Last time: 1Bord (0,1) category

- Objects: 0-manifolds
- Morphisms: classifying spaces for bordisms of 0-manifolds

Symmetric monoidal under \( \otimes \).

Theorem: If \( E \) is any symmetric monoidal (K1)-category,

\[ \rightarrow \text{Fun}^{\otimes} (1\text{Bord}, E) \rightarrow \text{Dutchess objects } C \in E \]

\[ F \rightarrow F(\xi) \]

Consider the special case: \( E \) is a Picard (00-)groupoid, all morphisms are invertible

& all objects two-sided unit \( \otimes \)

ie. \( E \) is an (00) category

on a space \( X \), with a "comm, assoc, multiple in

up to coherent homotopy: \( X \) is in

an infinit loop space, or spectrum.

1Bord is not a Picard groupoid, but any

map to \( E \) as above factors through

completion of 1Bord \( \text{we may insert} \)
all morphisms also a topological space
\[ |\mathbf{1}\text{Bord}| \quad \text{geometric realization of the}
\quad \text{simplicial space underlying } |\mathbf{1}\text{Bord}|
\quad \text{as a complete Segal space.}

\[ \text{Fun}^{\otimes}(|\mathbf{1}\text{Bord}|, \mathcal{C}) \cong \text{Fun}^{\otimes}(|\mathbf{1}\text{Bord}|, \mathcal{C}) \]

\[ \text{Thm} / \Sigma \cong \text{Hom}_{\text{spaces}}( |\mathbf{1}\text{Bord}|, \mathcal{C}) \]

\[ X = \text{all (co)limits}
\quad \text{objects in } \mathcal{C} \]

\[ \text{i.e. } |\mathbf{1}\text{Bord}| \text{ has universal property away}
\quad \text{infinite loop space: Theorem in this case}
\quad \text{just says } |\mathbf{1}\text{Bord}| \cong Q^\infty \xrightarrow{\text{def}} L_\Sigma S^n \]

\[ \text{the stable sphere: consequence of the work of}
\quad \text{Galatius- Madsen-Tillman-Weiss in the n=1}
\quad \text{computing homotopy type of bordism categories.} \]

\[ |\mathbf{2}\text{Bord}| (n \geq 2) \quad \text{category: all morphisms are}
\quad \text{2 are invertible,}
\quad \text{\& category under it (sym. monoidal).} \]
Objects: $O$-manifolds $M, N$

morphisms: bordisms of $O$-manifolds $B, N'$

Space of 2-morphisms: $\left( \frac{M}{B} \right) \rightarrow N$

$\text{2Hom}(B, B') = \text{classifying space}$

for bordisms from $B$ to $B'$, trivial along $M \times N'$

- probably can't straighten, need more systematic definition in flavor of complete segregated spaces (as bisimplicial topological space).

**Question:** can an $s$-manifold model $(0,2)$-dgy?

- describe $\text{Fun}^2(2\text{ Bord }, C)$

**Easier version:** case $C$ is a Frobeni"{u}s graded, in this case we're calculating $\text{12Bord}$ as an infinite loop space (cohom by $G\cdot M\cdot T\cdot W$).
Theorem (GMTW) \( \pi_{2} \mathrm{Bord} \simeq \sum_{i} \mathbb{MT} S^{4}(2) \)

i.e., \( \pi_{2} \mathrm{Bord} \) has a presentation as the cokernel of a map \( (\mathbb{OS})_{5} \rightarrow \mathbb{OS}^{0} \)

("coherel'') : generators coming from \( \mathbb{OS}^{0} = \mathbb{1} / \pi_{1} \mathrm{Bord} \). \( \pi_{2} \mathrm{Bord} \) has \( \mathbb{1} / \pi_{1} \mathrm{Bord} \) inside, these are generators, so need to impose some relations: kill off \( \mathbb{OS}^{1} \):

\[ \mathbb{OS}^{1} = \lim_{\longleftarrow} \mathbb{OS}^{n+1} \] (one relation in \( \mathbb{OS}^{0} \)):

the circle dies in \( \pi_{2} \mathrm{Bord} \)

since if bands ... in particular it bounds the disc. The restriction of the circle extends to the disc, so bounds \( S^{1} \)-equivariantly! so are generators & one relation.

Goal: give an analogous presentation of \( \pi_{2} \mathrm{Bord} \) itself by generators & relations...
Try to lift GMTU to representation of 2Bord itself.

Suppose \( F : 2\text{Bord} \to C \), 2d TQFT.
What do we get out of \( \text{dim} \)?
\( F(\emptyset) = C \) dualizable (dual: \( F(\emptyset) \))
\( \rightarrow \) determines \( F(\emptyset) = \text{dim} C \in \text{Hom}(1,1) \).

\[ F(\emptyset) = \eta \in \text{Hom}_C(\text{id}_1, \text{dim} C) \]

\[ \eta : \text{id}_1 \to \text{dim} C \] — not an isomorphism, just
a morphism from \( \text{id} \) to \( \text{dim} C \).

