D. Gaitsgory - Langlands duality for quantum groups

Note Title 3/14/2008

Geometric Langlands:
originates from $\mathrm{Perv}_{G[[t]]}(Gr_G) \simeq \mathrm{Rep}\,\check{G}$

- does not survive on derived level:
RHS semisimple, LHS not.

RHS deforms to $\mathrm{Rep}^q(\check{G}) := U_q\check{\mathfrak{g}}\text{-modules}_{f.d.}$

Want to deform LHS to perverse sheaves twisted by powers of the determinant line bundles... but this fails

$\mathrm{Perv}^q_{G[[t]]}(Gr_G)$ has only one object

Replace to get (even at $q=1$) a derived equivalence: Whittaker sheaves
$\mathrm{Perv}_{N((t)),\chi}(Gr) \simeq \mathrm{Rep}\,\check{G}$

Jacob Lurie conjectured $\mathrm{Perv}^q_{N((t)),\chi}(Gr) \simeq \mathrm{Rep}^q\check{G}$

- can prove at least $q \notin$ root of unity, probably in general.

Link

$$
\begin{tikzcd}
\mathrm{Perv}^q_{N((t)),\chi}(Gr) \arrow[rr, "\sim"', "?"] \arrow[dr] & & \mathrm{Rep}^q(\check{G}) \arrow[dl, "S"] \\
& FS^q &
\end{tikzcd}
$$

factorizable sheaves
Finkelberg - Schechtman
(at least for generic q)

Understand Bezrukavnikov-Finkelberg-Schechtman book using E_2 / Hopf algebra picture from Lurie's talk.

<u>Quantum geometric Langlands</u> $\quad q = \exp\left(\frac{\pi i}{c}\right)$

$$
\begin{tikzcd}
& KL^c_{\check{G}} \arrow[dl, leftrightarrow, "\text{basis for quantum Langlands}"'] \arrow[dd, "S"] \arrow[dr, leftrightarrow] & \\
\text{Whittaker category} \arrow[dr] & & \mathrm{Rep}^q \check{G} \arrow[dl, dash, "S"] \\
& FS^q &
\end{tikzcd}
$$

Kazhdan-Lusztig: reps of affine algebra + integrability

Today: discuss $FS^q \xleftarrow{\sim} \mathrm{Rep}^q \check{G}$
[but denote $\check{G} \rightsquigarrow G$: only one group involved today]

Λ = weight lattice
$q : \Lambda \times \Lambda \longrightarrow \mathbb{C}^\times$ pairing (level)
assume generic: q(roots) not root of unity

$\mathcal{O}_q$ - quantum category $\mathcal{O}$:
Objects are Λ-graded vector spaces with operators E_i $i \in I$ vertices of Dynkin diagram F_i raise & lower grading as usual,

$E_i F_j = F_j E_i \qquad E_i F_i - F_i E_i = \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$

$q_i = q(\alpha_i, \alpha_i)$
& $K_i \cdot v^\lambda = q(\alpha_i, \lambda) v^\lambda$ on λ-graded piece

+ quantum Serre relations (arise automatically)
+ modules have grading bounded above

We'll describe this category via perverse sheaves on config. spaces + factorization.

E, F, K form a Hopf algebra

(*) $\Delta F_i = K_i \otimes F_i + F_i \otimes 1 \quad \cdots$

Let $U_q^- =$ associative algebra generated by the F_i : not a Hopf algebra due to (*)

– consider instead braided monoidal category $Vect_q^\Lambda$: Λ-graded vector spaces but with braiding

$$v^\lambda \otimes w^\mu \longmapsto w^\mu \otimes v^\lambda \cdot q(\lambda, \mu)$$

$\Rightarrow$ U_q^- is a Hopf algebra in $Vect_q^\Lambda$.

$$\mathcal{O}_q \xrightarrow{\quad} Z(U_q^- \text{-mod})$$ restriction functor to modules for the Drinfeld centre

<u>Lemma</u> q generic : this functor is an equivalence

ie R matrix gives $M \otimes N \to N \otimes M$
for $M \in \mathcal{O}_q$ & $N \in U_q^-$-mod,

So restriction functor lands in the Drinfeld center of U_q^--modules.

