Remarque. In a dimension, superconformal group $G$ has even part $\text{Spin}(2n) \times H$.

Which of these can we get by reducing dimension? Pick $T : \text{Spin}(2n) \rightarrow \mu_{2n}$, fixing $\text{Spin}(2n)$. Denote

If we can lift $T$ to $GG = G^{2}$, $G^{2}$ = superconformal group in dimension.

However, the 5-dim group can't be obtained from the 6-dim group this way.

Let's see: for each $\text{ADE}$ group $H$ get superconformal theory with group $O_{Sp}(2,6)$. $SO(5,5) \times Sp(4)$.

Formation on 6-manifold $M$ with little extrudes:

$\text{Z}(6) = \text{Center of group } G \text{ of } \text{ADE} \text{ type}$
Have a form $\mathbb{Z}_{(c)} \times \mathbb{Z}_{(6)} \subset \mathbb{Q}$ such that

$$\phi: \mathbb{Z}_{(c)} \times \mathbb{Z}_{(6)} \rightarrow \mathbb{Q}$$
If \( M_5 = M_5 \times S^1 \), \[ \text{get a low energy/} \]
long distance description of our theory by

\[ \text{gauge theory on } M_5, \text{ } \& \text{ } \text{coherent invariance} \]

\[ \Rightarrow \text{coefficient of action } J = \frac{1}{m} \int \text{Tr} \text{ } F \wedge \ast F \]

is inverse radius ... opposite to dim

\[ \text{reduction of a gauge theory!} \]

Refinement: want to consider states of \( \mathbf{6} \)

\[ \text{theory not invariant under } S^1 \text{ rotation, } \]
\[ \text{i.e. states at different momenta } \]

\[ \text{hard without classical description ...} \]

A) \[ \text{Let } M_5 = R^4 \times M_4 \times S^1 \]

\[ \text{to get Hamiltonian interpretation (Hilbert space)} \]

Two operators are "energy" \[ H \rightarrow -\frac{\beta}{2} \]

\[ \text{momentum } P \rightarrow -i \frac{d}{\partial \theta} \]

\[ \theta \in [0, 2\pi] \text{ parameter on } S^1 \]

What does \( P \) correspond to in gauge theory
description?
\( H \) = Hilbert space (quantizing all the fields, correct in A etc. on \( M_4 \))

[SUSY] \[ 1^+ \cdot P = \sum Q_x^2, \quad 1^- \cdot P = \sum Q_x^{-2} \]

So \( H \) is a sum of squares:

Superconformal group on \( M_4 \) has odd generators, either by twist or if \( M_4 \)
is hyper-Kähler, \( \mathcal{N} \)Their commutators give contours of the even generators, which are only

\( H_4 \) generically as geometric symmetry &

\( \text{Sp}(4) \ R \)-symmetry; cuts an spinor point

Hilbert space comes from quantizing fields

on \( M_4 \), which breaks up as a direct sum

\[ H = \oplus H_n \]

\( H_n \) = quantization of connection on

bundley with \( C_2 = n \).

Parts by multiplicative \( 1 - \frac{1}{2} \cdot n \) (up to sign)

- moduli along circle matches up

with \( C_2 \) of bundles.
SUSY ⇒ $H = 1$ for all states

& $(H - P) |\psi\rangle = 0$

⇒ some SUSY $Q_i |\psi\rangle = 0$

$(H + P) |\psi\rangle = 0$ ⇒ some $Q_i |\psi\rangle = 0$

so these states are very special! BPS states

If $M_g$ is hyperkähler, then

$\{ y / p, y = \frac{1}{2m} n y \}$ & $(H + P) y = 0$

is the cohomology of the instanton moduli space. (We're suppressing the center of group / $w_2$ invariants)

$M_g$ hyperkähler ⇒ moduli of instantons is hyperkähler ⇒

cohomology of moduli space comes on

$Sp(4)$ action ("Lefschetz")
The local superconformal symmetry group is $\text{OSp}(3\vert 4) \supset \text{Sp}(4)$.

