz fibrations were popular in the aughts and were generalized to other singular surface bundles over surfaces, eventually giving rise to *trisections*; a trisection (or *Threegaard splitting*) of a 4-manifold is a sort of generalized Heegaard splitting where any closed, smooth, connected, oriented 4-manifold can be written as the union of three copies of $\natural_k S^1 \times B^3$ intersecting pairwise in 3-dimensional handlebodies, and with triple intersection a closed orientable surface.

Torus and Knot Surgery

Suppose we have an embedded torus $T = S^1 \times S^1 \hookrightarrow X$ of self-intersection 0, so $\nu(T) = T \times D^2$, $\partial\nu(T) = S^1 \times S^1 \times S^1$ where the homology classes generating $H_1(\partial\nu(T))$ of the three torus factors are labeled (respectively) a_1, a_2, and μ. It is a fact that $\mathrm{MCG}(T^3) \cong \mathrm{SL}_3(\mathbb{Z})$, so for any $A \in \mathrm{SL}_3(\mathbb{Z})$ we may consider the manifold

$$X(T, A) = (X \setminus \nu(T)) \cup_A (S^1 \times S^1 \times D^2)$$

Note that we are intentionally writing $S^1 \times S^1$ rather than T^2 to emphasize that a factorization of the torus is part of our input data.

Exercise 2.6.1

Show that $X(T, A)$ is determined up to diffeomorphism by $A(\mu)$. Hint: show that Dehn surgery $(S^3 \setminus \nu(K)) \cup_\varphi \nu(K)$ is determined by $\varphi(\mu)$.

Bonus

What can you say about how σ, χ, b_2, and π_1 change under torus surgery?

PROOF : Since $X(T, A)$ is $X \setminus \nu(T)$ with $\nu(T)$ glued back in along some boundary twist, the trick is to look at a relative handle diagram for $\nu(T)$;. Such a diagram starts with a Dehn surgery diagram of T^3, and $\nu(T) = T^2 \times D^2$ has a simple handle diagram (as in Exercise **??**), consisting of a single 0-handle, two 1-handles, and a 0-framed 2-handle. Because we are considering a relative handle diagram of $\nu(T)$, this becomes a 0-framed 2-handle (whose attaching circle is the belt of the original 2-handle), two 3-handles, and a 4-handle. By Laudenbach-Poénaru, the 3-handles are attached uniquely, as is the 4-handle, so $X(T, A)$ is determined by the attachment of the 2-handle. As noted, the attaching circle for this 2-handle is the meridian to the original attaching curve, thus the image of the meridian under A determines $X(T, A)$.

For the bonus, torus surgery preserves χ and σ (and therefore preserves b_2 when X is simply-connected and closed). To see this, it is useful to consider the notion of a *round* 2*-handle*, which in our case is $S^1 \times D^2 \times D^2$ attached along $S^1 \times S^1 \times D^2$. If we attach a round 2-handle to $X \times \{1\} \subseteq X \times I$, the resulting manifold is a cobordism between X and $X(T, A)$. Since signature is a cobordism invariant, this proves that σ is unchanged under torus surgery. Moreover, a round 2-handle may be decomposed as the union of a 5-dimensional 2-handle and 3-handle, and the contribution of these 5-dimensional handles to the Euler characteristic of $X(T, A)$ cancel out.

In general, an n-dimensional *round* k*-handle* is $S^1 \times D^k \times D^{n-k-1}$ attached along $S^1 \times S^{k-1} \times D^{n-k-1}$. A round k-handle can in general be decomposed as the union of ordinary k and $k+1$-handles.

Torus surgery, in general, can change π_1; this should seem reasonable if one regards torus surgery as a higher dimensional analogue of Dehn surgery, as Dehn surgery often alters π_1. ■

Since the image of μ prescribes a torus surgery, we define the (r, s, p) torus surgery $X_{(r,s,p)}(T)$ to be $(X \setminus \nu(T)) \cup_A (S^1 \times S^1 \times D^2)$ where $A\mu = (r, s, p)$, otherwise known as a *logarithmic transformation* (this is the nomenclature of algebraic geometers). The coefficient p is called the *multiplicity* of the log transform, and is generally the most important coefficient.

The multiplicity should be thought of as analogous to the numerator of a Dehn surgery coefficient, in that it largely controls the algebraic topology of the surgered manifold.

Example 2.6.2: Torus Surgery can be Violent

Let C denote the cusp manifold as in Example 2.5.4, defined to be $S^1 \times S^1 \times D^2$, together with a vanishing cycle along each S^1 factor. In said exercise, we showed that $C = X_0(\text{RHT})$. Let T be a regular fiber of the given Lefschetz fibration structure on C, then $C' := C_{(0,1,0)}(T) \underset{\text{sm}}{\cong} Z \# \mathbb{CP}^2$ for some 4-manifold Z — the relevant bit is that the surgered manifold admits a $\overline{\mathbb{CP}^2}$ summand,

and, therefore, $C \underset{\text{top}}{\ncong} C'$.

That C' admits a $\overline{\mathbb{CP}^2}$ summand follows from the existence of a smoothly embedded sphere in C' of self-intersection -1 (a neighborhood of which is $(\overline{\mathbb{CP}^2})^\circ$). The key point is that the meridian μ is mapped to $(0,1,0)$ which is a vanishing cycle in C, so the sphere of self-intersection -1 is given by the union of a meridional disc bounded by μ with the core disc of the corresponding -1-framed 2-handle. These two discs technically live on different fibers of the Lefschetz fibration but can be connected by an annulus above a path between the fiber basepoints. Note that the same would be true if we instead performed $(1,0,0)$ surgery.

Thus, torus surgery can radically alter the algebraic topology of a given 4-manifold. Since we are being coached to regard torus surgery as an analogue of Dehn surgery, this should cause no alarm, as Dehn surgery is also capable of violence. When the multiplicity of a log transform is 1, the algebraic topology can survive unscathed:

Lemma 2.6.3

Given $T = S^1 \times S^1 \subseteq X$ satisfying the criteria for torus surgery, we may embed T in $X \setminus \nu(T)$ by taking a pushoff of the boundary (since T itself had self-intersection 0) via an embedding ι.

If $\iota_* : H_1(T) \to H_1(X \setminus \nu(T))$ is the zero map, then $H^*(X_{(r,s,1)}(T)) = H^*(X)$. Moreover, if $\iota_* : \pi_1(T) \to \pi_1(X \setminus \nu(T))$ is the zero map, then $\pi_1(X_{(r,s,1)}(T)) = \pi_1(X)$.

Multiplicity one log transforms can result in a manifold with the same algebraic topology as the starting manifold, analogous to $\frac{1}{n}$ Dehn surgery.

Exercise 2.6.4

Show that the hypotheses of the above lemma hold for the cusp manifold C, with T a regular fiber.

Bonus

Let F be the *fishtail manifold*, the Lefschetz fibration over a disc with T^2 fiber and a single vanishing cycle. Which of the $F_{(r,s,p)}(T)$ are simply-connected (with T a regular fiber as above)?

Proof : It suffices to prove that ι_* is the zero map on π_1. This follows immediately since both S^1 factors of the regular T^2 fiber are vanishing cycles in C.

For the bonus, note that $\pi_1(F)$ is nontrivial (in fact, it is equal to $\mathbb{Z}$) since a single vanishing cycle can only kill one of the generators of $\pi_1(T^2)$. Suppose a_1, a_2 represent the two S^1 factors in the regular fiber, and assume that the vanishing cycle is homotopic to a_1. For $F_{(r,s,p)}(T)$ to be simply-connected, we need a_2 to bound a disc in the surgered manifold.

For example, $F_{(0,1,1)}(T)$ is simply-connected since the surgery coefficients force a_2 to bound a disc. ∎

Evidently, we have not given a complete answer to the bonus. I suspect that trivial π_1 is impossible in the positive multiplicity case.

Proposition 2.6.5

For $X = E(n)$, T a regular fiber, $r, s \in \mathbb{Z}$, $X \underset{\text{top}}{\cong} X_{(r,s,1)}(T)$.

PROOF : The cusp manifold C embeds into $E(n)$ (explicitly, in the handle decomposition given for $E(n)$ above, we see C if we stop after the first two 2-handles), so we may surger a torus fiber in a neighborhood of the cusp fiber (i.e., C) inside $E(n)$. By the above exercise and lemma, this surgery does not change π_1 or H^* (since surgery is a local operation, we only need to check that it doesn't change π_1 or H^* locally). Thus, by Freedman, $X \underset{\text{top}}{\cong} X_{(r,s,1)}(T)$ (where we are using the fact that $\pi_1(E(n)) = 1$ since it has enough vanishing cycles). ∎

Audrick notes that the above in fact gives a proof that $(r, s, 1)$ log transforms do not change the homeomorphism type of any manifold in which $C = X_0(\text{RHT})$ embeds (provided that the torus being surgered can be taken to be the regular fiber of C).

Any time we apply Freedman to show that some operation preserves the homeomorphism type of a smooth 4-manifold, we should be excited that we have a candidate procedure for producing exotica. Unfortunately, we are foiled yet again:

Theorem 2.6.6: Gompf '91[7]

In the above setting, $X_{(r,s,1)}(T) \underset{\text{sm}}{\cong} X$ and, in fact, $C_{(r,s,1)}(T) \underset{\text{sm}}{\cong} C$ (hence, the result holds for any 4-manifold containing C as above). The latter diffeomorphism can be taken to be rel-∂.

