0. Category - Longlands duality for quantum groups

Genetic logics:

origins from \( \text{Perv}_{G_{[1]}[1]}(G_{\mathfrak{n}}) \rightarrow \text{Perv} \tilde{G} \)

- does not survive on derived level:
  \( \text{RHS} \text{ semi-simple, LHS not.} \)

RHS deforms to \( \text{Rep}^y(C^x) := \{ \chi : y = \text{mod} y \}_{y} \)

Want to deform LHS to perverse sheaves twisted by powers of the determinant line

\( \text{Perv}_{G_{[1]}[1]}(G_{\mathfrak{n}}) \) has only one object.

Replace to get (even of \( q=1 \) a derived equivalence):

\( \text{Whitaker sheaves} \)

\( \text{Perv}_{N((H))_{\mathfrak{X}}} (G_{\mathfrak{n}}) = \text{Rep} \tilde{G} \)

Jade Lue conjectured \( \text{Perv}^\mathfrak{y}_{(H)_{\mathfrak{X}}}(G_{\mathfrak{n}}) = \text{Rep}_{\mathfrak{Y}} \tilde{G} \)
- can prove at least $q$ of not $d$ at all
probably in general.

\[
\text{LIne } \text{Pou } (G) \xrightarrow{n} \text{Rep } (G)
\]

\[
F_{S^2} \xrightarrow{s} \text{factorizable spaces}
\]

Enkelberg - Slechtma (at least for generic)

Understand Beuzekommer-Enkelberg - Slechtma

took using $\mathbb{F}_2$ / Had - algebra points from
Lurie's talk.

\[
\text{Quantum geometric stacks } \quad q = \exp (\mathbb{F}_2)
\]

basis for quantum algebra

Kitaev-Lesfog - reps of other algebra

Whitten category

\[
F_{S^2} \xrightarrow{s} \text{Rep } \mathbb{F}_2
\]
Today: discuss $F^G \subseteq \text{Rep}_k G$

[But denote $G^0 \rightarrow G$; only one group involved today]

$\Lambda = \text{weight lattice}$

$q: \Lambda \times \Lambda \rightarrow \mathbb{C}^*$ defining (level)

assume generic: $q$ (rational) not root of unity

$\mathcal{O}_G$ - quantum category $\mathcal{O}$:

- objects as $\Lambda$-graded vector spaces
- with operators $E_i$, $i \in I$ action of Dynkin diagram $F_i$: raise & lower grad

as usual,

$E_i, F_j: \mathcal{E}_i, \mathcal{F}_j \rightarrow \mathcal{E}_{i+j}, \mathcal{F}_{j+i}$

$E_i, F_i: \mathcal{E}_i, \mathcal{F}_i \rightarrow \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$

$q_i = q(x_i, \lambda)^{\Lambda_x}$

$K_i \cdot V^\Lambda = q(x_i, \lambda)^{\Lambda_x} V^\Lambda$ on $\Lambda$-graded place

+ quantum Serre relations (axioms automatically)
+ modules have grady bound as above
We'll describe this category via perverse sheaves on $\text{config.}$ spaces + factorization.

$\mathcal{E}, \mathcal{F}, \mathcal{K}$ form a Hopf algebra

\[ \Delta \mathcal{E} = K \otimes \mathcal{E} \oplus \mathcal{E} \otimes K \]

- Let $U^-$ be the associative algebra generated by $\mathcal{E}$
- The $\mathcal{E}$ is not a Hopf algebra due to $(\Delta)$

Consider instead braided monoidal category $\text{Vec}^+$

\[ \text{Vec}^+ : A \text{-graded vector spaces} \]

but with braiding

\[ U^- \otimes V^+ \overset{\rho}{\rightarrow} V^+ \otimes U^- \]

$U^-$ is a Hopf algebra in $\text{Vec}^+$

$O_{\mathcal{E}} \rightarrow \mathbb{Z}(U^-)$ functor to modify for the Dixmier coalgebra

\[ \text{Lema 2 shows: this functor is an equivalence} \]
Recap from Jacob’s talk:

Let $A$ be a Hopf algebra in the category $\text{A-mod}^L$ with left and right coactions.

$Z(\text{A-mod}^L)$ is the Drinfeld center of $\text{A-mod}^L$.

A is augmented as an algebra.

Take Koszul dual to $A$ as an algebra, call it $B$ (Bar or $A^!$).

$B$ has two compatible coalgebra structures, i.e., it is an $E_2$ coalgebra.

Koszul duality yields equivalence of left comodules $\text{A-mod}^L \rightleftharpoons \text{B-comod}^L$. 

- equivalence of monoidal categories
  as appropriately defined derived categories (assuming suitable finiteness)
  $M$ is normal for $k^0k = B$.

Therefore we obtain
\[
\mathbb{Z} (A_{\text{mod}}) \cong \mathbb{Z} (B_{\text{mod}})
\]
\[
= B_{E_2\text{-comodule}}.
\]

OTOH for $B$ an $E_2$-coring
the (cofiber $\Delta$) shears $B$ on $\text{Ran}(\mathbb{R}^2)$

with factorization:

\[
\begin{array}{ccc}
\text{Ran}(\mathbb{R}^2) \times \text{Ran}(\mathbb{R}^2) & \to & (\text{Ran} \times \text{Ran})_{\text{disj}} \\
\downarrow \phi & \searrow & \downarrow \phi \\
\text{Ran}(\mathbb{R}^2) & \ni & \text{Ran}(\mathbb{R}^2)
\end{array}
\]

\[
\phi : B \cong B \otimes B_{\text{disj}}
\]

factorization algebra
Pick $0 \in \mathbb{R}^2$.

