Inequalities in Harmonic Analysis

A modern panorama on classical ideas

William Beckner
The University of Texas at Austin
Nanjing — May 2012

Purpose: Development of models to rigorgously describe many-body interactions and behavior of dynamical phenomena has suggested novel multilinear embedding estimates and forms that characterize fractional smoothness. This framework increases understanding for genuinely n-dimensional aspects of Fourier analysis.

Goals: To have an understanding of the tools we use from first principles, and to gain insight for the balance between weighted inequalities that connect size estimates for a function and its Fourier transform.

Eli Stein (1967)

"we shall begin by studying the fractional powers of the Laplacian"

Hardy–Littlewood–Sobolev Inequality

$$f \longrightarrow |x|^{-\lambda} * f\ , \qquad 0 < \lambda < n$$

$$\big\| |x|^{-\lambda} * f \big\|_{L^q(\mathbb{R}^n)} \leq A \|f\|_{L^p(\mathbb{R}^n)} \qquad \frac{1}{q} = \frac{\lambda}{n} + \frac{1}{p} - 1$$

Hausdorff–Young Inequality

$$(\mathcal{F}f)(x) = \hat{f}(x) = \int e^{2\pi ixy} f(y)\, dy$$

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^n)} \leq A \|f\|_{L^p(\mathbb{R}^n)} \qquad \frac{1}{p} + \frac{1}{p'} = 1\ , \ \ 1 \leq p \leq 2$$

Sobolev Embedding

$$\int_{\mathbb{R}^n} |\hat{f}|^2 \, d\xi \leq c \left[\int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/2} f \right|^p dx \right]^{2/p}$$

$$\alpha = n \left(\frac{1}{p} - \frac{1}{2} \right) \geq 0 \,, \qquad 1 < p \leq 2$$

Uncertainty & Pitt's Inequality

$$\left[\int_{\mathbb{R}^n} |f|^2 \, dx \right]^2 \leq B_\alpha \int_{\mathbb{R}^n} |x|^\alpha |f|^2 \, dx \int_{\mathbb{R}^n} |\xi|^\alpha |\hat{f}|^2 \, dx$$

$$B_\alpha \simeq \left(\frac{4\pi}{n} \right)^\alpha$$

Paradigms & Principles

1. Characterization of smoothness
2. Rigorous description of many-body interactions
3. Establish sharp embedding estimates
4. Expand working framework for
    a. Fourier transform
    b. convolution
    c. Riesz potentials
    d. Stein-Weiss integrals (Hardy-Littlewood-Sobolev inequalities)
    e. weights & symmetrization
    f. analysis on Lie groups and manifolds with negative curvature

5. Gain new insight
    a. uncertainty
    b. restriction phenomena
    c. geometric symmetry
6. Effort for optimal constants
    a. new features for exact model calculations
    b. encoded geometric information
    c. precise lower-order effects
7. Symmetry determines structure
8. Multilinear analysis $\rightsquigarrow$ understanding for genuinely n-dimensional aspects of Fourier analysis

Objects of Study

$$\Lambda_\alpha = (-\Delta/4\pi^2)^{\alpha/2}\,, \quad \alpha > 0\,,\ \ 0 < \beta < 1 \ \text{ and } \ 1 \le p < n/\beta$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy \quad \rightsquigarrow \quad \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|(\nabla f)(x)-(\nabla f)(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy$$

$$\rightsquigarrow \quad \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|(\Lambda_\alpha f)(x)-(\Lambda_\alpha f)(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy$$

1. Aronszajn-Smith Formulas

Classical Formula: $0 < \alpha < 2$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}}\, dx\, dy = D_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{f}(\xi)|^2\, d\xi$$

Frank-Lieb-Seiringer: $0 < \alpha < \min(2, n)$; $g = |x|^\lambda f$, $0 < \lambda < n - \alpha$

$$D_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{f}(\xi)|^2\, d\xi = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x-y|^{n+\alpha}} \left[|x|\,|y|\right]^{-\lambda} dx\, dy$$

$$+ \Lambda(\alpha, \lambda, n) \int_{\mathbb{R}^n} |x|^{-\alpha} |f(x)|^2\, dx$$

