J. Lurie – Ring Spectra & Koszul Duality

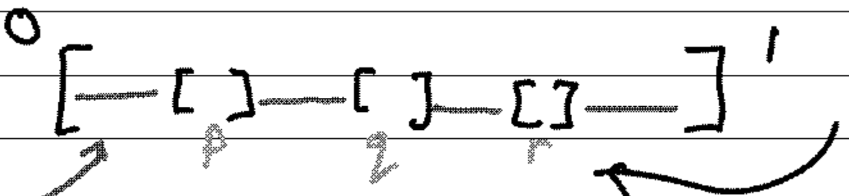
Note Title 3/14/2008

Motivation from topology: E_n-operads

(X,x) pointed space

$\Omega X = \mathrm{Map}\big(([0,1],\partial),(X,x)\big)$

"associative multiplication" – make sense of using operads:



basepoint

intervals embedded in $[0,1]$

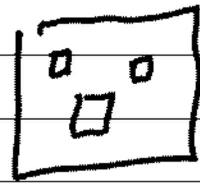
– defines map $\underset{p}{\Omega X}\times\underset{q}{\Omega X}\times\underset{r}{\Omega X}\longrightarrow\Omega X$

– ie a composition depending on a picture of little intervals.

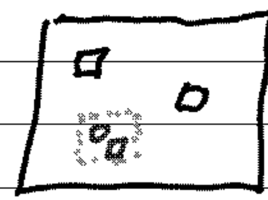
E_n-operad: (an operad in spaces): collection of spaces $\{\mathcal{O}(k)\}$

$\mathcal{O}(k)$ = configurations of k cubes (of dim n) inside $[0,1]^n$ (nonintersecting)

Ex. ($n=2$) $\mathcal{O}(3)$ is space of pictures



To compose with  draw inside one of the boxes:



Observation: X pointed space $\Longrightarrow$
E_n operad acts on $\Omega^n X$

$$\mathcal{O}(k) \times (\Omega^n X)^k \longrightarrow \Omega^n X$$

<u>May</u>: roughly, n-fold loop spaces

$$\Longleftrightarrow$$

spaces with action of E_n operad.

Can make sense of E_n actions on many other kinds of objects - today, <u>chain complexes</u>

K field, can make an E_n operad valued in chain complexes over k by taking singular chains:

m-ary operations : $C_*(\mathcal{O}(m); K)$

$\Rightarrow$ can talk about E_n-algebras $/K$

<u>Def</u> An E_n algebra A is a chain complex with an action of this operad, ie

$$C_*(\mathcal{O}(m), K) \otimes A^{\otimes m} \longrightarrow A$$

compatible with compositions.

If A is an E_n-algebra $\Rightarrow$ have multiplication maps $A^{\otimes m} \longrightarrow A$ labelled by points of $\mathcal{O}(m)$ which (up to homotopy) is the configuration space of m points in $\mathbb{R}^n$.

$n=1$: $A^{\otimes n} \longrightarrow A$ for every n points arranged on a line — ie <u>ordered</u>

$\Longleftrightarrow$ "associative" algebra :

E_1 algebra = A_∞ algebra = (dg) associative algebra / k.

$n \to \infty$: configuration spaces get more & more connected, in limit become contractible $\Rightarrow$ get E_∞ algebra : if char $k = 0$ this is just a commutative dg algebra (in general E_∞ is a good notion of commutative product on chain complexes).

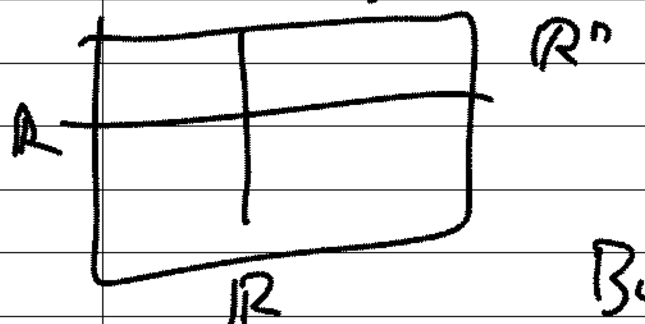
Case $n = 2$

Suppose A is an ordinary (ie dg = 0) vector space. For $n > 1$ config spaces in $\mathbb{R}^n$ are connected, so E_2-algebra = E_3 = ... = E_∞ = commutative algebra structure on A

... for chain complexes have a measure of commutativity given by n.

Suppose A is an E_n-algebra

$\Rightarrow$ in particular A is E_1: have an associative multiplication!



look at configurations that happen to lie on some line.

But many kinds of lines - eg horizontal & vertical:

In n dimensions get n "different" "compatible" assoc. products (for each axis e.g.) $m_1, m_2, \ldots m_n$ say.

Compatibility:

$$(A, m_1) \otimes (A, m_1) \xrightarrow[m_2]{} (A, m_1)$$

is a map of associative algebras

$\Rightarrow$ E_n-algebras $\Longleftrightarrow$ chain complexes with n compatible assoc alg. structures!

