Motivation from topology: $E_n$-operads

$(X,x)$ pointed space

$LX = \text{Map}((\overline{[0,1]},\overline{0}), (X,x))$

- associative multiplication
- make sense of using operads:

\[
[\text{inhabited }] \quad \text{embedded in } L[X]
\]

- defines map $LX \times LX \times LX \to LX$

- i.e. a composition depending on a picture of little intervals.

$E_n$-operad: (an operad in spaces):

collection of spaces \{C(X_i)\}

$C(k)$ = configuration of $k$ cubes (of dim $n$) inside $[0,1]^n$ (non-intersecting)
Ex (n=2) O(2) is space of pictures

To compare with draw inner one of the boxes:

Observation: X pointed space \( \Rightarrow \)
En operad acts on \( \mathcal{D}^X \)

\[ O(n) \times (\mathcal{D}^X)^k \rightarrow \mathcal{D}^X \]

May: roughly, n-fold loop space \( \Leftrightarrow \) spaces with action of En operad.

Can make sense of En acts on any other kinds of objects – today, chain complexes.
K field, can make an $E_n$ operad valued in chain complexes over $k$ by taking singular chains:

Many operations: $C_\ast(O(n), k)$

So can talk about $E_n$-algebras $A/k$

Def: An $E_n$ algebra $A$ is a chain complex with an extra of this operad, i.e.

\[
C_\ast(O(n), k) \otimes A^{\otimes n} \to A
\]

compatible with compositions.

If $A$ is an $E_n$ algebra we have multiplication maps $A^{\otimes n} \to A$

labelled by points of $O(n)$

which (up to homotopy) is the configuration space of $n$ points in $\mathbb{R}^n$.

$n=1: A^{\otimes n} \to A$ for every $n$ points arranged on a line - it ordered

$\iff$ "associative" algebra.
\[ E, \text{ algebra} = A_\infty \text{ algebra} = (dg) \text{ associative algebra over } k. \]

\[ n \rightarrow \infty, \text{ configuration spaces get more} \]
\[ L \text{ more connected in limit become} \]
\[ \text{contractible} \Rightarrow \text{ get } E\infty \text{ algebra}. \]
\[ \text{If } deg k = 0 \text{ this is just a commutative} \]
\[ \text{dg algebra (in general } E\infty \text{ is} \]
\[ \text{a good notion of commutative product} \]
\[ \text{on chain complexes).} \]

Case \( n = 2 \)

Suppose \( A \) is an ordinary (ie deg=0)
\[ \text{vector space. For } n \geq 1 \text{ config spaces} \]
\[ \text{in } \mathbb{R}^n \text{ are connected, so} \]
\[ E_2 \text{ algebra} = E_3 = \ldots = E\infty = \text{commutative} \]
\[ \text{algebra structure on } A \]
\[ \ldots \text{ for chain complexes have a measure} \]
\[ \text{of commutativity given by } n. \]
Suppose \( A \) is an \( E_n \)-algebra

\[ \rightarrow \text{in particular } A \text{ is } E_1 \text{ : have an associative multiplication.} \]

\[ \mathbb{R}^n \text{ look at configurations that happen to lie on same line.} \]

\[ \mathbb{R} \]

But many kinds of lines

- eg. horizontal, vertical, etc.

in \( n \) dimensions get \( n \) different "compatible" \( E_n \) products (for each axis \( e.g. \) \( m_1, m_2, \ldots, m_n \) say.)

Compatibility:

\[ (A, m_1) \otimes (A, m_2) \rightarrow (A, m_1 \cdot m_2) \]

is a map of associative \( E_n \) algebras

\[ \Rightarrow E_n \text{-algebras } \leftrightarrow \text{chain complexes with } n \text{ compatible } \langle e \rangle \text{ alg. structures!} \]
Modules over $E_n$ algebras

Two notions:

$n=1$: $A$ is a homogeneous left $A$-module
- $A$-bimodule

General case: $A$ is an $E_n$ algebra
- notion of left $A$-module
  (passing to underlying associative algebra)

Module structure $\Rightarrow$
- certain collection of maps
  \[ A^m \otimes M \rightarrow M \]

For left modules, these are parameterized
by configurations in the line
\[ a_m \ a_2 \ a_1 \ a_0 \]
For binodals use all configurations

In two dimensions: consider
\[ A \otimes \mathcal{M} \rightarrow \mathcal{M} \]
\[ \text{parametrized by configs in the plane} \]

Use full 2-dimensionality of \( A \)

A notion of \( E_n \)-module over an \( E_n \) algebra, where these are actions parametrized by an \( S^{n-1} \) roughly.

