

## Fractional Smoothness and Embedding Potentials

1. Characterization of smoothness
2. Rigorous description of many-body interactions
3. Establish sharp embedding estimates
4. Expand working framework for
  - a. Fourier transform
  - b. convolution
  - c. Riesz potentials
  - d. Stein-Weiss integrals (Hardy-Littlewood-Sobolev inequalities)
  - e. weights & symmetrization

5. Gain new insight
  - a. uncertainty
  - b. restriction phenomena
  - c. geometric symmetry
6. Effort for optimal constants
  - a. new features for exact model calculations
  - b. encoded geometric information
  - c. precise lower-order effects
7. Symmetry determines structure
8. Multilinear analysis  $\rightsquigarrow$  understanding for genuinely  $n$ -dimensional aspects of Fourier analysis

## OBJECTS OF STUDY

$$\Lambda_\alpha = (-\Delta/4\pi^2)^{\alpha/2}, \quad \alpha > 0, \quad 0 < \beta < 1 \quad \text{and} \quad 1 \leq p < n/\beta$$

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} dx dy &\rightsquigarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|(\nabla f)(x) - (\nabla f)(y)|^p}{|x - y|^{n+p\beta}} dx dy \\ &\rightsquigarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|(\Lambda_\alpha f)(x) - (\Lambda_\alpha f)(y)|^p}{|x - y|^{n+p\beta}} dx dy \end{aligned}$$

## 1. ARONSZAJN-SMITH FORMULAS

Classical Formula:  $0 < \alpha < 2$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} dx dy = D_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{f}(\xi)|^2 d\xi$$

Frank-Lieb-Seiringer:  $0 < \alpha < \min(2, n)$ ;  $g = |x|^\lambda f$ ,  $0 < \lambda < n - \alpha$

$$\begin{aligned} D_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{f}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x - y|^{n+\alpha}} [|x| |y|]^{-\lambda} dx dy \\ &\quad + \Lambda(\alpha, \lambda, n) \int_{\mathbb{R}^n} |x|^{-\alpha} |f(x)|^2 dx \end{aligned}$$

Beckner:  $0 < \beta < 2, \beta \leq \alpha < n; g = |x|^{(n-\beta)/2}(-\Delta/4\pi^2)^{(\alpha-\beta)/4}f$

$$C_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{f}(\xi)|^2 d\xi \geq \int_{\mathbb{R}^n} |x|^{-\alpha} |f(x)|^2 dx \\ + \frac{C_\alpha}{D_\beta} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x - y|^{n+\beta}} [|x| |y|]^{-(n-\beta)/2} dx dy$$

## 2. MULTILINEAR FRACTIONAL EMBEDDING (APRES GROSS & PITAEVSKI)

Pitt's inequality:  $n - \beta = mn - \alpha$ ,  $\alpha = \sum \alpha_k$ ,  $0 < \alpha_k < n$ ,  
 $(m - 1) < \alpha/n < m$

$$\int_{\mathbb{R}^n} |x|^{-\beta} |f(x, \dots, x)|^2 dx \leq C_\beta \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} |\Pi(-\Delta)k/4\pi^2)^{\alpha_k/4} f|^2 dx_1 \dots dx_m$$

Hardy-Littlewood-Sobolev inequality:  $mn - \alpha = 2n/q$

$$\left[ \int_{\mathbb{R}^n} |f(x, \dots, x)|^q dx \right]^{2/q} \leq F_\alpha \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} |\Pi(-\Delta_k/4\pi^2)^{\alpha_k/4} f|^2 dx_1 \dots dx_m$$

Similar results on  $S^n$

Key insight on “multilinear products”

$$F(x) = \int_{\mathbb{R}^{mn}} \Pi g_k(x - y_k) H(y) dy \quad ; \quad H \in L^p(\mathbb{R}^{mn}) \quad \rightsquigarrow \quad F \in L^q(\mathbb{R}^n)$$

### 3. RESTRICTION TO SUBMANIFOLD & UNCERTAINTY

“classical uncertainty principle”

$$c \int_{\mathbb{R}^n} |f|^2 dx \leq \int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/4} |x|^{\alpha/2} f(x) \right|^2 dx$$

Restriction to  $k$ -dimensional linear sub-variety

$$d \int_{\mathbb{R}^k} |\mathcal{R}f|^2 dx \leq \int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/4} |x|^{\beta/2} f(x) \right|^2 dx$$

with  $n - \alpha = k - \beta$ ,  $n \geq k > \beta > 0$

$$d = \pi^{-\alpha} \frac{\Gamma(\alpha/2)}{\Gamma(\beta/2)} \left[ \frac{\Gamma\left(\frac{k+\beta}{4}\right)}{\Gamma\left(\frac{k-\beta}{4}\right)} \right]^2$$

## 4. TRIANGLE INEQUALITY ESTIMATES

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^n} |g(y-x)f(x) - h(x-y)f(y)|^p dx dy \\ & \geq \int_{\mathbb{R}^n} \left| |g(y)| - |h(-y)| \right|^p dy \int_{\mathbb{R}^n} |f(x)|^p dx \end{aligned}$$

**Proof:**  $p \geq 1$

$$\begin{aligned} & \int_{\mathbb{R}^n} \left\{ \left( \int_{\mathbb{R}^n} |g(y)f(x) - h(-y)f(y)|^p dx \right)^{1/p} \right\}^p dy \\ & \geq \int_{\mathbb{R}^n} \left\{ \left| |g(y)| \|f\|_p - |h(-y)| \|f\|_p \right| \right\}^p dy \\ & = \int_{\mathbb{R}^n} \left| |g(y)| - |h(-y)| \right|^p dy \int_{\mathbb{R}^n} |f(x)|^p dx \end{aligned}$$