[7] Gompf, *Nuclei of elliptic surfaces*, §2

In the pursuit of producing closed simply-connected exotica from genus 1 Lefschetz fibrations via torus surgery, this is pretty definitive, since for our starting Lefschetz fibration to be simply-connected, it necessarily contains a copy of C (we need vanishing cycles that kill both generators of $S^1 \times S^1$ in π_1). Undeterred, we may generalize this in a number of directions. The mildest generalization is to consider other surfaces, also satisfying $\Sigma \cdot \Sigma = 0$. It turns out that, for $g \geq 2$, every diffeomorphism of $\Sigma_g \times S^1$ extends over $\Sigma_g \times D^2$, so surgering such surfaces cannot change anything.

This leaves us only with spheres of self-intersection 0, and (by a mapping class group calculation) the only nontrivial operation on such spheres is the Glück twist. This is unhelpful for our approach to exotic Lefschetz fibrations, since we want to have nonvanishing mixed invariants, which guarantee (by adjunction) that there are no such nontrivial spheres.

Of course, one can also surger trivial (i.e. nullhomotopic) spheres (and surfaces more generally) but since such spheres live in some B^4, the result of such a surgery is in general a homotopy 4-ball, so trying to produce exotica this way is equivalent to the Poincaré conjecture (and therefore too hard to be profitable at current theorem prices).

The other obvious direction in which to generalize is to allow for surfaces of nonzero self-intersection. This has the cost of making $\partial\nu(\Sigma)$ more complicated than before. For $\Sigma = S^2$, $\Sigma \cdot \Sigma < 0$ (for adjunction reasons as above), $\partial\nu(\Sigma) = L(n, 1)$. By the classification of mapping class groups of lens spaces (due to Bonahon), there's no interesting regluings of $\nu(\Sigma)$. In particular, for any surface in any 4-manifold, $(X \setminus \nu(\Sigma)) \cup_\varphi \nu(\Sigma)$ is determined by $\varphi(\mu)$ as above (the proof given above generalizes), and Bonahon showed the following:

Theorem 2.6.7: Bonahon

$\varphi(\mu) = \mu$ for all $\varphi \in \text{MCG}(L(n,1))$.

For the remaining cases, $g(\Sigma) \geq 1$ with $\Sigma \cdot \Sigma \neq 0$, work of Haken and Waldhausen implies that the same is true, i.e., $\varphi(\mu) = \mu$ for all $\varphi \in \text{MCG}(\partial\nu(\Sigma))$. Thus, we have systematically eliminated all hope for our present strategy.

Fintushel-Stern Knot Surgery[8]

[8] Fintushel and Stern, *Knots, links, and 4-manifolds*.

Having exhausted the possibility of producing verifiable exotica (in our specific setup and utilizing mixed invariants) by taking out a surface and gluing it in in a different way, another direction to pursue is to delete a surface and glue *something else* in (with the same boundary, as with classical surgery).

Let T be a square zero torus as before, with a chosen trivialization $\nu(T) = S^1 \times S^1 \times D^2$. The starting point for Fintushel-Stern knot surgery is the observation that $S^1 \times D^2$ is the complement of the unknot, but the boundary of all knot exteriors is precisely $S^1 \times S^1$, so we may define

$$X_K(T) := (X \setminus \nu(T)) \cup_\varphi S^1 \times (S^3 \setminus \nu(K))$$

for any knot $K \subseteq S^3$. The S^1 factors of $\nu(T)$ are labeled $a_1, a_2, \partial D^2$, and the S^1 factors of $S^1 \times (S^3 \setminus \nu(K))$ are x, μ, λ where μ is the meridian and λ the Seifert longitude of K. In order to ensure that $X_K(T) \underset{\text{top}}{\cong} X$, we choose $\varphi \in \text{MCG}(T^3)$ such that $x \mapsto a_1$, $\mu \mapsto a_2$, and $\lambda \mapsto \partial D^2$.

In the literature, Fintushel-Stern knot surgery generally does not fully specify a gluing as we have above. To show that $X_K(T) \underset{\text{top}}{\cong} X$ via Freedman, it suffices to enforce that $\partial D^2 \mapsto \lambda$ together with the condition on φ in the lemma below, and the other two choices are left unspecified (and any such manifold is abusively denoted $X_K(T)$). Fintushel and Stern's calculation of the Seiberg-Witten invariants of $X_K(T)$ are insensitive to the choices of image of a_1 and a_2, though, recently, examples have been constructed where it is possible to show that these choices affect the smooth structures on the surgered manifold.

Lecture 21: April 8th

Lemma 2.6.8

$H^*(X) = H^*(X_K(T))$. Moreover, if $\varphi(\mu_K) \subseteq \partial\nu(T)$ has $[\varphi(\mu_K)] = 0 \in \pi_1(X \setminus \nu(T))$, then $\pi_1(X_K(T)) = \pi_1(X)$.

Proof : To show that $H^*(X) = H^*(X_K(T))$, pick a basis for $H_2(X)$ containing $[T]$. Assuming that $\pi_1(X_K(T)) = \pi_1(X)$ (which will be shown next), the operation of knot surgery naturally turns this into a basis for $H_2(X_K(T))$. T gets mapped to the torus generated by x and μ, and for any surface Σ intersecting T transversely, knot surgery on T replaces a neighborhood of each point of intersection on Σ with T with a copy of a Seifert surface for K. One can easily check in the basis given by T' and such surgered surfaces Σ' for $H_2(X_K(T))$ that the intersection form of $X_K(T)$ agrees with that of X:

- $[T']^2 = 0$ since we may leave x fixed and pick a parallel meridian μ to obtain a pushoff of T' that doesn't intersect it

- For $[\Sigma] \neq [T]$ in our basis,

$$[\Sigma] \cdot [T] = [\Sigma'] \cdot [T']$$

  This follows from the construction of the $[\Sigma']$, where each geometric point of intersection between Σ and T corresponds to a geometric point of intersection between Σ' and T'.

- For $[\Sigma], [\Gamma] \neq [T]$ in our basis,

$$[\Sigma] \cdot [\Gamma] = [\Sigma'] \cdot [\Gamma']$$

  One can arrange for the intersections between Σ and Γ to live away from any intersections with T, so that knot surgery does not affect these intersections at all.

Perhaps the moral of this gluing is that, since ∂D^2 bounds in the manifold pre-surgery, whatever curve it is mapped to should also bound if we want to have some hope of preserving any of our algebraic topology.

As with Dehn surgery and torus surgery, it is interesting that one of the surgery coefficients always seems to primarily (if not completely) control the algebraic topology of the surgered manifold. I wonder if the same can possibly be true in high dimensions when there are arbitrarily many coefficients.

To see that knot surgery preserves the fundamental group when the given criterion is satisfied, we apply van Kampen:

$$\pi_1(X) = \pi_1(X \setminus \nu(T)) *_{\pi_1(T^3)} \pi_1(T^2 \times D^2) = \pi_1(X \setminus \nu(T))/\langle\!\langle \partial D^2 \rangle\!\rangle$$

and, assuming $\varphi(\mu_K)$ vanishes in $\pi_1(X \setminus \nu(T))$,

$$\pi_1(X_K(T)) = \pi_1(X \setminus \nu(T)) *_{\pi_1(T^3)} \pi_1(S^1 \times (S^3 \setminus \nu(K))) = \pi_1(X \setminus \nu(T))/\langle\!\langle \partial D^2 \rangle\!\rangle$$

since $\pi_1(S^3 \setminus \nu(K))$ is normally generated by the meridian μ_K, so the two groups are isomorphic. ■

Tye takes over here to discuss more Floer homology, towards the beautiful statement that the mixed invariants of $X_K(T)$ are determined by the mixed invariants of X and the Alexander polynomial $\Delta_K(t)$ of the knot K.

A Further Interlude on Floer Theoretic Calculations

Lefschetz Fibrations have Nonvanishing Mixed Invariants

Far above, in Theorem 2.5.3, we claimed that the mixed invariants of Lefschetz fibrations are nonvanishing (i.e., there exists at least one spin^c structure for which $\Phi_{X,\mathfrak{t}} \neq 0$). We now wish to prove this fact, starting with Lefschetz fibrations whose fiber genus is ≥ 2 for technical reasons.

Recall from Theorem 2.4.9 that

$$\mathrm{HF}^-(Y|\Sigma_g) := \bigoplus_{\mathfrak{s} \cdot \Sigma_g = 2g-2} \mathrm{HF}^-(Y, \mathfrak{s})$$

(since the Thurston norm of a genus g surface is, by definition, $2g-2$). By Theorem 2.3.8, $\mathrm{HF}^-(Y|\Sigma_g) = \mathbb{F}$ if and only if Y fibers over S^1 with fiber Σ_g — provided that $g > 1$, and that Σ_g is non-separating (Which replaces our previous requirement that Y be prime).

Using the refined surgery exact triangle given in Theorem 2.4.9, induced by a knot K lying on Σ_g inside Y (this also requires that $g \geq 2$), we may apply Lemma 2.4.10 to obtain an isomorphism $\mathrm{HF}^-(Y|\Sigma_g) \cong \mathrm{HF}^-(Y_\lambda|\Sigma)$ where λ_Σ is the surface framing of K on Σ.

