On Lie Groups and Beyond ⇝
Kunze – Stein Phenomena, $SL(2,\mathbb{R})$, the Heisenberg Group and the Role of Symmetry

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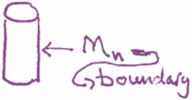
Analysis ⇝ Fundamental Direction

Basic Question: Prove Estimates that Encode Geometric Information

$\mathbb{R}^n \longrightarrow$ Geometric Manifold ⇝ Lie Group

Iwasawa Components — Product Manifolds

$$\boxed{\mathcal{M} = M_n \times \Sigma_m}$$



M_n = non-compact manifold of dimension n with non-positive curvature

Σ_m = compact manifold of dimension m

Break Euclidean Symmetry

ⓐ exponential growth of balls — negative curvature
hyperbolic space — Cartan — Hadamard manifold

ⓑ mixed homogeneity — include symplectic structure

Two intrinsic Lie Groups

$SL(2,\mathbb{R})$

most basic matrix group
— action by fractional linear transformations
on the upper-half plane

Heisenberg Group $\mathcal{H} \simeq \mathbb{C} \times \mathbb{R}$

simplest natural example that breaks the uniform homogeneity of the Euclidean manifold

group multiplication incorporates symplectic form

$$\boxed{\begin{array}{l} (z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}(\bar{z}_1 z_2)) \\ \text{metric} \quad d(p, \hat{0}) = \rho = (|z|^4 + t^2)^{1/4} \end{array}}$$

Tools

Convolution

$$\boxed{(f * g)(x) = \int_G f(y)\, g(y^{-1}x)\, dy}$$

Young's Inequality

Hardy–Littlewood–Sobolev Inequality

Riesz Potentials

Stein–Weiss Integrals

Metrics & Distance Functions

Rearrangement & Symmetrization

Riesz–Sobolev Rearrangement on $\mathbb{R}^n$

Brascamp–Lieb–Luttinger Rearrangement on $\mathbb{R}^n$

Riesz–Sobolev Rearrangement on $\mathbb{H}^n$

$$\boxed{\left| \int_{\mathbb{H}^n} (f * g)(w)\, h(w)\, d\nu \right| \le \int_{\mathbb{H}^n} (f^* * g^*)(w)\, h^*(w)\, d\nu}$$

f^* = equimeasurable radial decreasing rearrangement $|f|$

Two–Point Rearrangement on the Sphere

New Concepts

Coupled Metrics & Riesz Potential

$\mathcal{M} = M_n \times \Sigma_m \qquad d^2_{\mathcal{M}} = d^2_M + d^2_\Sigma$

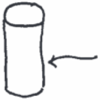
$$\boxed{\int_{\mathcal{M}\times\mathcal{M}} f(w)\, K\left[d^2(w,w')\right] f(w')\, dw\, dw'}$$

$K : [0,\infty) \longrightarrow [0,\infty)$ non-increasing

$f \in L^p(\mathcal{M})$ for $1 < p \leq 2$

Method — Break Group Symmetry

use Symmetry on the cylindrical Boundary



boundary $\simeq M_n$

Two Examples

$SL(2,\mathbb{R}) \rightsquigarrow$ Kunze–Stein Phenomena

Heisenberg Group $\mathcal{H} \simeq \mathbb{C} \times \mathbb{R}$
$\rightsquigarrow$ Hardy–Littlewood–Sobolev Inequality

Kunze–Stein Phenomena

$$\boxed{\|f*g\|_{L^2(G)} \leq A_p \|f\|_{L^p(G)} \|g\|_{L^2(G)}} \qquad 1 \leq p < 2$$

ⓐ Kunze & Stein — $SL(2,\mathbb{R})$ or any complex classical group

observation — analytic continuation of group representati[ons]

ⓑ Stein — holds for f bi-invariant on semi-simple Lie group

ⓒ Cowling — all semi-simple Lie groups (finite center)

use representation theory & interpolation

$G = SL(2,\mathbb{R})$ — 3 functions $f, g, h \geq 0$ $\quad g, h \in L^2(G)$

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Theorem

$$\int_G (f * g)(x)\, h(x)\, dx \lesssim c_p \, \|F\|_{L_{p,\infty}(\mathbb{H}^2)} \|g\|_{L^2(G)} \, \|h\|_{L^2(G)}$$

$$c_p = \pi \left[\frac{\Gamma(1/p')}{\Gamma(1/p)} \right] \left[\frac{\Gamma[(1 - 2/p')/4)]}{\Gamma[(1 + 2/p')/4)]} \right]^2$$

$$F = \left(\int_K |f_{\#}(w,k)|^p \, dk \right)^{1/p} \in L_{p,\infty}(\mathbb{H}^2)$$

Similar Argument: $SL(2,\mathbb{C})$, Lorentz Groups

Hardy–Littlewood–Sobolev Inequality on $\mathcal{H}$

$\mathcal{H} \simeq \mathbb{C} \times \mathbb{R} \Rightarrow SL(2,\mathbb{R}) \simeq M = \mathbb{H}^2 \times S^1$ $\quad 1 < p < 2\ , \ 1/p + 1/p' = 1$

Theorem 1

$$\boxed{\begin{aligned}&\int_{M\times M} f(w,\xi)\, k\left[d^2(w,w'),\ |\xi-\eta|^2\right] g(w',\eta)\, dm\, dm' \\ &\qquad \le C_p \|f\|_{L^p(M)} \|g\|_{L^p(M)}\end{aligned}}$$

$$d(w,w') = \frac{|w-w'|}{2\sqrt{yy'}} \qquad W = 1 + d^2(w,w')$$

$$k\left[d^2(w,w'),\ |\xi-\eta|^2\right] = \left[(\sqrt{W}-1)^2 + \sqrt{W}\,|\xi-\eta|^2\right]^{-2/p'}$$

Extremal Functions: $A(1+u)^{-2/p}$, $u = d^2(w,\hat{0})$

Theorem 2

$$\boxed{\|\rho^{-\lambda} * f\|_{L^{p'}(\mathbb{H})} \lesssim B_p \|f\|_{L^p(\mathbb{H})}}, \quad 1 < p < 2, \quad \lambda = Q/p' = 2N/p'$$

$\rho = (|z|^4 + t^2)^{-1/4}$ Extremal Functions: $A\left[(1+|z|^2)^2 + t^2\right]^{-N/p}$

Balance Symmetries

Theorem 1 $\Longrightarrow$ Theorem 2

- Hyperbolic Symmetry
- Inversion Symmetry
- Spherical Symmetry

$\mathbb{H} \approx \partial B$

$B =$ unit ball in $\mathbb{C}^{n+1}$

Paradigm

Kunze–Stein Phenomena

$G = SL(2,\mathbb{R}) \rightsquigarrow \mathcal{M} = \mathbb{H}^2 \times S^1$

Apply Riesz–Sobolev Rearrangement on Hyperbolic Space

Break Group Action of $SL(2,\mathbb{R})$

Replace Convolution Inequality by Stein–Weiss Integral

Hardy–Littlewood–Sobolev Inequality

$G = \mathbb{H} \rightsquigarrow M = \mathbb{H}^2 \times S^1$

Lift Riesz Potential to $SL(2,\mathbb{R})$

Break Intrinsic Symmetry of Heisenberg Group

Competing Symmetry — Hyperbolic Invariance, Inversion Symmetry, Spherical Representation