Of course $m_1, \ldots m_n$ all coincide up to homotopy but sometimes better to think of as different — but compatible & unital $\Rightarrow$ in fact E_n.

Modules over E_n algebras

Two notions.

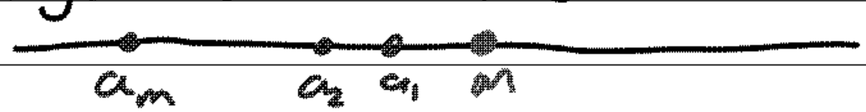
$n=1$: A assoc have • left A-modules
• A-bimodules

General case: A E_n algebra
$\Rightarrow$ notion of left A-module
(passing to underlying associative algebra)

Module structure $\longleftrightarrow$
certain collections of maps

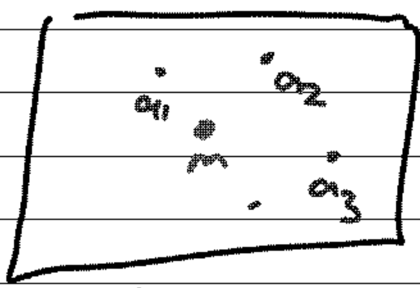
$$A^{\otimes m} \otimes M \longrightarrow M$$

For left modules, these are parametrized by configurations in the line



For bimodules use all configurations

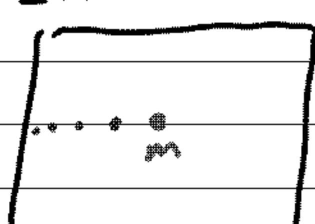
In two dimensions: can consider
$A^{\otimes n} \otimes M \to M$ parametrized
by configs in the plane



Use full 2-dimensionality of A

$\Rightarrow$ notion of E_n-module over an E_n algebra, where there are actions parametrized by an S^{n-1} roughly

Observation: Any E_n-module has an underlying left module



But have a lot more structure

Say A is an E_2-algebra,
m_1, m_2 compatible assoc. multiplications.

Left A-modules (wrt m_1 multiplication)
have functor
$$(\text{left } A\text{-mod}) \times (\text{left } A\text{-mod}) \xrightarrow[\otimes_A]{} (\text{left } A\text{-mod})$$
induced by $A \otimes A \xrightarrow[m_2]{} A$.

So if A is $E_2 \Rightarrow$ (left A-mod)
is a monoidal category.

More generally if A is $E_n \Rightarrow$
(left A-mod) is an E_{n-1} category:
$n-1$ compatible monoidal structures.

e.g. an E_1 category $\longleftrightarrow$ monoidal category
E_2 category $\longleftrightarrow$ braided monoidal category
etc.

What do E_n-modules form? $\rightsquigarrow$
get an E_n category:

eg $n=1$, A-bimodules form a monoidal category given by $\otimes_A$

Moreover it acts on the category of left A-modules

In general E_n-modules / A E_n acts on left modules / A
"central" action - compatibility with left structure.

eg $n=2$: left A-modules are a monoidal category $\mathcal{M}$.

E_2 - A-modules form a braided monoidal category = Drinfeld center of $\mathcal{M}$.

<u>Koszul duality</u> An augmentation on an E_n algebra A is a map $A \xrightarrow{\varepsilon} k$ of E_n algebras

$\Rightarrow$ A splits $\quad I = \ker \varepsilon$

$A = I \oplus k$

I is an E_n algebra w/o unit

$$\begin{tikzcd} \text{augmented } E_n\text{-algebras} \arrow[r, phantom, "\sim"] & \text{nonunital } E_n \text{ algebras} \arrow[l, "\text{adjoin unit}"] \end{tikzcd}$$

Koszul duality for $n=1$:

$A \to k$ augmented assoc $\Rightarrow$ makes k an A-module

<u>Def</u> The Koszul dual $A^\vee = R\mathrm{Hom}_A(k,k)$

$$\begin{tikzcd} A^\vee = R\mathrm{Hom}_A(k,k) \arrow[d, "\varepsilon^\vee"] \\ \mathrm{Hom}_k(k,k) = k \end{tikzcd}$$

$A^\vee, \varepsilon^\vee$ augmented associative algebra

Break this process in two:

$$\begin{tikzcd} \text{augmented assoc algebra } A, \varepsilon \arrow[rr, "{A \mapsto \mathrm{Hom}_A(k,k) = \mathrm{Hom}_k(k \otimes_A k, k)}"] \arrow[dr, "\text{bar construction}"'] & & \text{augmented assoc algebra} \\ & \text{coaugmented associative coalgebras} \arrow[ur, "\text{vector space dual}"'] & \end{tikzcd}$$