In this setting, one can prove something slightly stronger (apart from the genus restriction) than what is claimed in Theorem 2.5.3:

Theorem 2.7.1: LF $\implies \Phi \neq 0$, v2

If X is a closed Lefschetz fibration with $b_2^+(X) \geq 3$ and generic fiber Σ_g, $g \geq 2$, then

$$\Phi_{X|\Sigma} := \sum_{\mathfrak{t}\cdot\Sigma = 2g-2} \Phi_{X,\mathfrak{t}}$$

is nonzero.

The condition on b_2^+ can also be relaxed as before, at the cost of technical difficulty in the definition of $\Phi_{X,\mathfrak{t}}$.

Recall that the mixed invariants $\Phi_{X,\mathfrak{t}}$ are defined in terms of an admissible cut along some 3-manifold Y, and a pairing on $\mathrm{HF}_{\mathrm{red}}(Y)$. The idea we have to prove for the above result — restricting ourselves, for now, to the case where the base of the Lefschetz fibration is S^2 — is to split up the base into two discs and several annuli, one for each vanishing cycle of the fibration. Since $\Sigma_g \times S^1$ is evidently fibered, $\mathrm{HF}^-(\Sigma_g \times S^1|\Sigma_g) = \mathbb{F}$. A crucial fact that we will apply repeatedly below is the following:

Proposition 2.7.2

$$F^-_{\Sigma_g \times D^2|\Sigma_g}(1) \neq 0 \in \mathbb{F} = \mathrm{HF}^-(\Sigma_g \times S^1|\Sigma_g)$$

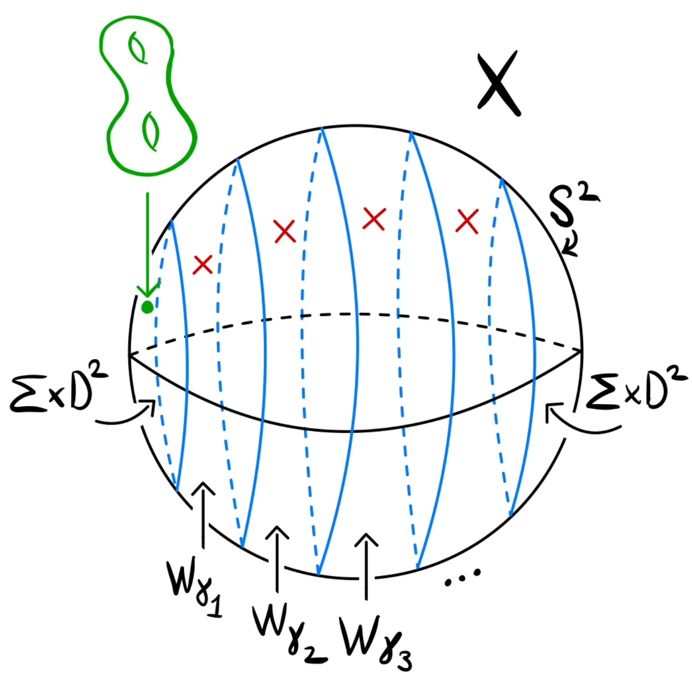


Figure 2.12: Breaking down a Lefschetz fibration X above two discs and many annuli. Each cobordism W_{γ_i} isolates a single vanishing cycle. The pairing associated to such a sequence of cuts (as in the definition of $\Phi_{X,\mathfrak{t}}$) is

$$\langle F^-_{W_{\gamma_n}|\Sigma} \circ \cdots \circ F^-_{W_{\gamma_1}|\Sigma} \circ F^-_{\Sigma \times D^2|\Sigma}(1), F^-_{\Sigma \times D^2|\Sigma}(1) \rangle$$

i.e. only one of the cuts is used to define the mixed invariants, but the factorization of X into annuli is useful for calculations.

There is a hitch in this plan, which is that these cuts are usually inadmissible (the annuli each containing a single vanishing cycle have no b_2^+, for example, though the cuts may also fail to be admissible because spin^c structures fail to restrict in the desired way). Moreover, we still have no way of dealing with Lefschetz fibrations whose generic fiber is a torus, as all cited results require $g \geq 2$.

Theorem 2.7.3

Let $X = X_1 \cup_Y X_2$ with $b_2^+(X) \geq 3$, spin^c structures $\mathfrak{t}_i$ on X_i restricting to the same (non-torsion) spin^c structure $\mathfrak{s}$ on Y. Then

$$\langle F^-_{X_1,\mathfrak{t}_1}(1), F^-_{X_2,\mathfrak{t}_2}(1)\rangle_{\mathrm{HF}_{\mathrm{red}}(Y,\mathfrak{s})} = \sum_{\substack{\mathfrak{t}|_{X_1}=\mathfrak{t}_1 \\ \mathfrak{t}|_{X_2}=\mathfrak{t}_2}} \Phi_{X,\mathfrak{t}}$$

Note that we did not require that Y be an admissible cut, so this significantly generalizes the applicability of our main tool. To deal with $b_2^+ = 1$, we need an auxiliary piece of data (this is the analogue of *wall-crossing* in Seiberg-Witten theory), $L = \mathrm{Im}(H_2(Y) \to H_2(X))$. L is the span of some surface $\Sigma \hookrightarrow X$ with $\Sigma \cdot \Sigma = 0$, so for any $t \in \mathrm{Spin}^c(X)$ with $\mathfrak{t} \cdot \Sigma \neq 0$, we may define the following invariant of the data $(X, \mathfrak{t}, L)$:

$$\Phi_{X,[\mathfrak{t}],L} := \langle F^-_{X_1,\mathfrak{t}_1}(1), F^-_{X_2,\mathfrak{t}_2}(1)\rangle_{\mathrm{HF}_{\mathrm{red}}(Y,\mathfrak{s})} = \sum_{\substack{\mathfrak{t}'|_{X_1}=\mathfrak{t}_1 \\ \mathfrak{t}'|_{X_2}=\mathfrak{t}_2}} \Phi_{X,\mathfrak{t}',L}$$

There appears to be a gap in the literature for $b_2^+ = 2$, though there are no wall-crossing phenomena in this case.

Exercise 2.7.4

Let $X = \Sigma_g \times S^2$, $g \geq 2$, $\mathfrak{t} = (2g-2)[S^2] + 2[\Sigma_g]$. Compute $\Phi_{X,[\mathfrak{t}],\mathrm{span}[\Sigma_g]}$ and $\Phi_{X,[\mathfrak{t}],\mathrm{span}[S^2]}$.

The point of this exercise is to give an easy example where the choice of L matters.

PROOF : To compute $\Phi_{X,[\mathfrak{t}],\mathrm{span}[\Sigma_g]}$, we split X along $\Sigma_g \times S^1$ which gives us $X_i = \Sigma_g \times D^2$. $\mathfrak{t}$ restricts to $2g-2 \in H_2(\Sigma_g) = \mathbb{Z}$ on either side so

$$\Phi_{X,[\mathfrak{t}],\mathrm{span}[\Sigma_g]} = \langle F^-_{\Sigma_g \times D^2|\Sigma_g}(1), F^-_{\Sigma_g \times D^2|\Sigma_g}(1)\rangle \neq 0$$

The pairing is nonzero by Proposition 2.7.2.

To compute $\Phi_{X,[\mathfrak{t}],\mathrm{span}[S^2]}$, we split along $S^1 \times S^2$ for some separating S^1 in Σ_g (since $g \geq 2$), so $\Phi_{X,[\mathfrak{t}],\mathrm{span}[S^2]}$ vanishes since $\mathrm{HF}_{\mathrm{red}}(S^1 \times S^2)$ vanishes for all spin^c structures. ■

We are now finally ready to state the final form (at least, until we generalize HF to have perturbed coefficients to deal with $g = 1$) of our "Lefschetz fibrations have nonvanishing mixed invariants" theorem:

Theorem 2.7.5: LF $\implies \Phi \neq 0$, v3

If X is a closed Lefschetz fibration with $b_2^+(X) \geq 3$ and generic fiber Σ_g, $g \geq 2$, then

$$\Phi_{X|\Sigma} := \sum_{\mathfrak{t}\cdot\Sigma=2g-2} \Phi_{X,\mathfrak{t}}$$

is nonzero.

Moreover, if $b_2^+(X) = 1$, then

$$\sum_{\mathfrak{t}\cdot\Sigma=2g-2} \Phi_{X,\mathfrak{t},\text{span}[\Sigma_g]} \neq 0$$

By working in the Σ-refined setting, we are implicitly using the fact that Σ can be isotoped to live in either $\Sigma_g \times D^2$ or any of the annuli.

Proof : We split X up as in Figure 2.7, i.e.,

$$X = (\Sigma_g \times D^2 \cup W_{\gamma_1} \cup \cdots \cup W_{\gamma_n}) \cup_{\Sigma_g \times S^1} \Sigma_g \times D^2$$

giving rise to the invariant

$$\langle F^-_{W_{\gamma_n}|\Sigma} \circ \cdots \circ F^-_{W_{\gamma_1}|\Sigma} \circ F^-_{\Sigma\times D^2|\Sigma}(1), F^-_{\Sigma\times D^2|\Sigma}(1)\rangle \neq 0$$

valued in $\text{HF}_{\text{red}}(\Sigma_g \times S^1|\Sigma_g) = \mathbb{F}$. That this expression is nonzero follows from the fact (referenced in the above discussion on the refined surgery exact triangle) that each W_{γ_i} induces an isomorphism on HF, together with Proposition 2.7.2.

