Link homology: \( L = S^3 \rightarrow H(L) \)

\[ H(\mathbb{S}^1; H(L)) \rightarrow H(L) \]

... simple version of 4d TQFT:
only see cobordisms \( S = S^3 \times [0,1] \).

Extended to foxtails:
2-category of foxtail cobordisms

- objects: n-tuples of points in plane
- 1-morphisms: foxtails \( \in \mathbb{R}^2 \times [0,1] \)

2-morphisms: foxtail cobordisms \( S = \mathbb{R}^2 \times [0,1]^2 \)

Consider 2-functors
\( n \)-tangle \( \rightarrow \) complex of maps over a ring \( H^n \)
[everything oriented]

Case of trivial link $\emptyset$ with simplest cobordism $\emptyset \to \emptyset$ etc.
give 2d TQFT $F$

$F: \emptyset \to R$ commutes with $\emptyset$

$F: \emptyset \to \text{free R-module}$, which inherits structure of a commutative Frobenius $R$-algebra

$A = A^* \otimes \text{Hom}(A, R)$

Let $\mathcal{B}^n = \text{cursingless matchings of } 2n \text{ points}$

\[ |\mathcal{B}^n| = \frac{1}{n!} \binom{2n}{n} \]

Given $a, b \in \mathcal{B}^n$ a $\circlearrowleft \cdots \circlearrowright \omega$ reflect

$w(1)$ to get closed 1-manifolds $w(\theta)$

$F(\psi(b, a)) = A^* \otimes \text{cycles}$
Let $H^n = \bigoplus_{a,b \in B^n} F(u(a))$

Claim: $H^n$ is an associative triple

$\mathbb{R}$-algebra: product

$F(u(d) c) \otimes F(u(a))$ product is zero

if $c = b$,

$F(u(d) b) \otimes F(u(b) a) \rightarrow F(u(a))$

$\Rightarrow$ gives a composition of form $c \rightarrow 1/a$
Associativity: subaddition we set are different.

\[ a \in \mathbb{B} \quad F(w(a)c) = A^{\otimes n} = 1^{\otimes n} = 1_a \]

\[ x \in F(w(b)x) \Rightarrow x1_a = x, a \in \mathbb{C} \]

\[ A^{\otimes 2} \quad \Rightarrow \quad C \quad \Rightarrow \quad 0 \]

\[ A^{\otimes 2} \quad \delta \quad A \quad \Delta \quad A^{\otimes 2} \]

Given \( H \to \) produce irreps of C6b algebra

\[ \text{Let their subalgebras: flat map be in plane} \]

\[ \text{(proper endity of exoixed) require \& \text{wnts} \text{r}) \]

\[ T \quad \text{to T assign} \quad (H^m, H^n) - \text{module} \]

\[ \text{F(T) =} \quad (4) \quad F(w(b)Ta) \quad \text{of} \]

\[ \text{be} \quad \text{of} \quad \text{be} \quad \text{of} \quad \text{be} \quad \text{of} \]

\[ \text{be} \quad \text{of} \quad \text{be} \quad \text{of} \quad \text{be} \quad \text{of} \]
Suppose all ways to close up $T$ with $a \otimes b$.

$F(\pi(\Theta_{T^c}) \otimes F(u(c)) \rightarrow F(u(b) T^c))$

etc.

$S \quad T = 11111 \quad F(T) = H^n.$

$a \in R^n \rightarrow \rho_a = F(x) \text{ get } H^n$-module

$= \odot^\oplus F(u(c))$

project $H^n$-module.

$\rho = \odot^\oplus \rho_a$ : complete list of irreducible $H^n$-modules.

Composition of functors $\Rightarrow$ tensor of $H^n$-modules

Cohomology of functors $\Rightarrow$ maps of $H^n$-modules

$\Rightarrow F$ 2-functor from target cohomology

to the 2-category of $H^n$-homology

objects = rings  morphisms = $H^n$-modules
This inverse cares about the topology of the embeddings but not their trivial embeddings in $\mathbb{R}^3$.

Different picture: forget about $\mathbb{R}^3$, $\mathbb{R}^2$ embedding.

Work with oriented 1-manifolds with corners.

A 2-manifolds with corners.