The neat point of the description of 1Bord
is that \( \text{dim} C \) has an action of \( S' \)

- close map is in fact equivalent.
\[ \eta \in \text{Hom}_C(\text{id}_1, \text{dim} C) \]

Then \( \text{Fun}^\text{op}(2\text{Bord}, C) \to \{ (\eta) : \text{CC dualizable} \} \)
\[ \eta \in \text{Hom}(1, \text{dim} C) \] non-degenerate
Nondegeneracy: analog for 2-Bnd of
dulitility of C. not extra data
just a condition

2-Bnd^0: objects: 0-manifolds
          maps: bundles
          2-morph: 2-hom(\(B, B'\)) =
          \{ surfaces \( \Sigma \) \( s \cdot \alpha \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \}
          \{ \Sigma \( \Sigma \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \() \}
          \{ \Sigma \( \Sigma \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \() \}
          in \( \Sigma \( \Sigma \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \() \) \( \Sigma \( \Sigma \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \() \) \( \Sigma \( \Sigma \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \() \) \( \Sigma \( \Sigma \) \( \Sigma \) \( \Sigma \) \( \Sigma \) \() \)

The Fun \( \otimes (2-Bnd^0, C) \) = same MKS
with easier nondegeneracy condition.

2-Categories
Example: Col: Obj: category
          maps: functors
          2-morph: natural transformations

Idea: notion of adjoint functors
$F : C \to D$ is an adjunction if we have a canonical isomorphism

$$\forall c \in C, \; \exists_{d \in D} \; \text{Hom}_D(c, gd) \cong \text{Hom}_D(fc, d)$$

$\forall c \in C, \; \exists_{d \in D} \; \text{Hom}_D(c, gd) \cong \text{Hom}_D(fc, d)$

A definition in purely 2-categorical terms:

If we have an adjunction, can we have

$$\text{id}_{GD} \in \text{Hom}_D(GD, GD) \Rightarrow \text{Hom}_D(FGD, D)$$

$$\Rightarrow \text{natural transformation } \eta : F \circ G \to \text{id}_D$$

which makes our isomorphism (A):

$$\text{Hom}_C(c, gD) \xrightarrow{F} \text{Hom}_D(Fc, FGd) \xrightarrow{\eta} \text{Hom}_D(Fc, D)$$

To see this, this is a natural map in other contexts: unit $\eta : \text{id} \to G \circ F$,

+ compatibility with $\eta$.

Can make sense of adjunctions in any (co)category:

Given objects $c, d$ and

morphisms $f : c \to d$ & $g : d \to c$

Can ask if these are adjoints.
\[ \exists \text{ unit map } \eta : 1 \rightarrow \text{unit} \]

\[ \text{a counit } \epsilon : \text{unit} \rightarrow 1\]

(\text{space of comonoids for which there is a unit}).

If \( C \in \text{co} \Rightarrow 1 \),

\[ \eta \circ \epsilon = \text{id} \]

\[ \epsilon \circ \eta = \text{id} \]

\[ \text{dim } C = 1 \]

\text{ie the unit } \eta : \text{id} \rightarrow \text{evcoev}.

\[ \text{Can cut if this actual trash is broken} \]

\[ \text{is an adjunction.} \]

\[ \text{Theorem } \text{Fun}^{\text{op}}(\text{Set}^{\text{op}}, \text{Set}) = \{ \eta, \epsilon : C \in \text{Set}^{\text{op}} \}
\]

\[ \eta \in \text{Hom}(\text{id}, \text{unit}) \]

\[ \epsilon \text{ is the unit of } \eta \text{ in an adjunction} \]

\text{These nondegeneracy conditions force}

\text{into existence lots of objects from above!}

\[ \exists \text{ C \in \text{co} \Rightarrow 1 } \]

\[ \exists 1 \Rightarrow \text{co} \]

\text{above how can it...}
What is the cube? $\text{cube} \rightarrow \text{id}$

Draw in 2D:

\[ \emptyset \quad \downarrow \quad 2 \text{map} \\
1 \quad \text{cube} \quad 1 \quad \text{map} \\
\hline
\text{cube} \quad \text{cube} \quad \text{id} \\
\hline

What is compatibility of unit and counit?
2-morph in 2Bord:

Given surface cut into pieces with
a Morse function

Only need Morse indices 0, 1 in 2Bord
- only need \( \emptyset, \{ \} \) saddle

(in full 2Bord would need also \( \{ \} \))

So given Morse function can decompose
into the operations we had before.

Any 2 Morse functions are related
by basic moves as above (compatibility
at int & cant) - Morse/Cerf Move
To get (4.7) categorical into however would need to use categorical dimensions families of Morse functions & their

... 

Actual proof: use ribbon graph model for moduli spaces of Rezn

surfaces, following work of K. Costello

Fraud only!

Fun \circ (2\mathcal{B} \mathcal{E} \mathcal{O}, \mathcal{E}) = \text{Ob} \mathcal{E} \mathcal{C} + 

... i.e., 2\mathcal{B} \mathcal{E} \mathcal{O} is a free span monoidal

(4.2) - category on a monoidal algebra...

[Baez-Olsz conjecture]

Replace 2\mathcal{B} \mathcal{E} \mathcal{O} by 2\mathcal{B} \mathcal{E} \mathcal{O}; need another nondegeneracy control, multiply

\circ as well --- corresponds to demanding certain maps are adjoint.

Next the: how to apply this in practice?

need \& trace on algebra.