Denoting by X_1, X_2 the two halves of our decomposition of X, the above may be rewritten as

$$\langle F^-_{X_1|\Sigma}(1), F^-_{X_2|\Sigma}(1)\rangle = \langle \sum_{\mathfrak{t}_1\cdot\Sigma=2g-2} F^-_{X_1,\mathfrak{t}_1}(1), \sum_{\mathfrak{t}_2\cdot\Sigma=2g-2} F^-_{X_2,\mathfrak{t}_2}(1)\rangle = \sum_{\mathfrak{t}_i\cdot\Sigma=2g-2} \langle F^-_{X_1,\mathfrak{t}_1}(1), F^-_{X_2,\mathfrak{t}_2}(1)\rangle = \sum_{\mathfrak{t}\cdot\Sigma=2g-2} \Phi_{X,\mathfrak{t}} = \Phi_{X|\Sigma}$$

where we use the definition of $F^-_{W|\Sigma}$, bilinearity of the pairing, and the definitions of $\Phi_{X,\mathfrak{t}}$ given in Theorem 2.7.3.

For the case $b_2^+(X) = 1$, note that $\text{Im}(H_2(\Sigma_g \times S^1) \to H_2(X)) = \text{span}(\Sigma)$, and the proof is otherwise similar. ■

Extending to Genus 1 Fibrations

Lecture 22: April 10th

Unfortunately, nothing we have done so far applies to genus 1 Lefschetz fibrations, including our favorite simple examples, the elliptic surfaces $E(n)$. In particular, we hope to establish that $\Phi_{\text{K3},\mathfrak{t}} \neq 0$ for some $\mathfrak{t}$, which will imply that $\text{K3}\#\overline{\mathbb{CP}^2} \underset{\text{sm}}{\ncong} \#_3\mathbb{CP}^2\#_{20}\overline{\mathbb{CP}^2}$ and also that K3 admits infinitely many exotic copies via Fintushel-Stern knot surgery. Accomplishing this computation will require us to further refine our definition of mixed invariants, using *perturbed coefficients* for Heegaard Floer homology.

Example 2.7.6

1. $\mathrm{HF}^-(S^1 \times S^2) = \mathbb{F}[U] \oplus \mathbb{F}[U]$
2. $\mathrm{HF}^-(S^3_0(\mathrm{RHT})) = \mathbb{F}[U] \oplus \mathbb{F}[U]$ (with different gradings than above)
3. $\mathrm{HF}^-(S^3_0(4_1)) = \mathbb{F}[U] \oplus \mathbb{F}[U] \oplus \mathbb{F}[U]/U$
4. $\mathrm{HF}^-(T^3) = (\mathbb{F}[U])^6$

Since RHT and 4_1 are fibered and admit genus 1 Seifert surfaces, their 0-surgeries are genus 1 fiber bundles over S^1 (as is T^3). That $\mathrm{HF}^-(S^3_0(\mathrm{RHT})) = \mathrm{HF}^-(S^1 \times S^2)$ as ungraded rings tells us that HF^- cannot tell the difference between sphere bundles and torus bundles over S^1 (and genus information cannot be recovered from the gradings).

That the latter three examples are all genus 1 fiber bundles over S^1 tells us that the naïve approach to extending Theorem 2.3.8 to $g = 1$ fails, since for two out of three of these fiber bundles, $\mathrm{HF}_{\mathrm{red}} = 0$.

Not sure why we're using $E(4)$ as our example as the same reasoning seems to imply that $E(2) = E(1)\#_{T^3}E(1)$ or, more generally, any fiber sum (to be defined below) has vanishing mixed invariants.

Using the fact that $\mathrm{HF}_{\mathrm{red}}(T^3) = 0$, we may note another troubling fact: consider $E(4)$, which is a genus 1 Lefschetz fibration with monodromy $(ab)^{24}$, which we may cut along a T^3 into two copies of $E(2) = \mathrm{K3}$ with monodromy $(ab)^{12}$ on each side (i.e., $E(4) = E(2)\#_{T^3}E(2)$). $\Phi_{E(4),\mathfrak{t}}$ then vanishes if computed along this cut, since $\mathrm{HF}_{\mathrm{red}}(T^3) = 0$. Since $\Phi_{X,\mathfrak{t}}$ is independent of the choice of cut, we may conclude that $E(4)$ has vanishing mixed invariants. This is only a problem inasmuch as we would like $E(n)$ to have nonvanishing invariants, so we are inspired to complicate our lives slightly.

The appropriate setting in which Theorem 2.3.8 extends to spheres and tori, and in which the analogous definition of $\Phi_{X,\mathfrak{t}}$ behaves as desired, is the following:

Definition-Proposition 2.7.7: HF with Perturbed Coefficients

Let Y^3 be a closed 3-manifold, $\eta \in H^2(Y^3)$ non-torsion, $\mathfrak{s}$ a spin^c structure; then there exists $\underline{\mathrm{HF}^\circ}(Y, \mathfrak{s}, \eta)$, a module over $\Lambda = \mathbb{F}[[z]][z^{-1}]$ (i.e. power series with finitely many Laurent terms) that is an invariant of the input data. If η is torsion, then $\underline{\mathrm{HF}^\circ}(Y, \mathfrak{s}, \eta) = \mathrm{HF}^\circ(Y, \mathfrak{s}) \otimes \Lambda$ where the tensor product is taken over $\mathbb{F}$ or $\mathbb{F}[U]$ depending on the flavor.

Moreover, if $W : Y_1 \to Y_2$ is a cobordism, with $\omega \in H_2(W)$ restricting to η_i on Y_i, then there are cobordism maps

$$\underline{F}^\circ_{W,\mathfrak{t},\omega} : \underline{\mathrm{HF}^\circ}(Y_1, \mathfrak{s}_1, \eta_1) \to \underline{\mathrm{HF}^\circ}(Y_2, \mathfrak{s}_2, \eta_2)$$

that are well-defined up to multiplication by a power of z.

Technically, HF^- is a module over $\Lambda[U]$.

Theorem 2.7.8: Ai-Peters, Ai-Ni

Let $\Sigma \hookrightarrow Y$ be a torus with $\eta(\Sigma) \neq 0$ (and suppose that Y is prime), then $\underline{\mathrm{HF}^-}(Y, \eta) = \Lambda$ if and only if Y fibers over S^1 with fiber Σ. Moreover, $\underline{\mathrm{HF}^-}(Y, \eta) = 0$ if and only if there exists a non-separating $S^2 \hookrightarrow Y$ such that $\eta(S^2) \neq 0$.

Thus, roughly, $\underline{\mathrm{HF}^-}$ detects spherical and torus fibrations as HF^- detects higher genus fibrations.

Theorem 2.7.9: Jabuka-Mark '08, Zemke '18

Let $X = X_1 \cup_Y X_2$ be a closed, smooth, 4-manifold, with $b_2^+(X) \geq 3$, and $\eta \in H^2(X)$. If $b_2^+(X_i) > 0$, or $\eta|_Y$ is non-torsion, or if there exists some $\mathfrak{t} \in \mathrm{Spin}^c(X)$ such that $\mathfrak{t}|_Y$ is non-torsion, then

$$\langle \underline{F}^-_{X_1,\mathfrak{t}_1,\eta_1}(1), \underline{F}^-_{X_2,\mathfrak{t}_2,\eta_2}(1) \rangle_{\underline{\mathrm{HF}}} = \sum_{\mathfrak{t}'|_{X_i} = \mathfrak{t}_i} z^{\eta(\mathfrak{t}')} \Phi_{X,\mathfrak{t}'}$$

is well-defined up to multiplication by z^n.

We are using above that there exists a nondegenerate pairing

$$\underline{\mathrm{HF}^\circ}(Y, \mathfrak{s}, \eta) \otimes \underline{\mathrm{HF}^\circ}(-Y, \mathfrak{s}, \eta) \to \Lambda$$

In fact, many of the results we have developed HF° carry over to the perturbed coefficients setting. For example, given K a nontrivial curve on $T^2 \hookrightarrow Y$, α a loop with $\alpha \cdot T \neq 0$ and $\alpha \cap K = \emptyset$, there is a surgery exact triangle:

$$\begin{array}{ccc} \underline{\mathrm{HF}^-}(Y, \mathrm{PD}[\alpha]) & \xrightarrow{\underline{F}^-_{W_1,\mathrm{PD}[\alpha\times I]}} & \underline{\mathrm{HF}^-}(Y_\lambda(K), \mathrm{PD}[\alpha]) \\ {\scriptstyle \underline{F}^-_{W_3,\mathrm{PD}[\alpha]}} \nwarrow & & \swarrow {\scriptstyle \underline{F}^-_{W_2,\mathrm{PD}[\alpha\times I]}} \\ & \underline{\mathrm{HF}^-}(Y_{\lambda+\mu}(K), \mathrm{PD}[\alpha \times I]) & \end{array}$$

where λ is a framing for K as before.