$[n] \mapsto$ bigger version: look at all possible ways that all oriented 1-manifolds are sitting two oriented 1-manifolds.

\[ \mathbb{B}_n : \]

\[ \begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\end{array} \]

Look at all complexes, $n!$, such...

\[ \Rightarrow \text{for bigger } n, \] minima extension of a 2d TQFT

$(\mathbb{R}, \mathbb{A})$ to a 2d TQFT with corners (calibrated TQFT)

... restrict to zero manifolds with same number of $+\circ$ and $-\circ$. 

2. Category:
- balanced versus oriented
- oriented cobordism
- cobordism of cobordism

3d : bimodules \( \rightarrow \) 4d: complexes of bimodules

3d TQFTs can be extended with semi-simple or abelian categories

4d TQFTs : expect to need triangulated categories.

closed surface \( \rightarrow \) triangulated category

3d cobordism \( \rightarrow \) exact functor

4d cobordism \( \rightarrow \) natural transformation of functors

Why? S surface, lift S to category C(S)

If S semi-simple \( \Rightarrow \) MCG(S) \( \rightarrow \) moduli space of simple algebras, no moduli etc...
Tangle $T$ complex b (mp) bimodules: write plane projection of $T$

$\begin{array}{c}
\text{II} \\
\text{X} \\
\text{III} \\
\text{D}
\end{array}$

$\text{II} \times \text{III} \Rightarrow \text{D}$, flip angles

Have a saddle point rotation $D \cong \text{D}^*$.

So can assign $b \in D$ the complex

$F(D) \xrightarrow{\text{Rs}} F(D)$

To an arbitrary $D$, decompose into crossings with at most one crossing

Need to check if $\text{Rs}$ is independent of the decomposition up to quasim.

$\Rightarrow \text{and } A/R \cong R$ as $R$-mod
\[ 121 \rightarrow 121 \]

\[ F(D) \otimes A \rightarrow F(D) \]

\[ F(D) \otimes R \cong F(D) \]

\[ x \rightarrow F(D) \otimes A/R \cong F(D) \]

So can add \( f = R/1 \oplus R \cdot x \)

get invariant under \( f \)

get \( F(0) \) on invariant of underlying field of \( R \).

Two interesting cases:

1. \( R = \mathbb{Z} = H^0(\mathbb{R}, \mathbb{Z}) \quad A = \mathbb{Z}[x]/x^2 = H^0(\mathbb{S}^2, \mathbb{Z}) \)

2. \( R = \mathbb{Q}[\ell] = H^0_{SU_2}(\cdot, \mathbb{Q}) \)

\[ A = \mathbb{Q}[x]/x^2 = H^0_{SU_2}(\mathbb{S}^2, \mathbb{Q}) \]

\( R, A \) are graded \( \deg \cdot = 2 \quad \deg x = 4 \).

\( R \) acts on \( A \).

Note all rings \( H^* \) are fields, \( F(0) \) are...
Theorem \( F(\mathbb{S}) \) is an involution of \( S \).

Inverse under maps, (18 sub.)

\[
\begin{align*}
\text{Braid } (\mathbb{S}) & \rightarrow F(\mathbb{S}) \text{ is an involution on }
\mathbb{S}^2 \\
\text{F}(\mathbb{S})^{-1} \ast \mathcal{F}(\mathbb{S}) & \cong \mathbb{S} \\
\text{Hom} (\mathcal{F}(\mathbb{S}), \mathcal{F}(\mathbb{S})) & = \text{Hom} (\mathbb{S}^2, \mathbb{S}^2) = \mathbb{Z}(\mathbb{S}^2)
\end{align*}
\]

Degree zero part is only \( \pm 1 \)......

So Redoncker move maps give inverse up to sign.

\[
\mathbb{Z}(\mathbb{S}^2) = H^* (\text{Springer fiber or partition } (m, n))
\]

Integral cochains are all points of \( \mathbb{S}^2 \), labeled by \( \mathbb{S}^2 \).
(Scott Mazur, Kevin Keller: get rid of signs by paying careful attention to orientations)

If our knot is a knot $K$

\[ \Rightarrow \quad F(K) \text{ is just a complex of graded abelian groups,} \]

...other polynomial join the Jones polynomial!

\[ g^2 J_K(\tau^2) - 2^{-2} J_K(\tau^2) = (2 \cdot 2^{-1}) J_K(\bar{p}) \]

\[ J(p) = 2 \tau^{-1} \quad J(\bar{u}) = \chi(F(L)) \]

\[ J(\bar{u}) = \chi(F(L)) \]