A $B$-factorization module is a sheaf $F$ on $\text{Rep}(\mathbb{R}^2)$ such that:

$$F \left( \text{Rep}(\mathbb{R}^2) \right) \supset F \otimes B$$

Pairs of subsets which are disjoint (second subset does not contain $0$)

Data must be associative with structure on $B$.

Theorem: $B$-factorization module $\iff$ $E_\phi$-module for $B$

Both sides are in some sense factorization or $E_\phi$ categories.

Contravariant braiding on $V_{E_\phi}$ will mean we get anti-symmetry but structures twisted by some gerbe...
\( \Lambda \geq \Lambda^0 \) negative weights: negative comb of simple roots.

We'll take \( X = A^1 \), but \( U^- \) has a ribbon structure \( \implies \text{compact on any Riemann surface} \).

\[
X^\lambda = \Lambda^- \text{-valued divisors on } X
\]

\[
\sum_{\lambda} x^\lambda = s, \quad \forall \lambda \in \Lambda^-, \quad \sum_{\lambda} x^\lambda = x.
\]

Since everything is good we'll be effectively living on the many strata of Riemann space --- strong torsors.

\( x_0 \in X \leadsto X_{x_0}^\lambda : \text{divisor } \sum_{\lambda} x^\lambda \) for \( \lambda \in \Lambda^- \) for \( x \neq x_0 \),

with any \( \lambda \) of \( x_0 \).

Addition of divisors:

\[
X^\lambda \times X^\mu \to X^{\lambda + \mu}
\]
\[ X_{x_0} \times X^1 \to X^0 \]

and have open subsets of disjoint

Claim: The data of \(\mathcal{G}\) defines a \(\mathcal{C}^0\)

gerbe \(\mathcal{P}_{X_{x_0}}\) on \(X^1_{x_0}\)

which factorizes:

\[
\mathcal{P} \times \mathcal{P} \to \mathcal{P} 
\]

Recall: \(Y\) a topological space, \(\mathcal{G}\) a \(\mathcal{C}^0\)

kerbe \(\mathcal{P} = g_{Y,y}\) :

specify a category locally with single transitive

category of sheaves on \(Y\) with

monodromy = \(\mathcal{G}\) \(\mathcal{C}^0\)
For $\Sigma \times x_i \subset X^{x_i}$ the line by \[ U_g \to B = \{ B^\Lambda \} \]

which we'll twist is \[ \otimes_{x_i} \varphi(\lambda, \lambda; 2p) \]

(cohesive \( \Lambda \))

(\( p \) has to do with twistiness of ribbon structure)

\[ \mathcal{B}^\Lambda \]

is a \( P_{X^{x_i}} \) - twist perverse sheaf

\[ \mathcal{B}^\Lambda \bigg/ (x_i^\ast x_i^\ast)^{\lambda_{ij}} \]

Let \( X^i \supset x_i \) : \( \Sigma x_i x_i \) where each \( \lambda^i \) is the negative of \( a \) simple root

\( P_{X^i} \bigg/ x_i \) acquires a canonical trivialization \[ q(\lambda_i, \lambda; 2p) = 1 \] for \( \lambda_i \) simple root...
easy to take power series on $X^*$

$Y^*_X = \text{sign local system}$

Key fact ($Y$ not nodal):

$Y^*_X = j_! j_* (Y^*/j^*X)$

-- this is where the same relation /
Caran dihydral appears...

if we use $j_!$ get Drinfeld double
of free Hopf algebra on $F_i$

$j_! \rightarrow \text{cofree algebra}$

Some relations enforce $j_! \rightarrow j^*_X$:

$\mathcal{V}_q = \text{Im} (\text{free} \rightarrow \text{cofree})$

on $F_i$ ...
Def: A factorizable sheaf \( F \) is a collection
\[
\{ F^i, i \in I \} \quad F^i \in \text{Div}(X^i)_{\mathbb{Z},\mathbb{Z}}
\]
endowed with a system of isomorphisms
\[
F^i_{x^i, x^j} \cong F^j \otimes \mathcal{L}^3
\]
\[
\implies O_{x_i}.
\]

Construction of sheaf \( P \):
on \quad X_i \ldots \quad X_{i-1}, x_i \quad \ldots \quad X_n

the sheaf is \( \bigotimes_{x_i} \mathcal{O} \log z(x_i, x_{i-2}) \)
\[
\otimes \quad \mathcal{O}(\sum_i \log z(x_i, x_i))
\]
What happens at a root of unity?

codimension \( \text{Codim} \) of \( \text{Luzhi} \)
there won’t be a lot of entanglement, finite
number of essentially irreducible group reps.

Enough to match with Whittaker

Is $G$ at rest of unity and small quantum group

Fibres of $Y$: homologies of
our Hopf algebras, geometry encodes
some relations!

In good sign abeit modern
category we get chiral edge
and a chiral algebra in this category
- that’s our $P$. 