As before, this will allow us to prove that $\underline{F}^-_{W_\gamma,\mathrm{PD}[\alpha\times I]}$ is an isomorphism, which is an important step in the proof of the theorem above. In particular, setting $\lambda = \lambda_T - \mu$, the bottom term in this triangle is $\underline{\mathrm{HF}^-}(Y_{\lambda_T}(K), \mathrm{PD}[\alpha])$ which is zero since it contains a non-separating sphere (the result of surface framed surgery on T) hitting α, so $\mathrm{PD}[\alpha](S^2) \neq 0$ (as in Theorem 2.7.8).

There is also an analogue of Proposition 2.7.2 in this setting:

Proposition 2.7.10

$$\underline{F}^-_{T^2\times D^2,\mathrm{PD}[D^2]} = \frac{1}{1-z} \in \underline{\mathrm{HF}^-}(T^3,\mathrm{PD}(S^1)) = \Lambda$$

These are the essential ingredients that we need to prove our main result:

Theorem 2.7.11: LF $\implies$ $\Phi \neq 0$, v4

If X is a Lefschetz fibration with torus fibers, and $b_2^+(X) \geq 3$, then

$$\sum_{\mathfrak{t}} \Phi_{X,\mathfrak{t}} z^{\mathrm{PD}[\text{section}]\cdot\mathfrak{t}} = \sum_{\mathfrak{t}\cdot\text{fiber}=0} \Phi_{X,\mathfrak{t}} z^{\mathrm{PD}[\text{section}]\cdot\mathfrak{t}} \neq 0$$

for a generic section and fiber of the fibration.

The proof is (apparently) substantially similar to the unperturbed proof above, and is therefore omitted. That the first and second sum are equal follows from adjunction: $\Phi_{X,\mathfrak{t}} \neq 0$ implies

$$|\mathfrak{t}\cdot\text{fiber}| + [\text{fiber}]\cdot[\text{fiber}] \leq 2g-2 = 0$$

but a generic fiber has a disjoint pushoff so this implies that $\mathfrak{t}\cdot\text{fiber} = 0$.

Corollary 2.7.12

$\Phi_{\mathrm{K3},\mathfrak{t}} \neq 0$ for some $\mathfrak{t}$.

In fact, $\Phi_{\mathrm{K3},0} \neq 0$ and $\Phi_{\mathrm{K3},\mathfrak{t}} = 0$ for all $\mathfrak{t} \neq 0$. Theorem 2.5.14, that $\mathrm{K3}\#\overline{\mathbb{CP}^2} \underset{\mathrm{sm}}{\not\cong} \#_3\mathbb{CP}^2\#_{20}\overline{\mathbb{CP}^2}$, immediately follows. We can also repeat this argument for higher elliptic surfaces to get more exotica. Note that, for n odd, $E(n)$ is non-spin, so we automatically get an exotic copy of $\#_a\mathbb{CP}^2\#_b\overline{\mathbb{CP}^2}$, while for n even, $E(n)$ is spin, so we have to blow up once for the same result.

The Invariants of Knot Surgery

Theorem 2.7.13: Mark, Juhász-Zemke

Let $T \hookrightarrow X$ be an embedded torus with $[T]\cdot[T] = 0$, $[T] \neq 0 \in H_2(X;\mathbb{R})$. Let $b_1,\cdots,b_n \in H^2(X;\mathbb{R})$ be a basis such that $b_1(T) = 1$, $b_i(T) = 0$ for $i > 1$ (i.e. $b_1 = \mathrm{PD}[T]$), and assume that $b_2^+(X) \geq 3$. Define

$$\Phi_{X,b} = \sum_{\mathfrak{t}} \Phi_{X,\mathfrak{t}} z_1^{\frac{b_1(\mathfrak{t})}{2}} \cdots z_n^{\frac{b_n(\mathfrak{t})}{2}} \in \mathbb{F}[z_1^{\pm1},\cdots,z_n^{\pm1}]$$

Then $\Phi_{X_K(T),b} = \Delta_K(z_1)\cdot\Phi_{X,b}$. where Δ is the Alexander polynomial of the knot K.

This is one of my favorite results in 4-manifold topology, its proof uses the skein relation for the Alexander polynomial and builds $X_K(T)$ by starting with

$$X = (X\setminus\nu(T)) \cup_{T^3} S^1\times(S^3\setminus\nu(U))$$

and doing crossing changes on U by $+1$ log transforms, which are basically a circle's worth of $+1$ Dehn surgeries (which we know can change crossings), until it turns into K.

Example 2.7.14

Consider K3 with T a regular fiber. $\pi_1(\mathrm{K3}\setminus T) = 0$, so K3 and $\mathrm{K3}_K$ are both simply-connected and have the same intersection form, and so are homeomorphic by Freedman. Moreover, $\Phi_{\mathrm{K3},b} = 1$, so $\Phi_{\mathrm{K3}_K,b} = \Delta_K(z_1)$ and we obtain infinitely many exotic copies of K3 by using knots with distinct Alexander polynomials (modulo some fussing about how important the choice of basis b is).

Smaller Exotica via Lefschetz Fibrations

Lecture 23: April 15th

Donaldson's original example of exotica in 1983 was an exotic copy of $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$. Shortly thereafter, the 9 was lowered to 8 by Kotchick (1986) and then to 7 by Park (2005). We will discuss how to construct these examples via Lefschetz fibrations (an exotic $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$ can already be achieved by knot surgery on a regular fiber of $E(1)$; the argument is similar to the one given above for K3, but using the $b_2^+ = 1$ mixed invariant). We will follow the exposition of Baykur-Korkmaz.[9]

[9] R. İ. Baykur and Korkmaz, *Small Lefschetz fibrations and exotic 4-manifolds*.

Recall that a Lefschetz fibration is of type (n, s) if there are n non-separating vanishing cycles and s separating vanishing cycles.

Theorem 2.8.1: Kas, Moishezon

$g = 1$ Lefschetz fibrations over S^2 are classified by (n, s). Moreover, if $s = 0$, then the total space of the fibration is $E(\frac{n}{12})$.

Note that separating vanishing cycles on a torus are necessarily nullhomotopic loops. To produce a Lefschetz fibration that isn't an elliptic surface, we have to increase the generic genus to 2. For our discussions of genus 2 Lefschetz fibrations, we will use the following standard basis of curves on Σ_2:

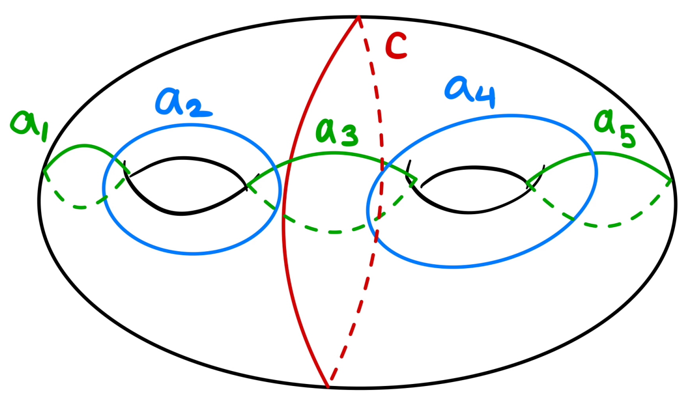


Figure 2.13: The curves a_i are all non-separating, and c is the only (nontrivial) separating curve on Σ_2.