On a hyperkähler manifold this survives

By "cohomology" we mean space of $L^2$ harmonic forms: it acts on relaiton

$H \cdot \omega \in \Omega_{\text{harm}}$ : any such annihilated by $H \cdot \omega$ must be annihilated by all the $Q_i$.

On hyperkähler manifold a harmonic form is annihilated by 8 different commuting linear transforms in fundamental rep of $\text{Sp}(4)$.

Problems with instanton moduli:

My noncompact $\rightarrow$ instanton moduli

noncompact $\rightarrow$ big instantons.

Also have problems with small instantons:

real $\rightarrow$ very singular, can make sense for
types $A,D$: type $A$ use noncommutative deformation of (Milnor) instantons (Nekrasov etc.)
On ALE space can use ADHM construction
formally to define our space of harmonic forms
by symplectic reduction: quantum
reduction is our definition of the
cohomology.

B. Instead of \( M_5 = M_5 \times S^1 \) take a
fibration \( S^1 \to M_5 \)
\[ \downarrow \]
\[ M_5 \]

Describe the confined structure via a Kählerian
connection on \( M_5 \) & on \( S^1 \) (i.e. a radius \( r \) )
& a \( SU(2) \) connection on the circle bundle.

[Assume available for today]

Still get a 5-dimensional gauge theory on \( M_5 \)
but with extra term in the action
let \( f = 2-form \) at the \( S^1 \) limit.
\[ C(A) = \frac{i}{4\pi} \text{Tr}(A \wedge A + \frac{2}{3} A \wedge A \wedge A) \]

= Chern-Simons 3-form of A

\[ J = \frac{i}{2} \int_{M_5} \text{Tr}(F \wedge F \wedge \ldots) + \int_{M_5} f \wedge C(A) \]

\[ \Rightarrow \text{Exp } iJ \text{ is well defined} \]

(due to integrality of f)

In second the path integral has a factor
\[ \text{Exp } (i \int_{M_5} f \wedge C(A)) \]

-- have a U(1)xG gauge theory on M_5
where U(1) acts on extra circle
1 have an invariant form in Lie (U(1)xG)
\[ \Rightarrow \text{get a well defined Chern-Simons 5-form} \]

if G to invariant depends not just on curve but on holonomy of circle bundle
Now allow $S^1$ to degenerate at some point: consider $M^r$ generically an $S^1$ fibration $M^r \to \Sigma$ e.g. if $M^r$ has a $M^r$ generically free $U(1)$ action, $M^r = M^r / U(1)$

Two cases of interest:

1. $G/Grid\times M^r$

2. $S^1$ action on $M^r = \mathbb{C}^2$ by complex $e^{i\theta}$

- isolated fixed point at origin (this $U(1) = U(1)$ which commutes with an $SU(2)$, roving three complex structure)
Our $U(1)$ is left multiplied by $i$
on $HH$, counts with right $i$ job
multiplied.

Quadratic by $U(1)$ is $\mathbb{R}^2$ 
& hyperkähler moment map is 
$\mathbb{R}^4 / U(1) \rightarrow \mathbb{R}^3$ 

$[ x = \cos(s) \exp \frac{\tau}{2} \exp \frac{2z}{\sqrt{2}} \text{ in lens of} 
\text{SU}(2) \text{ generator}]$

$[ \text{We'll consider} \quad M_5 = HH \times \Sigma$
\[ M_5 = \mathbb{R}^3 \times \Sigma \quad \text{3} \]

\[ \begin{array}{c}
\mathbb{R}^4 \quad \rightarrow \\
\mathbb{R}^3 \\
\mathbb{R}^4 \cdot 0 \sim \mathbb{S}^3 \\
\text{Singularity point}
\end{array} \]

\[ \mathbb{S}^2 \sim \mathbb{R}^3 \cdot 0 \quad (\text{boundary}) \]
In standard hyperbolic metric on $R^4$, 
the ends of $S^1 \to \infty$ 

But there another hyperbolic metric 
where the radius is fixed at $\infty$.

$TN=TN-LMT$ : let $V = 1 + \frac{1}{1/N}$

$ds^2 = \frac{1}{V} (d\phi + \omega_1 \cdot d\phi^1)^2 + Vd^2$ 

$x = v$-manifold $R^3$

$\omega = \text{correction one form on the V(a) bundle}$

restrict to each fiber is $d\phi \wedge d\phi$

$d\omega = \pi \times f$.

radius $= \text{circum} \frac{1}{V} \text{ so has fixed}$

value at $v$.