Observation: Any \( E_n \)-module has an underlying left module

But have a lot more structure
Say $A$ is an $E_2$-algebra, $m_1, m_2$ compatible assoc. multipliers.

Left $A$-modules (wrt $m$, multiplication) have functor $(\text{left } A\text{-mod}) \times (\text{left } A\text{-mod}) \rightarrow (\text{left } A\text{-mod})$

induced by $A \otimes A \rightarrow A$.

So if $A$, $E_2 \Rightarrow$ (left $A$-mod) is a monoidal category.

More generally if $A$, $E_n \Rightarrow$ (left $A$-mod) is an $E_n$ category:

- $n$-compatible monoidal structures.

E.g on $E_1$ category $\leftrightarrow$ monoidal category

$E_2$ category $\leftrightarrow$ braided monoidal category

etc.

What do $E_n$-modules for $n \rightarrow$ get an $E_n$ category?
E.g. n=1: A-bimodules for a monoidal category \( \mathcal{C} \) act on 
moreover it acts on the category of left A-modules.

In general, \( E_n \)-modules/A act on left modules/A.
"cofree" action - compatibly with left structure.

E.g. n=2: left A-modules are a monoidal category \( \mathcal{M} \).

\( E_2 \)-A-modules form a braided monoidal category = Drinfeld center of \( \mathcal{M} \).

Koszul duality: An augmentation on an \( E_n \) algebra \( A \) is a map \( A \to k \) of \( E_n \) algebras.
\( \Rightarrow \) \( A \) splits \( I = \ker e \)
\( A = I \oplus k \).
I is an En algebra w/o unit

\[ A \rightarrow k \] augmented associative algebra
\[ \xrightarrow{\text{unit}} \] En algebra

Koszul duality for \( n = 1 \):
\[ A \rightarrow k \] augmented associative algebra makes \( k \) an \( A \)-module

Def: The Koszul dual \( A^\vee = \text{RHom}_A(k, k) \)

\[ A^\vee, \varepsilon^\vee \] augmented associative algebra \( \text{Hom}_k(k, k) = k \)

Break this process in two:

\[ A \rightarrow \text{Hom}(k, k) \] augmented associative algebra
\[ \xrightarrow{\text{unit}} \text{Hom}_k(k, k) \]
\[ \text{vector space dual} \]

constructed on \( A \) to \( k \) constructed on \( A \) to \( k \)
\begin{align*}
&\{\text{Augmented } E_n\text{-algebras}\} \\
&\xrightarrow{\text{Koszul duality}} \\
&\{\text{augmented } E_n\text{-algebras}\} \\
&\{\text{complexes w/ n assoc.} \}
\uparrow \quad \text{vector space} \\
&\{\text{complexes with } + \varepsilon\} \\
&\downarrow \quad \text{Bar} \\
&\{\text{Bar} : c \otimes f\text{-factor}\} \\
&\{\text{En category}\} \\
&\downarrow \\
&\{\text{En category}\} \\
&\downarrow \quad \text{Bar} \\

\text{[really suppressing shift of } n \text{ - will matter if we want to take the limit]}

\text{n=0 : get just vector space duality}

\text{Under strong finiteness conditions this is an equivalence of categories, and we'll discuss this...}

\text{n=0 : K-theory = vector space duality}

\text{So } A(\text{algebra}) \rightarrow A \otimes A^* \rightarrow \text{pairing map of vector space}
\[ n=1 : \quad A \otimes A^\nu \cong A \otimes \text{Hom}_A(\varepsilon, t) \rightarrow k \]

augmentation on \( A \otimes A^\nu \)

restricting to the two augmentations
gives a sense in which \( A, A^\nu \) are dual

In general \( A \otimes A^\nu \rightarrow 1 \) induces
augmentations on \( A, A^\nu \)
\( A^\nu \) is universal with this property.

