G. Vezzosi - Background in DA6

2 appendices: J. Lurie

"functions at points" R. Toën-Vezzosi

Plan: what are derived spaces?
- geometric/algebraic derived spaces
- tangents & cotangents

$k = \text{comm. ring}$ (for today usually field of char 0)
- $k$-alg: comm. $k$-algebras
- $k$-alg $\to$ spaces, objects
  - schemes, sheaves
  - stacks $\to$ groupoids
  - $\text{sk Sets} \to \mathbf{SSets}$, simplicial sets

Think of $\mathbf{SSets}$, $\text{sk Sets}$ as oo-categories
- concretely, as model categories
<table>
<thead>
<tr>
<th>AG</th>
<th>DAG</th>
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<tbody>
<tr>
<td>$(k\text{-mod}, \otimes)$</td>
<td>sk mod or $C^\infty(k\text{-mod})$</td>
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<tr>
<td>$\text{Comm}(k\text{-mod}, \otimes)$</td>
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<tr>
<td>$(\text{Aff}^\text{op})^\text{op} = (\text{Aff}_k^\text{op})^\text{op}$</td>
<td>$\text{dAff}^\text{op}$, a model category</td>
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<tr>
<td>Commutative rings $\rightarrow$ affine schemes</td>
<td>simplicial k-modules $\rightarrow$ nonneg. comm. dgas.</td>
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<tr>
<td>PreStacks $\mathcal{P}(\mathcal{B}) = {\text{Aff}^\text{op} \rightarrow \mathcal{S}et}$</td>
<td>Derived prestacks $\mathcal{D}(\mathcal{P}(\mathcal{B})) = {\text{Aff}_k^\text{op} \rightarrow \mathcal{S}et}$</td>
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<tr>
<td>Model category (projective model structure)</td>
<td>Model category: add weak equivalences from same $h_m \rightarrow$ by weak eq on representable functors when $X \rightarrow Y$ is a weak equival.</td>
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Remark: Fibrant objects in \( dPst(\mathbb{A}) \) are

\[ F : dAff_k \to \text{Sets} \]

1. are objectwise fibrant
2. preserve weak equivalences

\[ \Rightarrow Ho(dPst(\mathbb{A})) \cong Ho(dAff_k \to \text{Sets}) \]

full subcategory map structure (i.e., satisfying 2. measure on both)

\[ \]

\( \mathbb{T} \) Gottlieb topology

on \( Aff_k \) (e.g., Grothendieck topology on \( dAff_k \)

\( \mathbb{T} \) = "model" topology

Remark: For \( A \in \text{sets} \), one shall that

of \( A \) as "given" (up to weak equivalence)

by a comm. \( k \)-algebra \( Ho(A) \to \mathbb{T}_\geq(\mathbb{A}) \)
Consider Spec \( \mathcal{T}_0 A \hookrightarrow \text{PCSpec } A \) (closed embedding) as a derived formal thickening.

"Derived geometry on \( A \)" =
"usual geometry on \( \mathcal{T}_0(A) \) + deformation theory from \( A \)."

Example: \( A \rightarrow B \) map of \( \text{Sp} \mathcal{G}_k \),

it is an isomorphism in the homotopy category iff \( \mathcal{T}_0(A) \cong \mathcal{T}_0(B) \)

and the relative cotangent complex \( L_{B/A} \rightarrow 0 \)

is trivial.

**Def.** \( A \rightarrow B \) is strongly étale (likely smooth) and quasi-étale

- if

1. \( \mathcal{T}_0(A) \rightarrow \mathcal{T}_0(B) \) is strongly étale

2. \( \mathcal{T}_n(A) \otimes \mathcal{T}_n(B) \rightarrow \mathcal{T}_n(B) \)

**Def.** \( \{ A \rightarrow k \} \) is a strongly étale covering if

1. \( A \rightarrow k \) is strongly étale
ii) \( \{ \text{spec } \mathcal{O}_U^A \to \text{spec } \mathcal{O}_V^A \} \) is an étale cover.

Examples: let's work with monos graded comm. algebras \( A^i \in \mathcal{O}_{x_1}^{A^i} \) (char \( k=0 \))

1. Standard strongly étale maps:

\[
A^i \to B^i = A^i[x_1, \ldots, x_n] \langle \xi_1, \ldots, \xi_n \rangle
\]

\[
\delta \xi_i = \xi_i \in \mathcal{O}_{x_1}^{A^i}[x_1, \ldots, x_n]
\]

\[
\text{deg } x_i = 0, \quad \text{deg } \xi_i = 1
\]

Jacobian criterion: \( \text{det } \begin{pmatrix} \frac{\partial \xi_i}{\partial x_j} \end{pmatrix} \in \mathcal{O}_{x_1}^{A^i}[x_1, \ldots, x_n] \)

is a unit.

Then \( A^i \to B^i \) is strongly étale, &

exists locally any strongly étale map is of this form. (analog of standard local description of étale maps.)
2. \( A \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} = \times^g \)

Think about the structure of \( T \). Notice that \( T \) is a set of \( \{1, 2, \ldots, g\} \). Slowly, the structure of \( T \) becomes evident.

Note: This is a story about open invasions, on the one hand, and principles on the other.
\[ F(x) \rightarrow \text{Rolyn}(F(0)) \]

\[ A: \text{16}(\text{PG}) \rightarrow \text{Mo}(\text{set}) - \text{cofinitization factor} \]

\[ \text{Rota}(E(0)) \rightarrow \text{def}(E(1)) \]

\[ \text{Map}(E(0)) \rightarrow \text{Rota}(E(1)) \]
\text{Recall} \quad \text{Ho} \left( \text{St}(A) \right) \xrightarrow{i} \text{Ho} \left( \text{St}_{\text{top}}(A) \right) \\
\text{to be defined}

i \text{ is fully faithful}

\text{Ho} \left( \text{Spec} \mathbb{R} \right) \xrightarrow{i} \text{Ho} \left( \text{Spec} \mathbb{R} \right)
\text{to} \ (\text{Spec} A) \xrightarrow{i} \text{Spec} \pi_0(A)

\text{to preserve holim/hocolim & \text{IR}Hom}

\text{does not preserve holim or \text{IR}Hom!}

Notion of geometric stuff

- always have a cotangent complex
- its dual evaluated at a point is the target space \text{IR}Hom \left( H^{\mathbb{R} \mathbb{E}^0}_2, X \right)
- \text{IR}Hom \text{ goes geometric near by to a cotangent complex}
- of a scheme as dual to target space of \text{IR}Hom \left( H^{\mathbb{R} \mathbb{E}^0}_2, X \right)

- to i preserves geometricity