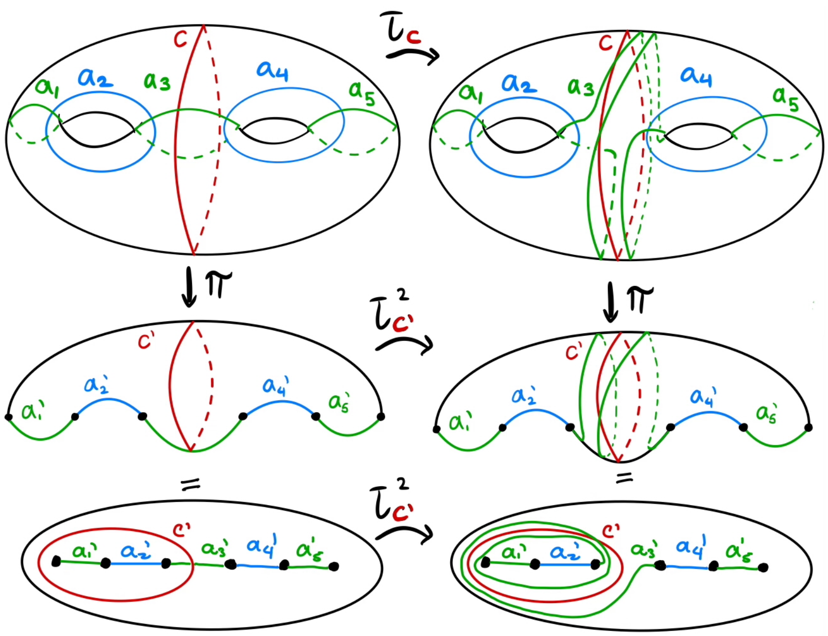
Exercise 2.8.2

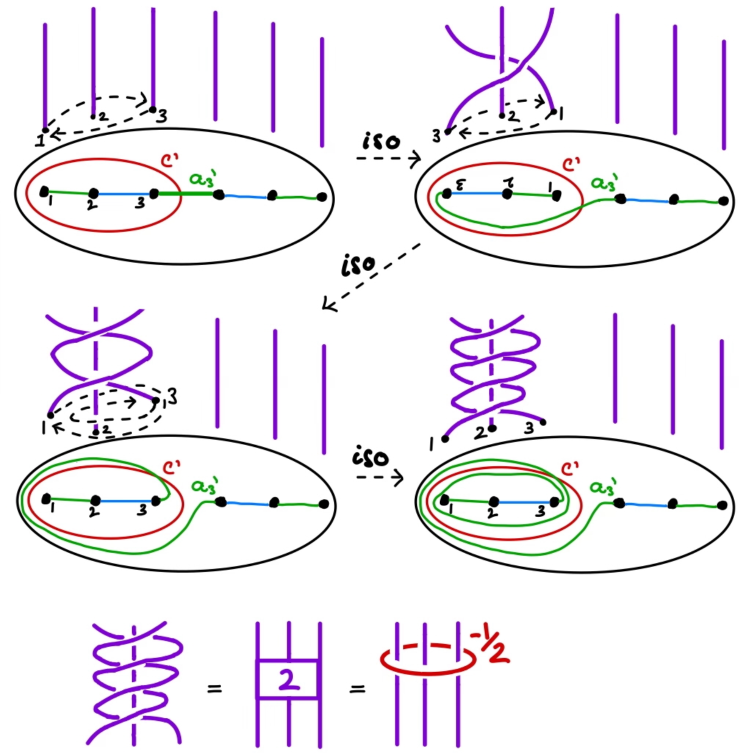
What does the Dehn twist τ_c along the separating curve c correspond to in B_6^S, the spherical braid group on 6 strands? Draw the mapping torus associated to the resulting braid.

Bonus

For a genus 2 Lefschetz fibration over S^2 of type (n, s), show that the Euler characteristic is given by $\chi = n + s - 4$.

Proof :





Above left, we see how τ_c induces two full twists on the first three strands (or last three strands) in the braid group. The drawing above right fills in more details.

Notably, τ_c achieves the braid $(a_1a_2)^6$ that would naïvely require twelve elementary Artin generators (corresponding to twelve separating vanishing cycles in the associated Lefschetz fibration). So, for $g \geq 2$, we have a much cheaper way to achieve (some) full twists.

For the bonus, note that $\chi(S^2 \times \Sigma_2) = -4$ and each 2-handle clearly adds one to the Euler characteristic. ■

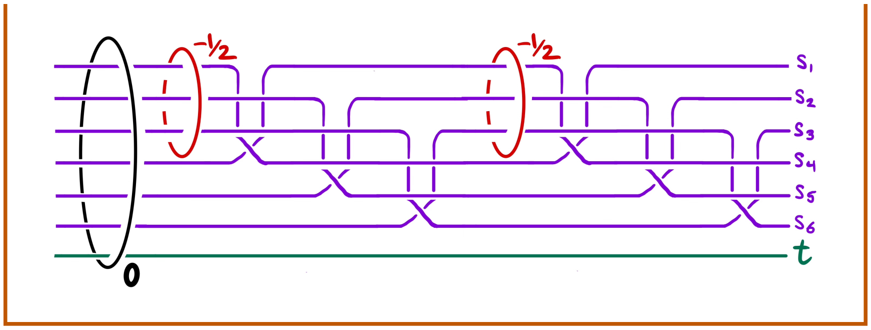
We proved the χ half of this above. The signature of a Lefschetz fibration is, in general, a little trickier, and one approach involves summing up the *local signatures* where the local contributions are indexed by the separating vanishing cycles.

Lemma 2.8.3

For a genus 2 Lefschetz fibration X over S^2, $\chi(X) = n + s - 4$ and $\sigma(X) = -\frac{1}{5}(3n + s)$.

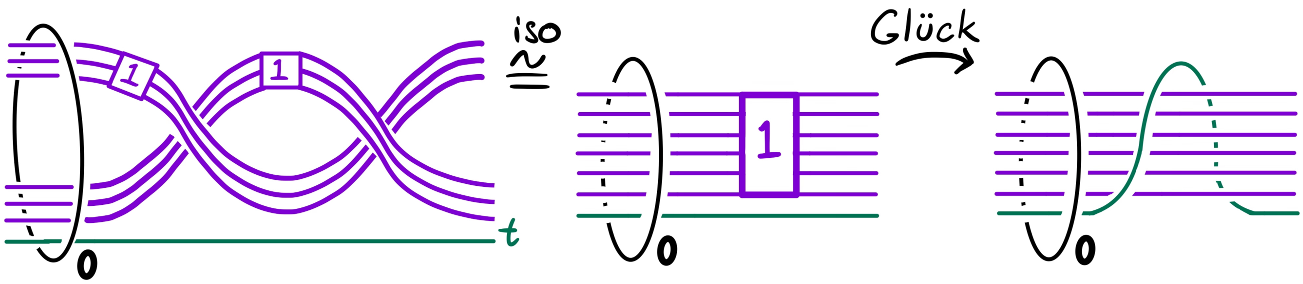
Theorem 2.8.4: Matsumoto '96

The branched double cover of $S^1 \times S^2$ along the braid β (drawn below) is diffeomorphic to $\Sigma_2 \times S^1$.



The type of the associated Lefschetz fibration is $(6, 2)$, corresponding counter-respectively to the 2 red curves and the 6 "generalized braid moves" where two purple strands exchange places.

Proof : Redraw the braid as below left:



To see that the branched double cover along this braid is diffeomorphic to $\Sigma_2 \times S^1$, we apply Corollary 2.5.17; there is an evident isomorphism φ between the exterior of the braid above right with the exterior of the trivial braid in $S^1 \times S^2$, and we need only check that $f \circ \varphi_* = f$ where $f : H_1(S^1 \times S^2 \setminus \nu(\beta)) \to \mathbb{Z}/2$ is the covering homomorphism.

Then, as before, note that $\varphi_*(s_i) = s_i$ and $\varphi_*(t) = t + s_1 + \cdots + s_6$, so since $f(s_i) = 1$ and $f(t) = 0$, $f(\varphi_*(t)) = 0 + 6 = 0 \pmod 2$ and the covering criterion is satisfied. Thus we may conclude that the branched double cover along β is indeed diffeomorphic to $\Sigma_2 \times S^1$. ■

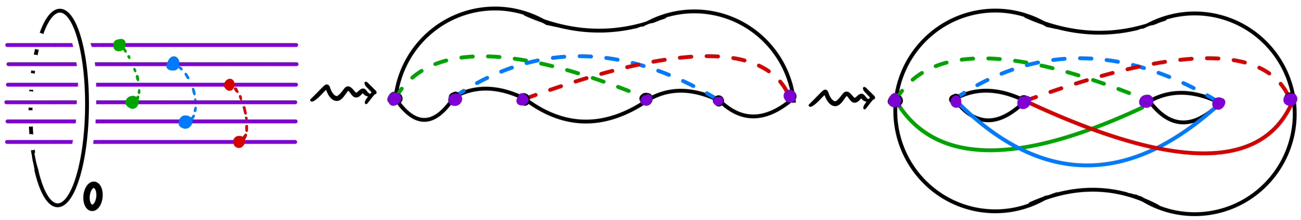
Exercise 2.8.5

Describe the lift of β to $\Sigma_2 \times S^1$ as a product of Dehn twists.

Bonus

Calculate χ, b_i, and σ of the resulting Lefschetz fibered 4-manifold.

Proof : We know that the $-\frac{1}{2}$-framed curves inducing two full twists on the braid lift to Dehn twists along c. What remains is to find Dehn twist curves that induce the generalized braid moves in β. For example, to swap strands 1 and 4 in the above picture, we need a Dehn twist curve intersecting the associated fixed points (of the hyperelliptic involution) in Σ_2. There is an obvious choice of such curve:



For the bonus, we know χ and σ from Lemma 2.8.3, so it remains only to determine $b_1 = b_3$ (which, together with χ, will determine b_2). The vanishing cycles attached along c do not affect b_1 since c is already nullhomologous in Σ_2, so we only have to consider the non-separating vanishing cycles. Recall that $H_1(\Sigma_2)$ has basis a_1, a_2, a_4, a_5, and we are attaching two vanishing cycles apiece along $a_2 + a_5$, $a_4 + a_1$, and $a_2 + a_4 + a_1 + a_5$. The first two relators each bring b_1 down by one, and the third is clearly redundant as the sum of the other 2, so $b_1 = 2$. ■

While this is a new and interesting Lefschetz fibration given by a nontrivial braid that is BR-equivalent to the identity braid, it is not simply-connected.

More generally, you can fiber sum any two 4-manifolds containing square zero surfaces of the same genus.

Definition 2.8.6: Fiber Sums

A *fiber sum* of two genus g Lefschetz fibrations X_1, X_2 is the manifold $X_1 \#_{F,\varphi} X_2$ obtained by deleting a neighborhood of a regular fiber F in both X_i, and gluing the resulting boundaries together along $\varphi \in \mathrm{MCG}(F)$. Explicitly,

$$X_1 \#_{F,\varphi} X_2 = (X_1 \setminus \nu(F)) \cup_\Phi (X_2 \setminus \nu(F))$$

where $\Phi : S^1 \times F \to S^1 \times F$ is given by $(\theta, x) \mapsto (\theta, \varphi(x))$.