Beckner: $0 < \beta < 2, \beta \leq \alpha < n; g = |x|^{(n-\beta)/2}(-\Delta/4\pi^2)^{(\alpha-\beta)/4} f$

$$C_\alpha \int_{\mathbb{R}^n} |\xi|^\alpha |\widehat{f}(\xi)|^2 \, d\xi \geq \int_{\mathbb{R}^n} |x|^{-\alpha} |f(x)|^2 \, dx$$

$$+ \frac{C_\alpha}{D_\beta} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x - y|^{n+\beta}} \left[|x| \, |y| \right]^{-(n-\beta)/2} dx \, dy$$

2. Multilinear Fractional Embedding (apres Gross & Pitaevski)

Pitt's inequality: $n - \beta = mn - \alpha$, $\alpha = \sum \alpha_k$, $0 < \alpha_k < n$, $(m-1) < \alpha/n < m$

$$\int_{\mathbb{R}^n} |x|^{-\beta} |f(x, \cdots, x)|^2 \, dx \le C_\beta \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} |\Pi(-\Delta)k/4\pi^2)^{\alpha_k/4} f|^2 dx_1 \cdots dx_m$$

Hardy-Littlewood-Sobolev inequality: $mn - \alpha = 2n/q$

$$\left[\int_{\mathbb{R}^n} |f(x, \cdots, x)|^q \, dx \right]^{2/q} \le F_\alpha \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} |\Pi(-\Delta_k/4\pi^2)^{\alpha_k/4} f|^2 dx_1 \cdots dx_m$$

Similar results on S^n

Key insight on "multilinear products"

$$F(x) = \int_{\mathbb{R}^{mn}} \Pi g_k(x - y_k) H(y) \, dy \quad ; \quad H \in L^p(\mathbb{R}^{mn}) \quad \rightsquigarrow \quad F \in L^q(\mathbb{R}^n)$$

"classical uncertainty principle"

$$c \int_{\mathbb{R}^n} |f|^2 \, dx \le \int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/4} |x|^{\alpha/2} f(x) \right|^2 dx$$

Restriction to k-dimensional linear sub-variety

$$d \int_{\mathbb{R}^k} |\mathcal{R}f|^2 \, dx \le \int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/4} |x|^{\beta/2} f(x) \right|^2 dx$$

with $n - \alpha = k - \beta, \quad n \ge k > \beta > 0$

$$d = \pi^{-\alpha} \, \frac{\Gamma(\alpha/2)}{\Gamma(\beta/2)} \, \left[\frac{\Gamma\left(\frac{k+\beta}{4}\right)}{\Gamma\left(\frac{k-\beta}{4}\right)} \right]^2$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |g(y-x)f(x) - h(x-y)f(y)|^p \, dx\, dy$$

$$\geq \int_{\mathbb{R}^n} \Big| |g(y)| - |h(-y)| \Big|^p dy \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

Proof: $p \geq 1$

$$\int_{\mathbb{R}^n} \left\{ \left(\int_{\mathbb{R}^n} |g(y)f(x) - h(-y)f(y)|^p \, dx \right)^{1/p} \right\}^p dy$$

$$\geq \int_{\mathbb{R}^n} \Big\{ \Big| \, |g(y)| \, \|f\|_p - |h(-y)| \, \|f\|_p \Big| \Big\}^p \, dy$$

$$= \int_{\mathbb{R}^n} \Big| \, |g(y)| - |h(-y)| \, \Big|^p dy \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

This proves for $0 < \beta < 1$ and $1 \le p < n/\beta$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} \, dx \, dy \ge D_{p,\beta} \int_{\mathbb{R}^n} |x|^{-p\beta} |f(x)|^p \, dx$$

$$D_{p,\beta} = \int_{\mathbb{R}^n} \left| 1 - |x|^{-\lambda} \right|^p |x - \eta|^{-n-p\beta} \, dx$$

for $\lambda = (n - p\beta)/p$ and $\eta \in S^{n-1}$

5. Surface Convolution (apres Klainerman & Machedon)

$$\int_S \frac{1}{|w-y|^\lambda} \frac{1}{|y|^\mu} \, d\sigma$$

$w \in \mathbb{R}^m$ and $S =$ smooth submanifold in $\mathbb{R}^n$

$$(g * f_1 * \cdots * f_m)(w) \, , \quad g \in L^1(\mathbb{R}^n) \, , \quad f_k \in L^{n/\alpha_k}(\mathbb{R}^n)$$
$$\alpha = \sum \alpha_k = n(m-1) \, , \qquad 0 < \alpha_k < n$$