Recap from Jacob's talk

Let A be a Hopf algebra $\Rightarrow$
categories $A\text{-mod}^l$ left modules, monoidal
$Z(A\text{-mod}^l)$ braided monoidal

A is augmented as an algebra
$\Rightarrow$ take Koszul dual to A as an algebra, call it B (Bar of A)

$\Rightarrow$ has two compatible coalgebra structures, ie an E_2 coalgebra.

Koszul duality induces equivalence of left (co)mods
$$A\text{-mod}^l \longleftrightarrow B\text{-comod}^l$$

- equivalence of monoidal categories.
as appropriately defined derived categories (assuming suitable finiteness)

$M \otimes_A k$ comod for $k \otimes_A k = B$.

Therefore we obtain

$$Z(A\text{-mod}^{\ell}) \simeq Z(B\text{-comod}^{\ell})$$
$$= B\text{-}E_2\text{-comodules}.$$

OTOH for B an E_2-coalgebra
$\rightsquigarrow$ (complex of) sheaves $\mathcal{B}$ on $Ran(\mathbb{R}^2)$
with factorization:

open subset consisting of disjoint unions

$$\begin{tikzcd}
Ran(\mathbb{R}^2) \times Ran(\mathbb{R}^2) \arrow[d, "\sqcup"] & (Ran \times Ran)_{disj} \arrow[l, phantom, "\supset"] \arrow[dl, "j"] \\
Ran(\mathbb{R}^2) &
\end{tikzcd}$$

$$\boxed{j^* \mathcal{B} \simeq \mathcal{B} \boxtimes \mathcal{B}|_{disj}}$$

factorization algebra

Pick $0 \in \mathbb{R}^2$.

A $\mathcal{B}$-<u>factorization module</u> is a sheaf $\mathcal{F}$ on $\mathrm{Ran}(\mathbb{R}^2)$ s.t.

$$\boxed{\mathcal{F}\big|_{(\mathrm{Ran}\times\mathrm{Ran})_{disj}'} \simeq \mathcal{F} \boxtimes \mathcal{B}}$$

||

pairs of subsets
which are disjoint
& second subset doesn't contain 0

+ data must be associative wrt structure on $\mathcal{B}$.

<u>Theorem</u> $\mathcal{B}$-factorization module $\iff$ E_2-module for $\mathcal{B}$

[both sides are in some sense factorization or E_2 categories]

Nontrivial braiding on $\mathrm{Vect}_q^\wedge$ will mean we get not sheaves but sheaves twisted by some gerbe ...

$\Lambda \supset \Lambda^{neg}$ negative weights : negative combos of simple roots.

We'll take $X = \mathbb{A}^1_{\mathbb{C}}$ but U_q^- has a ribbon structure ⇝ compat on any Riemann surface.

X^λ = Λ-valued divisors on X

$$\sum_{x_i \in X} \lambda_i x_i \quad \text{s.t. } \forall \lambda_i \in \Lambda^{neg}$$
$$\& \ \sum \lambda_i = \lambda.$$

Since everything is graded we'll be effectively living on fin. many strata of Ran space --- strong finiteness

$x_0 \in X$ ⇝ $X^\lambda_{x_0}$: divisors $\sum \lambda_i x_i$ where $\lambda_i \in \Lambda^{neg}$ for $x_i \neq x_0$, can have any λ at x_0.

Addition of divisors: $X^{\lambda_1} \times X^{\lambda_2} \longrightarrow X^{\lambda_1 + \lambda_2}$

$$X_{x_0}^{\lambda_1} \times X^{\lambda_2} \longrightarrow X_{x_0}^{\lambda_1+\lambda_2}$$

+ have open subsets of disjoint finite subsets

Claim The data of q defines a $\mathbb{C}^*$ gerbe $\mathcal{P}_{X^\lambda_{x_0}, q}$ on $X^\lambda_{x_0}$ which factorizes:

$$\mathcal{P}_{X_{x_0}^{\lambda_1+\lambda_2}} \Big|_{(X_{x_0}^{\lambda_1} \times X^{\lambda_2})_{disj}} = \mathcal{P}_{X^{\lambda_1}_{x_0}, q} \boxtimes \mathcal{P}_{X^{\lambda_2}, q}$$