Evidently, the fiber sum of Lefschetz fibrations is a Lefschetz fibration. By a judicious choice of φ, it turns out that we may fiber sum two copies of the Matsumoto Lefschetz fibration given above and get a simply-connected manifold.

Lecture 24: April 17[th]

Lemma 2.8.7

$$\pi_1(X_1 \#_{F,\varphi} X_2) = \pi_1(\Sigma_2)/\langle \delta_i, \varphi_*(\delta_i), c, \varphi_*(c)\rangle$$

where the X_i are two copies of the Matsumoto Lefschetz fibration, and where the δ_i are its non-separating vanishing cycles as drawn above. Moreover, there exists a choice of φ so that $\pi_1(X_1 \#_{F,\varphi} X_2) = 1$.

Set $Z := X_1 \#_{F,\varphi} X_2)$ for some such φ chosen to kill π_1, which is a Lefschetz fibration of type $(12,4)$, with $\chi(Z)=12$, $\sigma(Z)=-8$, $b_2=10$ (hence $b_2^+=1$ and $b_2^-=9$), and with monodromy

$$(c\delta_1\delta_2\delta_3)^2\varphi((c\delta_1\delta_2\delta_3)^2)$$

Since Z is smooth but its signature is not divisible by 16, by Rokhlin's theorem, Q_Z is odd. We also know that Q_Z is indefinite, hence $Q_Z = (1)\oplus 9(-1)$ by the classification of indefinite forms.

Theorem 2.8.8

$Z \underset{\text{top}}{\cong} \mathbb{CP}^2\#_9\overline{\mathbb{CP}^2} =: Z'$ and $Z \underset{\text{sm}}{\ncong} Z'$.

Proof : We know Q_Z, and that $\pi_1(Z)=1$, so the homeomorphism claim follows immediately by Freedman. To obstruct a diffeomorphism, we need to calculate mixed invariants. Since $b_2^+(Z)=1$, the calculation of mixed invariants requires (in addition to a choice of spin^c structure) a choice of $\Sigma\in H_2(Z)$, $\Sigma\cdot\Sigma=0$, to define $\Phi_{Z,[\mathfrak{t}],\text{span}[\Sigma]}$. By Theorem 2.7.5, if we take Σ to be the generic fiber, then

$$\sum_{\mathfrak{t}\cdot\Sigma=2g-2}\Phi_{X,\mathfrak{t},\text{span}[\Sigma]}\neq 0$$

It will suffice to show that there exists $\Sigma'\in H_2(Z')$, with $\Sigma'\cdot\Sigma'=0$ such that

$$\sum_{\mathfrak{t}\cdot\Sigma=2g-2}\Phi_{X,\mathfrak{t},\text{span}[\Sigma']}=0$$

(this is not automatic, since Z' cannot be decomposed as a sum of manifolds with positive b_2^+).

A priori, this is not enough to show that $Z \underset{\text{sm}}{\ncong} Z'$ as we the above sum for Z could vanish for some other choice of surface; we will remedy this concern with a small argument at the end.

Our Σ' will be $\mathbb{CP}^1\#\overline{\mathbb{CP}^1}\hookrightarrow Z'$ i.e. the H_2-generators of the first two summands of Z' tubed together. $\mathbb{CP}^1$ has self-intersection $+1$ and $\overline{\mathbb{CP}^1}$ has self-intersection -1, so $\Sigma'\cdot\Sigma'=0$ as claimed. If we set $Y=\partial\nu(\Sigma')=S^1\times S^2$ to be our admissible cut, then the mixed invariant sum automatically vanishes since $\text{HF}_{\text{red}}(S^1\times S^2)=0$.

Now we need to argue that this implies $Z \underset{\text{sm}}{\ncong} Z'$. Suppose we have a diffeomorphism $G: Z\to Z'$; we will argue that we can modify G to produce $F: Z\to Z'$ satisfying $F_*(\Sigma)=\Sigma'$. This, together with the above, implies $Z \underset{\text{sm}}{\ncong} Z'$ since F would carry the nonzero sum of mixed invariants in Z to 0 in Z', which is a contradiction.

To produce F from G, we need the following results:

An automorphism of a quadratic form on the level of its matrix Q is of the form $P^TQP=Q$ where P is an invertible integer matrix.

Lemma 2.8.9

There exists an automorphism γ of $Q_{Z'}$ such that $\gamma(G_*(\Sigma))=\Sigma'$.

This is just linear algebra and relies only on the fact that Σ and Σ' are primitive (i.e., they are not multiples of any other element of their respective lattices).

I'm not sure to what extent this lemma applies. I suspect when our lattice is more complicated than diagonal, the analogous result fails wildly.

Theorem 2.8.10: Wall '64[10]

Let W be a closed, smooth, oriented, simply-connected 4-manifold with $b_2(W) \leq 8$ or Q_W indefinite. Then every automorphism of $Q_{W\#(S^2\times S^2)}$ is realizable by a diffeomorphism.

[10] Wall, *Diffeomorphisms of 4-manifolds*, Theorem 2

Note that $Z' \underset{\text{sm}}{\cong} (S^2 \times S^2)\#_8\overline{\mathbb{CP}^2}$ since $S^2 \times S^2\#\overline{\mathbb{CP}^2} \underset{\text{sm}}{\cong} \mathbb{CP}^2\#_2\overline{\mathbb{CP}^2}$, so if $\Gamma : Z' \to Z'$ is a diffeomorphism of Z' inducing γ on $Q_{Z'}$ as in the above lemma, $F := \Gamma \circ G : Z \to Z'$ satisfies $F_*(\Sigma) = \Sigma'$, giving the desired contradiction. ■

The Wall result generalizes in our immediate context of producing exotic copies of $\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ with n as small as possible:

Corollary 2.8.11

For $n \in \{2, \cdots, 9\}$, every automorphism of $\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ is realized by a diffeomorphism.

For $n \geq 10$, this corollary no longer holds by work of Friedman and Morgan; see Ruberman-Strle[11] for discussion and the original references.

[11] Ruberman and Strle, *Wall's stable realization for diffeomorphisms of definite 4-manifolds*

Corollary 2.8.12

If X admits a Lefschetz fibration structure with genus $g \geq 2$, $\pi_1(X) = 1$ and $2 \leq b_2(X) \leq 10$, $\sigma(X) = 2 - b_2(X)$, then X is an exotic copy of $\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ where $n = b_2(X) - 1$.

The proof is essentially identically to the one given above, so we can potentially produce even smaller exotica for free if we can come up with smaller genus 2 Lefschetz fibrations. Towards this goal, we will now consider the geography of genus 2 Lefschetz fibrations[12] to see how small we can get without increasing g again.

[12] Nakamura, *Geography of genus 2 Lefschetz fibrations*, Hook 'Em.

Lemma 2.8.13: Özbağcı '02

For a genus 2 Lefschetz fibration over S^2 of type (n, s), $n + s \geq 7$.

Lemma 2.8.14: Baykur-Korkmaz '15[13]

$2n - s \geq 3$, $n + 7s \geq 20$. Moreover, $n + 2s \equiv 0 \pmod{10}$.

[13] R. İ. Baykur and Korkmaz, *Small Lefschetz fibrations and exotic 4-manifolds*, Lemma 5

These conditions together imply that $n \geq 4$, and $n = 4$ implies $s \geq 3$ so the minimal type is $(4, 3)$, which we will see is achieved.

Baykur and Korkmaz go on to prove that the only such (irreducible, i.e., not a fiber sum of simpler fibrations) Lefschetz fibration types with $n + 7s < 30$ are the following:

(4, 3) corresponding to $(S^2 \times T^2)\#_3\overline{\mathbb{CP}^2}$

(6, 2) corresponding to $(S^2 \times T^2)\#_4\overline{\mathbb{CP}^2}$

(18, 1) corresponding to $\mathbb{CP}^2\#_{12}\overline{\mathbb{CP}^2}$

(20, 0) corresponding to $\mathbb{CP}^2\#_{13}\overline{\mathbb{CP}^2}$

Note that the manifold Z we constructed above was a fiber sum of two Lefschetz fibrations of type $(6, 2)$, so the fiber sum of "standard" manifolds can be non-standard.

For different choices of φ that kill π_1, do we get different exotic copies? Assuming that there is more than one choice that works. We're only using the mixed invariants on the level of zero vs. nonzero so it doesn't seem possible to say without further calculations.

Exercise 2.8.15

Suppose X_1 is a Lefschetz fibration of type $(4, 3)$ type and X_2 is of type $(6, 2)$. What are χ and σ of $X_1\#_{F,\varphi}X_2$ and $X_1\#_{F,\varphi}X_1$?

Bonus

Prove that there exists gluing maps φ making either of the above fiber sums simply-connected, and conclude that there are exotic copies of $\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ for $n \in \{8, 7\}$.