Replace f_k's by Riesz potentials; constrain multivariable integration to hyperbolic surface

$$|w|^\sigma \int_{\mathbb{R}^n \times \cdots \times \mathbb{R}^n} \delta \left[\tau \sum{}' |x_k|^2 - |x_m|^2 \right] \delta \left(w - \sum x_k \right) \Pi |x_k|^{-\alpha_k} \, dx_1 \cdots dx_n$$

Objective: Multilinear Embedding Estimates

$$\left[\int\left[\int\left|\sum x_k\right|^{\lambda}|\hat{f}|^r\,d\nu\right]^q dw\,d\tau\right]^{p_*/(rq)} \le c\Lambda_{p_*}(f;\{\beta\})$$

$$\Lambda_{p_*}(f;\{\beta\}) = \int_{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}\left|\prod_{n=1}^{m}\left(-\Delta_k/4\pi^2\right)^{\beta_k/2}f\right|^{p_*}dx_1\ldots dx_m$$

6. Apres Bourgain-Brezis-Mironescu Theorem

Theorem

For $f \in \mathcal{S}(\mathbb{R}^n)$, $0 < \beta < 1$ *and* $1 \leq p < n/(\alpha + \beta)$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|(\Lambda_\alpha f)(x) - (\Lambda_\alpha f)(y)|^p}{|x-y|^{n+p\beta}} \, dx \, dy \geq c \left(\int_{\mathbb{R}^n} |f|^{q^*} \, dx \right)^{p/q^*} , \tag{1}$$

$$q^* = \frac{pn}{n - p(\alpha + \beta)}$$

Proof: (1) Set $g = \Lambda_\alpha f$ and apply Symmetrization Lemma

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x)-g(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy \geq \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g^*(x)-g^*(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy$$

(2) Apply "triangle inequality estimate"

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g^*(x)-g^*(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy \geq D_{p,\beta}\int_{\mathbb{R}^n} |x|^{-p\beta}|g^*(x)|^p\,dx$$

g^* non-negative & radial decreasing

$$g^*(x) \leq c|x|^{-n/q}\ , \quad q = pn/(n-p\beta)$$

(3) $$\int_{\mathbb{R}^n} |x|^{-p\beta}|g^*(x)|^p dx \geq c\left[\int_{\mathbb{R}^n} |g^*(x)|^q dx\right]^{p/q} = c\left[\int_{\mathbb{R}^n} |\Lambda_\alpha f|^q dx\right]^{p/q}$$

(4) $$\left[\int_{\mathbb{R}^n} |\Lambda_\alpha f|^q \, dx\right]^{p/q} \geq c\left[\int_{\mathbb{R}^n} |f|^{q^*} \, dx\right]^{p/q^*}$$

since $$\left\|\frac{1}{|x|^{n-\alpha}} * f\right\|_{L^{q*}(\mathbb{R}^n)} \leq c\|f\|_{L^q(\mathbb{R}^n)}$$

for $q^* = np/(n - p(\alpha + \beta))$

Tools — Symmetrization Lemma

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy$$

$$\geq \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f^*(x)-f^*(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy$$

for $p \geq 1$ and $0 < \beta < 1$

$f^* =$ radial equimeasurable decreasing rearrangment of $|f|$

General Symmetrization Lemma

$$M \quad \rightsquigarrow \quad \mathbb{R}^n,\ S^n,\ H^n \quad \text{(hyperbolic space)}$$

$$\int_{M\times M} \varphi\left[\frac{|f(x)-f(y)|}{\rho[d(x,y)]}\right] K\left[d(x,y)\right]\,dx\,dy$$

$$\geq \int_{M\times M} \varphi\left[\frac{|f^*(x)-f^*(y)|}{\rho[d(x,y)]}\right] K\left[d(x,y)\right]\,dx\,dy$$