Another POV: deformation theory

Say \( A \rightarrow k \) augmented \( \mathbb{E}_n \) algebra

"Spec \( A' \)" is finer

\[ \text{comm. algebra (i.e. } \mathbb{E}_n) \rightarrow \text{Hom}_{\mathbb{E}_n}(A, R) \]

"Spec \( R \) valued point of Spec \( A' \"

\( \mathcal{E} \) gives a "point" \( \text{Spec } k \rightarrow \text{Spec } A \)

\( \Rightarrow \) take forget scheme (can target complex).
this is \( A^\nu \), up to shift.
Clear = 0: philosophy says n-nilpotent space is formally deformation by algebra.

Note we can think of Spec $A$ as a functor on all En algebras.

$\Rightarrow$ tangent space is an augmented En algebra, $E$ as such is $A^n$.

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Geometric interpretation via configuration spaces:

A En-coalgebra

$\Rightarrow A \rightarrow A \otimes A$ labeled by pairs of points in $\mathbb{R}^n$...

Put one at 0, get $S^{n-1}$ worth of maps $A \rightarrow A \otimes A$.

Encode this by a sheaf $F$ on $\mathbb{R}^n$.

s.t. stalk of $F$ at 0 is labeled with $A$ & stalk at any other point is labeled with $A \otimes A$. 
- need specialization map new origin
\[ \Rightarrow \text{sum of maps } A \rightarrow A \otimes A \]

More symmetrically:
\[ \text{Def: The Ran space of } \mathbb{R}^n \text{ is the space of nonempty finite subsets } S \subset \mathbb{R}^n \]
\[ \Rightarrow \text{filtered by cardinality of } S. \]

Given an Enr-algebra \( A \), can build a sheaf
\[ \mathcal{A} \text{ on the Ran space of } \mathbb{R}^n \text{ with stalks } A_S = \bigotimes_{x \in S} A \]
\[ \Rightarrow \text{describes the sheaf as a locally constant sheaf of any given cardinality} \]

Properties:
\[ 1. \ A_{|\text{Ran}^< (\mathbb{R}^n)} = \mathbb{R}^n \text{ is constant} \]
\[ 2. \text{If } S, S' \text{ are disjoint, } \]
\[ A_S \otimes A_{S'} = A_S \otimes A_{S'} \]
- Very closely analogous to factorable algebras where we have algebraic version & drop condition 1.

**Dictionary**

<table>
<thead>
<tr>
<th>alg. topology</th>
<th>sheaves on $\mathbb{R}^n$ space</th>
<th>chiral algebras</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_n$ (col)alg.</td>
<td>&quot;factorizable&quot; sheaf $\mathcal{A}$ [\text{ie satisfy (1,2)}]</td>
<td>chiral object [\mathfrak{A} = 2, \text{curve } \mathcal{A}]</td>
</tr>
<tr>
<td>$E_n$ module</td>
<td>sheaf or similar space</td>
<td>chiral module</td>
</tr>
<tr>
<td>left module</td>
<td>choose direction in $\mathbb{R}^n$ [\mathfrak{A}]</td>
<td>(our chiral algebra is not a <em>Drinfeld cubic</em> of sorts)</td>
</tr>
<tr>
<td>Koszul duality</td>
<td>Verdier duality</td>
<td></td>
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<tr>
<td>$A$ in deg 0</td>
<td>$A$ is in deg 0</td>
<td></td>
</tr>
<tr>
<td>$\text{Bar}(A)$ is Hopf</td>
<td>$A$ is perfect [\text{not all sheaves even dimensional...}]</td>
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</tbody>
</table>
Koszul duality induces equivalence on category of En modules (under certain hypotheses).

left $E_n$ modules are comodules over $Hom_{A}(-, Dk(A))$

$E_n$ modules over $A$ are comodules over Drinfeld double of $R/k$

Any $n!$-Rim space has strata whose all strata have dim multiplicity of $n$ in natural numeracy but also all rim constructions in our Koszul duality, and ask any one of them to be a plain vector space.

Condition 1 too strong for $n!$-algebras:
strong equivalence, stronger than tiny away from a vertex algebra.