Related $L^2$ harmonic 2-form on $TN-LMT$ 
(unlike on $R^4$ of standard metric) where

is $\pi \omega = \frac{1}{\sqrt{V}}$

$[ t = x \cdot \frac{1}{\ell}]$
$L^2$ cohomology is one dim! mondrie

The set of some integer periods of $\alpha$ to get

Integral cohomology:

$H^2_{L^2 \text{ tensor}} (TN, \mathbb{Z})$:

- generator $(x,x) \in \mathbb{H}$ (looks like $\mathbb{H}$) would suffice

The metric is made of 0

Look in two ways of $M^6 = \mathbb{R} \times S^1 \times TN$:

A) $M^6 \rightarrow \mathbb{R} \times TN$, forget $S^1$ factor

- $\Rightarrow$ gauge theory on $\mathbb{R} \times TN$

  BPS states $= H^2_{L^2 \text{-form}} (\text{Mink}(TN,G))$

B) $M^6 \rightarrow (\mathbb{R} \times S^1) \times \mathbb{R}^3$ via

  - monodromy map $TN \rightarrow \mathbb{R}^3$

  $\Rightarrow$ gauge theory on $\mathbb{R} \times S^1 \times \mathbb{R}^3$ with

  same correction on $\mathbb{R} \times S^1 \times \{0\}$ from singularity.
$M_\Sigma$

$
\downarrow

M_\Sigma \to \Sigma = \text{2-manifold of } \text{"best points"}
$

$= M^{\Sigma \nu\mu}_{\Sigma

\text{gauge theory on } M_\Sigma \text{ corrected along } \Sigma.

\text{Factor } \exp(-i \int \text{CS}(A)) \text{ is}

\text{not gauge invariant in our singular situation! } \int f = \tfrac{i}{4} \text{not closed on } \Sigma.

[We're assuming bundle is topologically trivial!]

\text{Under gauge transform } \Sigma

\text{CS} (A) \to \text{CS} (A) + d(Tr \text{ } \text{d}A)

\text{exp} (-i \int \text{CS}(A)) \text{ not well defined as a number but is defined as a section}

\text{of a line bundle } \tilde{\Sigma} \to \text{space of connections}

(f = \text{det bundle}) \text{ on } \Sigma
So along Σ "there are degrees of freedom that cancel the anomaly" -

ie there is a quantum theory w/ 6 symsy
whose partition function is 1-valued (curly)
*** can twist by any bundle with anomaly
        to the correction Σ

[U(n) case: free fermions have this see kind
     of anomaly.]

The current algebra of level 1 is such a system
      - level 1 WZW model

Hamiltonian version

\[ R^3 \times S^1 \]
\[ 5 \times R^3 = \text{Space} \]
\[ U \]
\[ 5' \times pt \]

\[ \Rightarrow \text{see representation theory of loop group} \]

of level 1:

\[ (\text{Gauge theory on } R^3 \times S^1) \times (\text{rep of GL at level 1}) \]
Space of BPS states comes from the second factor: cohomology of instanton moduli space = level 1 rep of LG

Hamiltonian picture: On $S^1 \times \mathbb{R}^3$ have
- gauge fields coincide & rep of LG at $S^1 \times 0$
- This gives the Hilbert space of a two dim hyper which makes sense on any surface $\Sigma$: if better

Since any $\Sigma$ can arise from our construction, $(\rightarrow \text{2d CFT})$.

Space of lowest energy/BPS states doesn't feel the $\mathbb{R}^3$, just get the rep of the loop group: see a chiral

CFT on the BPS states:

\[ \frac{2}{\pi} \left( H - P \right) (\text{Rep of LG}) = 0 \]
The way we take SUSY mean we get
only a chiral term.

On Torb: NUI: TN not simply connected as:
should look at instants with
fixed holonomy
to get a good problem, & problem is best
behaved if holonomy is regular
(centralizer is a torus).