Recall: Y a topological space, $\mathcal{L}$ a line bundle$_{\mathbb{C}}$ & $q \in \mathbb{C}^\times \rightsquigarrow$ gerbe $\mathcal{P} = \mathcal{L}^{\otimes \log q}$:

specify a category locally with simple transitive action of line bundles:
category of sheaves on $\mathcal{L}^\times$ with monodromy $= q \in \mathbb{C}^*$

For $\sum \lambda_i x_i \in X^{\lambda}_{x_0}$ the line by which we'll twist is

$$\bigotimes_i \omega_{x_i}^{\otimes \log q(\lambda_i, \lambda_i + 2\rho)}$$

(cotangent lines)

(ρ has to do with twistedness of ribbon structures)

$U_q^- \rightsquigarrow \mathcal{B} = \{\mathcal{B}^\lambda\}$

$\mathcal{B}^\lambda$ is a $P_{X^\lambda, q}$ - twisted perverse sheaf

$$\mathcal{B}^\lambda \Big|_{\substack{(X^{\lambda_1} \times X^{\lambda_2})_{disj} \\ \lambda_1 + \lambda_2 = \lambda}} \simeq \mathcal{B}^{\lambda_1} \boxtimes \mathcal{B}^{\lambda_2}$$

Let $X^\lambda \underset{j}{\supseteq} \overset{\circ}{X}^\lambda$: $\sum \lambda_i x_i$ where each λ_i is the negative of a simple root

$P_{X^\lambda, q} \big|_{\overset{\circ}{X}^\lambda}$ acquires a canonical trivialization

$q(\lambda_i, \lambda_i + 2\rho) = 1$ for λ_i simple root...

$\rightsquigarrow$ easy to define "twisted" perverse sheaves on $\mathring{X}^\lambda$

$\mathcal{B}^\lambda|_{\mathring{X}^\lambda}$ = sign local system

Key fact (q not root of unity):

$$\mathcal{B}^\lambda = j_{!*}(\mathcal{B}^\lambda|_{\mathring{X}^\lambda})$$

-- this is where the Serre relation / Cartan diagram appears --

if we use $j_!$ get Drinfeld double of free Hopf algebra on F_i

j_* $\rightsquigarrow$ cofree algebra.

Serre relations encode $j_! \to j_*$:

$V_q^- = \mathrm{Im}$ (free $\to$ cofree) on F_i ...

Def A factorizable sheaf $\mathcal{F}$ is a collection
$\{ \mathcal{F}^\lambda, \lambda \in \Lambda \}$ $\mathcal{F}^\lambda \in \mathrm{Perv}_{\mathcal{P}_{X^\lambda_{x_0}, q}} (X^\lambda_{x_0})$
endowed with a system of isomorphisms
$$\mathcal{F}^\lambda |_{X^{\lambda_1}_{x_0} \times X^{\lambda_2}_{disj}} \simeq \mathcal{F}^{\lambda_1} \boxtimes \mathcal{B}^{\lambda_2}$$
$$\longleftrightarrow \mathcal{O}_q .$$

Construction of gerbe $\mathcal{P}$:
on $\underbrace{X \times \cdots\cdots X}_{n}$ & weights $\lambda_1 \ldots \lambda_n$
the gerbe is $\boxtimes\, \omega^{\log q(\lambda_i, \lambda_i - 2\rho)}$
$$\otimes \bigotimes_{i \neq j} \mathcal{O}(\Delta_i)^{\log q(\lambda_i, \lambda_j)}$$

What happens at a root of unity?
fact sheaves $\longrightarrow$ Drinfeld double of Lusztig's U_q^-

these won't be $!*$ extensions, functor
is fully faithful to quantum group reps,
enough to match with Whittaker

$!*$ at root of unity $\longleftrightarrow$ small quantum group

Fibers of $\mathcal{B}$: homologies of
our Hopf algebra. geometry encodes
Serre relations!

In general given a braided monoidal
category $\rightsquigarrow$ get chiral category
& a chiral algebra in this category
— that's our $\mathcal{P}$.