This proves for  $0 < \beta < 1$  and  $1 \leq p < n/\beta$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} dx dy \geq D_{p,\beta} \int_{\mathbb{R}^n} |x|^{-p\beta} |f(x)|^p dx$$

$$D_{p,\beta} = \int_{\mathbb{R}^n} \left| 1 - |x|^{-\lambda} \right|^p |x - \eta|^{-n-p\beta} dx$$

for  $\lambda = (n - p\beta)/p$  and  $\eta \in S^{n-1}$

## 5. SURFACE CONVOLUTION (APRES KLAINERMAN & MACHEDON)

$$\int_S \frac{1}{|w - y|^\lambda} \frac{1}{|y|^\mu} d\sigma$$

$w \in \mathbb{R}^m$  and  $S =$  smooth submanifold in  $\mathbb{R}^n$

$$(g * f_1 * \cdots * f_n)(w), \quad g \in L^1(\mathbb{R}^n), \quad f_k \in L^{n/(n-1)}(\mathbb{R}^n)$$

Replace  $f_k$ 's by Riesz potentials; constrain multivariable integration to hyperbolic surface

$$|w|^2 \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \delta \left[ \tau \sum' |x_k|^2 - |x_n|^2 \right] \delta \left( w - \sum x_k \right) \prod |x_k|^{-(n-1)} dx_1 \cdots dx_n$$

## 6. APRES BOURGAIN-BREZIS-MIRONESCU THEOREM

### Theorem

For  $f \in \mathcal{S}(\mathbb{R}^n)$ ,  $0 < \beta < 1$  and  $1 \leq p < n/(\alpha + \beta)$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|(\Lambda_\alpha f)(x) - (\Lambda_\alpha f)(y)|^p}{|x - y|^{n+p\beta}} dx dy \geq c \left( \int_{\mathbb{R}^n} |f|^{q^*} dx \right)^{p/q^*}, \quad (1)$$
$$q^* = \frac{pn}{n - p(\alpha + \beta)}$$

**Proof:** (1) Set  $g = \Lambda_\alpha f$  and apply Symmetrization Lemma

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^p}{|x - y|^{n+p\beta}} dx dy \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g^*(x) - g^*(y)|^p}{|x - y|^{n+p\beta}} dx dy$$

(2) Apply “triangle inequality estimate”

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g^*(x) - g^*(y)|^p}{|x - y|^{n+p\beta}} dx dy \geq D_{p,\beta} \int_{\mathbb{R}^n} |x|^{-p\beta} |g^*(x)|^p dx$$

$g^*$  non-negative & radial decreasing

$$g^*(x) \leq c|x|^{-n/q}, \quad q = pn/(n - p\beta)$$

$$(3) \quad \int_{\mathbb{R}^n} |x|^{-p\beta} |g^*(x)|^p dx \geq c \left[ \int_{\mathbb{R}^n} |g^*(x)|^q dx \right]^{p/q} = c \left[ \int_{\mathbb{R}^n} |\Lambda_\alpha f|^q dx \right]^{p/q}$$

$$(4) \quad \left[ \int_{\mathbb{R}^n} |\Lambda_\alpha f|^q dx \right]^{p/q} \geq c \left[ \int_{\mathbb{R}^n} |f|^{q^*} dx \right]^{p/q^*}$$

since  $\left\| \frac{1}{|x|^{n-\alpha}} * f \right\|_{L^{q^*}(\mathbb{R}^n)} \leq c \|f\|_{L^q(\mathbb{R}^n)}$

for  $q^* = np/(n - p(\alpha + \beta))$

## Tools — Symmetrization Lemma

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} dx dy \\ & \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f^*(x) - f^*(y)|^p}{|x - y|^{n+p\beta}} dx dy \end{aligned}$$

for  $p \geq 1$  and  $0 < \beta < 1$

$f^*$  = radial equimeasurable decreasing rearrangement of  $|f|$

## 7. HAUSDORFF-YOUNG INEQUALITY FOR FRACTIONAL DERIVATIVES

Aronszajn-Smith

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\beta}} dx dy = D_\beta \int_{\mathbb{R}^n} |\xi|^{2\beta} |\widehat{f}(\xi)|^2 d\xi$$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} dx dy \rightsquigarrow \int_{\mathbb{R}^n} \left[ |\xi|^\beta |\widehat{f}(\xi)| \right]^{p'} d\xi$$

**Theorem**

$$0 < \beta < 1, \quad 1 < p < \infty, \quad 1/p + 1/p' = 1$$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} dx dy \geq c \left[ \int_{\mathbb{R}^n} \left[ |\xi|^\beta |\widehat{f}(\xi)| \right]^{p'} d\xi \right]^{p/p'} \quad 1 < p \leq 2$$

$$\leq c \left[ \int_{\mathbb{R}^n} \left[ |\xi|^\beta |\widehat{f}(\xi)| \right]^{p'} d\xi \right]^{p/p'} \quad 2 \leq p < \infty$$

## 8. SMOOTHNESS FUNCTIONALS AND SIZE ESTIMATES

**Example:**

$$\begin{aligned} c \int_{\mathbb{R}^n} |f|^2 dx &= \int (\nabla f)(x) \cdot (\nabla f)(y) |x - y|^{-(n-2)} dx dy \\ &\leq d \left[ \int |\nabla f|^p dx \right]^{2/p}, \quad p = 2n/(n+2) \end{aligned}$$

Interesting examples:  $q \neq 2$  ?