$\mathbb{R}^3 \times S^1$ picture: can fix holonomy around $S^1$

$\rightarrow$ get representation of a

finite loop group, with an inner twist (i.e.

still $\sim$ keep you).

$g(\theta + 1) = h \cdot g(\theta) \cdot h^{-1}$.

We've broken G symmetry on the representation $\mathbb{R}$
to the maximal torus (preserves the twisted

loop group): centralizer of h acts

& we're assuming $Z_\infty(h) = 1$
Now decompose rep. \( R \) into \( T \) weights:

\[ R = \bigoplus R_{\alpha} \text{ but } \alpha \text{ only lies in a torus, i.e. affine version of the weight lattice when we avoid a loop under shift in labeling.} \]

**Generalization** \( \mathbb{R}^4 \rightarrow \mathbb{R}^4/\mathbb{Z}^4 \) hyperbolic manifold

\[ \mathbb{R}^4/\mathbb{Z}^4 \rightarrow \mathbb{R}^5 \text{ hyperbolic manifold map} \]

Metric still has to have the form

\[ ds^2 = \frac{1}{V} (df_1^2 + df_2^2) = \cdots \]

for some \( V \) \[ f = x \, dv \]
We'll take \( V = 1 + \sum_{j=1}^{n} \frac{1}{||x-x_j||} \)

Get smooth 4-manifold, singularities of \( V \) all look locally like the n+1 example.

If we erase the "1" in \( V \) we get the ALE space, whose radius at the circle increases at \( oo \), while in \( TN_0 \) radius is fixed at \( oo \).

\[ H^2_{ord} = \bigoplus_{\mu} \text{Hom}(\mathfrak{g}, \mathfrak{t}) \] 
not uniréductive

\[ H^2_{l^2-bilinear} = \bigoplus_{\mu} \text{Hom}(U(n)) : \]

\[ \langle x_i, x_j \rangle : (x_i, x_j) = \delta_{ij} \]

uniréductive but not \( \text{semisimple} \)

Now run same story as before:

\( \mathcal{M} = \mathbb{R} \times S^3 \times TN_0; \ x_1, \ldots, x_n; \ x_1, \ldots, x_n \in \mathbb{R}^3 \)
A. \[ M_3 \rightarrow \mathbb{R} \times TN_{x_1, x_2, \ldots, x_n} \]

BPS states \( = H^2_{L^2}(M_3, \text{TN}_{x_1, x_2, \ldots, x_n}) \)

B. \[ M_3 \rightarrow \mathbb{R} \times S^1 \times \mathbb{R}^3 \text{ via } \text{TN}_{x_1, x_2, \ldots, x_n} \rightarrow \mathbb{R}^3 \]

\[ \Rightarrow \mathbb{R} \times S^1 \times \mathbb{R}^3 \text{ via } \mathbb{R}^3 \text{ with k distinguished pts} \]

On \( \mathbb{R} \times S^1 \times \{x\} \text{ are a current algebra} \)

at level 1:

\[ \mathcal{H}^1 \left( \text{quantize} \right) \otimes \bigoplus_{i=1}^{k} R_{(i)} \]

\[ R_{(i)} \text{ reps of } LG \text{ ... which are?} \]

remember we have a finite number of

only from \( H^2(M_3, \mathbb{Z}(1)) \) \( \oplus H^2(M_3, \mathbb{Z}(6)) \)

In our case:

\[ H^2_{L^2}(TN) \oplus \mathbb{Z}(6) \]
So our polarization ends up telling us which reps of low group to take

... our 6-dm space $U^2 (\mathbb{C})$

\[ H^1 (\Sigma, \mathbb{R}_\mathbb{C}) \otimes^n \]

So 6-d problem reduces to a central of the $U(1)$ story.

\[ L_0 = C_2, \text{ also have relative first Chern class giving rest of torus action.} \]

When we're inside get higher level reps of those points!

$x \mapsto x^g \otimes \text{level 2 modulo for } G$

\[ x \mapsto x^g \quad \text{clearing level 1 @ level 1} \]

\[ \Rightarrow \text{level 2 @ reps of coset } \]

So banner of reps = at class of $x$