Sketch : Lemma 2.8.3 tells us the Euler characteristic and signature so the bonus is the main problem to deal with. We know that $X_1 \underset{\text{sm}}{\cong} (S^2 \times T^2)\#_3\overline{\mathbb{CP}^2}$ so $\pi_1(X_1)$ is abelian on two generators. The idea is that, whatever the monodromy of X_1, its non-separating vanishing cycles had to kill 2 generators of $\pi_1(\Sigma_2)$ to get down to the two generators of $\pi_1(X_1)$, so we should be able to kill another two generators using a clever repositioning of the vanishing cycles. The same logic holds for X_2. $n = 8$ and $n = 7$ for the two fiber sums follows from the calculation of χ and σ. ■

Lecture 25: April 22nd

For our purposes (constructing Lefschetz fibrations starting from a braid, and never going the other way), we may as well just accept an ordering of the vanishing cycles as part of the input data for a Lefschetz fibration.

Remark 2.8.16: Does Order Matter?

In our discussion on Lefschetz fibrations, we have assumed throughout that the data of a positive factorization of the identity both defines and is defined by a Lefschetz fibration. However, in order to obtain such a factorization from a given Lefschetz fibration, there are some choices to be made since there is no natural ordering on the vanishing cycles giving rise to such a factorization.

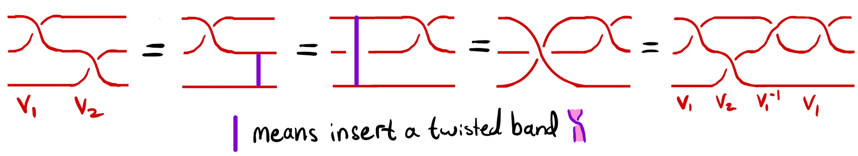
The point is that our description of the boundary depends on a choice of ordering. For example, suppose we're starting with some Lefschetz fibration which has as a substring of its monodromy, $\tau_{v_1}\tau_{v_2}$, and suppose that we want to swap the order of these Dehn twists. Isolating each Dehn twist along an annulus as we're used to by now, we have three copies of $\Sigma \times I$, where the first copy is glued to the second along τ_{v_1}, and the second to the third along τ_{v_2}. To change the ordering, we need $t \in \text{MCG}(\Sigma)$ such that $t \circ \tau_{v_1} = \tau_{v_1} \circ \tau_{v_2}$ i.e. $t = \tau_{v_1} \circ \tau_{v_2} \circ \tau_{v_1}^{-1}$. This is bad for two reasons: it exchanges a single Dehn twist for three (and we are being conservative with our b_2), and it introduces a negative twist (and now we are in the regime of achiral Lefschetz fibrations). There is a remedy to both problems.

Lemma 2.8.17

$$t = \tau_{\tau_{v_1}^{-1}(v_2)}$$

I.e., t as above can be written as a single, positive Dehn twist.

PROOF :



■

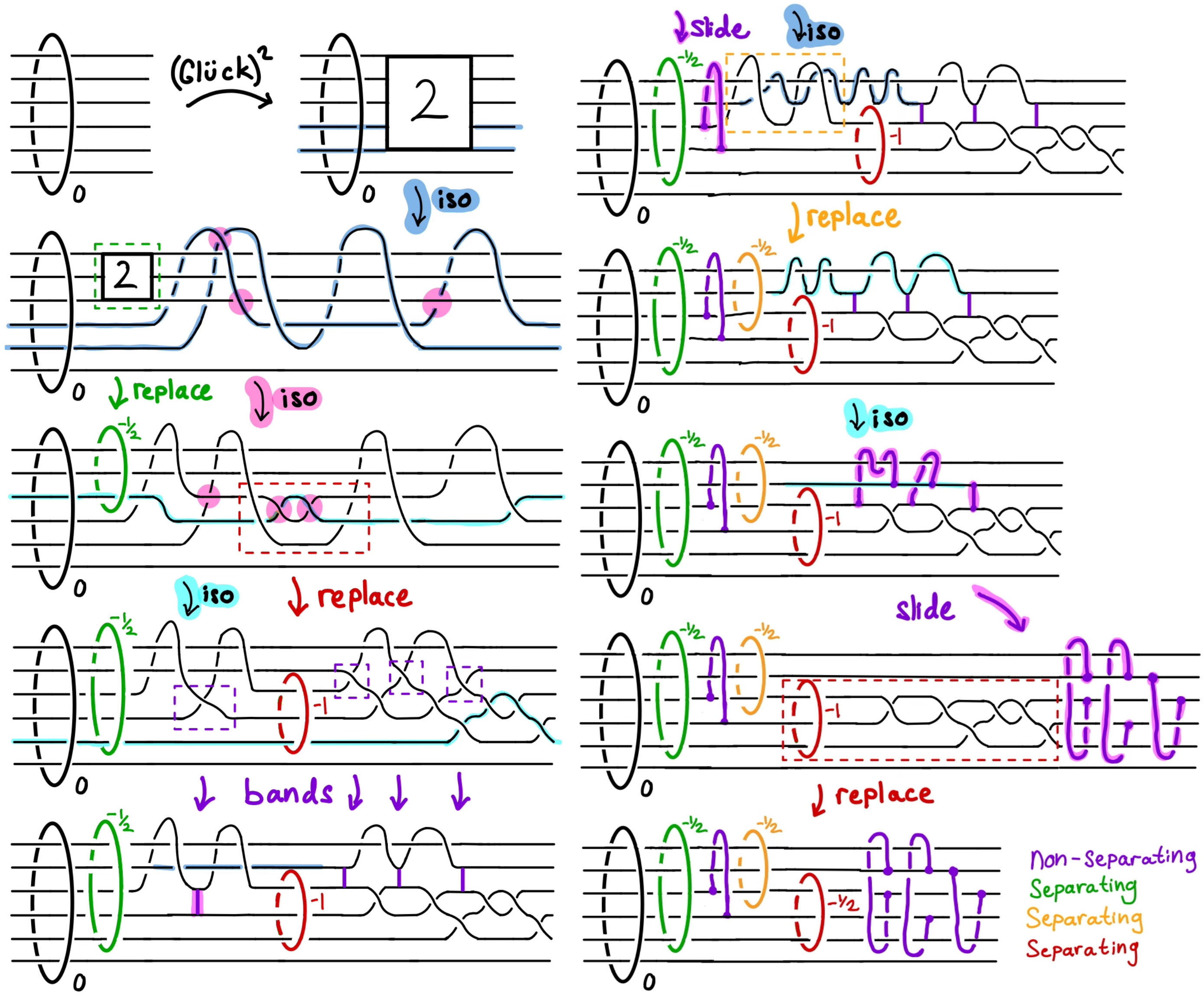
If this discussion seems tangential at best to what we're actually doing, that's because this was all a response to a question I asked.

Theorem 2.8.18: Baykur-Korkmaz '15[14]

There exists a braid in $S^1 \times S^2$ whose corresponding genus 2 Lefschetz fibration is of type $(4, 3)$, realizing the lower bound.

[14] R. İ. Baykur and Korkmaz, *Small Lefschetz fibrations and exotic 4-manifolds*, Theorem 7

PROOF :



■

It is easy to verify that the usual condition on homology is satisfied by the given (final) braid.

Going Smaller

Having done a few examples in full detail, here we will briefly sketch some related techniques for building even smaller exotica:

- Since we've exhausted what $g = 1, 2$ have to offer us, we can try $g = 3$. Baykur-Hamada-Simone[15] produce an exotic $\mathbb{CP}^2 \#_5 \overline{\mathbb{CP}^2}$ in this way. The trouble, of course, is that higher genus brings more π_1 to kill, which takes more 2-handles, so at high enough g, our exotica will necessarily start getting bigger rather than smaller.

- We could also increase the genus of the base (so far we have only considered $B = S^2$). This brings its own π_1 troubles, but also offers us some new and interesting tori: if a is a curve in the base and c a curve in

[15] R. Baykur, Korkmaz, and Simone, *Geography of symplectic Lefschetz fibrations and rational blowdowns*.

the regular fiber, then T_{ac} is a square zero torus that we can do torus surgery on with coefficients $(a, c, \mu) = (0, 1, 1)$. This will kill c in H_1, and we can do more surgery to kill π_1 of the base, but getting rid of π_1 completely is tricky here. Another wrinkle in this approach is that the monodromy of the fibration is more complicated, due to the base having its own π_1. Baykur-Korkmaz[16] produce an exotic $\mathbb{CP}^2\#_4\overline{\mathbb{CP}^2}$ and Baykur-Hamada[17] an exotic $\#_7 S^2 \times S^2$ using such techniques.

- Another approach, implicitly considered above, is to start with a surface bundle over a surface and kill its fundamental group using (Luttinger) surgeries as opposed to 2-handle attachments along vanishing cycles for Lefschetz fibrations. This approach has been applied by Akhmedov-Park[18] to build an exotic $\mathbb{CP}^2\#_3\overline{\mathbb{CP}^2}$ and a cohomology $S^2\times S^2$. Lidman-Piccirillo[19] have additionally constructed a cohomology $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ using related techniques. The fundamental group of the latter two manifolds is nontrivial.

[16] R. İ. Baykur and Korkmaz, *Small Lefschetz fibrations and exotic 4-manifolds*.

[17] R. I. Baykur and Hamada, *Exotic 4-manifolds with signature zero*.

[18] Akhmedov and B Doug Park, *Exotic smooth structures on small 4-manifolds*; Akhmedov and B. Doug Park, *Exotic smooth structures on $S^2\times S^2$*.

[19] Lidman and Piccirillo, *Distinguishing closed 4-manifolds by slicing*.

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