$\varphi, K, \rho \geq 0$ on $[0,\infty)$

(i) $\varphi(0)$, φ convex and monotone increasing, $\varphi''(0) \geq 0$ and $t\varphi'(t)$ convex

(ii) K monotone decreasing, ρ monotone increasing

(iii) $d(x,y)$ = distance between x and y

(iv) ρ constant $\Longrightarrow$ remove last hypothesis on φ

7. Hausdorff-Young Inequality for Fractional Derivatives

Aronszajn-Smith

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\beta}}\,dx\,dy = D_\beta \int_{\mathbb{R}^n} |\xi|^{2\beta}|\widehat{f}(\xi)|^2\,d\xi$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy \quad \rightsquigarrow \quad \int_{\mathbb{R}^n} \left[|\xi|^{\beta}|\widehat{f}(\xi)|\right]^{p'}\,d\xi$$

Theorem

$0<\beta<1, \quad 1<p<\infty, \quad 1/p+1/p'=1$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p\beta}}\,dx\,dy \geq c\left[\int_{\mathbb{R}^n} \left[|\xi|^{\beta}|\widehat{f}(\xi)|\right]^{p'}\,d\xi\right]^{p/p'} \quad 1<p\leq 2$$

$$\leq c\left[\int_{\mathbb{R}^n} \left[|\xi|^{\beta}|\widehat{f}(\xi)|\right]^{p'}\,d\xi\right]^{p/p'} \quad 2\leq p<\infty$$

8. Smoothness Functionals and Size Estimates

Example:

$$c \int_{\mathbb{R}^n} |f|^2 \, dx = \int (\nabla f)(x) \cdot (\nabla f)(y) \; |x - y|^{-(n-2)} \, dx \, dy$$

$$\leq d \Big[\int |\nabla f|^p \, dx \Big]^{2/p} , \qquad p = 2n/(n+2)$$

Interesting examples: $\quad q \neq 2$?

9. New Objects of Study

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} K(x-y) \big| f(x)(\nabla f)(y) - f(y)(\nabla f)(x) \big|^p \, dx\, dy$$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} K(x-y) \big| f(x)g(y) - f(y)g(x) \big|^p \, dx\, dy$$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} K(y) \big| f(x+y) + f(x-y) - 2f(x) \big|^p \, dx\, dy$$

$\rightsquigarrow$ role of convolution

n-dimensional Euclidean space

$\rightsquigarrow$ manifold with non-positive sectional curvature

homogeneous under action of non-unimodular Lie group

hyperbolic space $\mathbb{H}^n$ — $L_s = -\Delta_H + s(s-n+1)\mathbf{1}, \quad s \geq (n-1)/2$

potentials $\rightsquigarrow$ fundamental solutions

$$\left[\|F\|_{L^q(\mathbb{H}^n)}\right]^2 \leq A_q \int_{\mathbb{H}^n} F(L_s F)\, d\nu \,, \qquad q > 2$$

Question: When can you compute optimal values for A_q?

Model embedding structure from Euclidean framework.

References

W. Beckner, *Weighted inequalities and Stein-Weiss potentials*, Forum Math. **20** (2008), 587–606.

W. Beckner, *Pitt's inequality with sharp convolution estimates*, Proc. Amer. Math. Soc. **136** (2008), 1871–1885.

W. Beckner, *Pitt's inequality and the fractional Laplacian: sharp error estimates*, Forum Math. **24** (2012), 177–209.

W. Beckner, *Multilinear embedding estimates for the fractional Laplacian*, Math. Res. Lett. **19** (2012), 1–15.

W. Beckner, *Multilinear embedding — convolution estimates on smooth submanifolds*, arXiv: 1204.5684

W. Beckner, *Embedding estimates and fractional smoothness*, (in preparation)

W. Beckner, *Analysis on Lie groups — embedding potentials*, (in preparation)