$A \mapsto k \otimes_A k$

$$\begin{tikzcd}
\{\text{Augmented } E_n\text{-algebras}\} \arrow[r, "\text{Koszul duality}"] \arrow[d, phantom, "\simeq" rotate=90] & \{\text{augmented } E_n \text{ algebras}\} \\
\{\text{complexes w/ } n \text{ assoc multiplications, } + \varepsilon\} \arrow[d, "\text{Bar : a } \otimes \text{ functor}"] & \\
\{\text{associative coalgebra w/ } n-1 \text{ assoc. multiplications}\} \arrow[r, dashed, "\text{Bar} \; \cdots \; \text{Bar}"] & \{E_n \text{ coalgebra w/ coaugmentation}\} \arrow[uu, "\text{vector space dual}"']
\end{tikzcd}$$

[really suppressing shift of n - will matter if we want to take the limit]

$n=0$: get just vector space duality

Under strong finiteness conditions this is an equivalence of categories, and we'll assume this...

$n=0$: K.duality = vector space duality

So A 0-algebra $\Rightarrow$ $A \otimes A^\vee \longrightarrow k$ pairing

map of vector spaces

$n=1$: $A \otimes A^\vee \simeq A \otimes \mathrm{Hom}_A(k,k) \longrightarrow k$
augmentation on $A \otimes A^\vee$
restricting to the two augmentations
— gives a sense in which $A, A^\vee$ are dual

In general $A \otimes A^\vee \longrightarrow k$ inducing
augmentations on $A, A^\vee$ &
$A^\vee$ is <u>universal</u> wrt this property.

<u>Another POV</u> : deformation theory

Say $A \xrightarrow{\varepsilon} k$ augmented E_n algebra

"Spec A" : functor
comm. algebra (ie E_∞) $R \longrightarrow \mathrm{Hom}_{E_n\text{-alg}}(A, R)$

"Spec R valued point of Spec A"

ε gives a "point" Spec $k \to$ Spec A
$\Rightarrow$ take tangent space (or tangent complex)!
this is $A^\vee$, up to shift.

Char $=0$: philosophy every moduli space is formally determined by a dga.

Note we can think of Spec A as a functor on all E_n algebras
$\Rightarrow$ tangent space is an augmented E_n algebra, & as such is $A^\vee$.

Geometric interpretation via configuration spaces

A E_n-coalgebra
$\Rightarrow$ $A \longrightarrow A \otimes A$ labelled by pairs of points in $\mathbb{R}^n$ ....
put one at 0, get S^{n-1} worth of maps $A \to A \otimes A$.

Encode this by a sheaf $\mathcal{F}$ on $\mathbb{R}^n$ s.t. stalk of $\mathcal{F}$ at 0 is identified with A & stalk at any other point is identified with $A \otimes A$.

— need specialization map near origin
$\Longleftrightarrow$ S^{n-1} of maps $A \longrightarrow A \otimes A$

More symmetrically:
Def The Ran space of $\mathbb{R}^n$ is the space of nonempty finite subsets $S \subset \mathbb{R}^n$
... filtration by cardinality of S.

Given an E_n-algebra A can build a sheaf $\mathcal{A}$ on the Ran space of $\mathbb{R}^n$ with stalks $\mathcal{A}_S \simeq \bigotimes_{s \in S} A$
... describes the sheaf as a locally constant sheaf of any given cardinality

Properties:
1. $\mathcal{A}|_{\mathrm{Ran}^{\leq 1}(\mathbb{R}^n) = \mathbb{R}^n}$ is constant

2. If S, S' are disjoint,
$\mathcal{A}_{S \cup S'} = \mathcal{A}_S \otimes \mathcal{A}_{S'}$

- very closely analogous to factorization algebras where we have algebraic version & drop condition 1.

Dictionary

alg. topology	sheaves on Ran space	chiral algebra
E_n (co)algebra $\longleftrightarrow$	"factorizable" sheaf $\mathcal{A}$ (ie satisfy 1,2)	chiral algebra ($n=2$, curve is $\mathbb{A}^1$)
E_n module $\longleftrightarrow$	sheaf on similar space	chiral module
left module $\longleftrightarrow$	choose directions in $\mathbb{R}^n$	? (our chiral algebra is not a Drinfeld double of something)
Koszul duality	Verdier duality	
A in deg 0	$\mathcal{A}$ is in deg 0	
Bar(A) is Hopf algebra in $\underline{\text{Vect}}$ not complexes	$\mathcal{A}$ is perverse ($n=2$: all strata even dimensional...)	

Koszul duality induces equivalence on category of E_n modules (under restrictive hypotheses)

left E_2-modules over A $\rightsquigarrow$ comodules over Hopf algebra Bar A

E_2-modules over A $\longleftrightarrow$ comodules over Drinfeld double of Bar A

Any n! Ran space has stratification where all strata have dim multiples of n $\longleftrightarrow$ $n+1$ natural perversity

but also $n+1$ Bar constructions in our Koszul duality, and ask any one of them to be a plain vector space.

Condition 1 too strong for vertex algebras!
strong equivariance, stronger than
than coming from a vertex algebra