A. Beilinson – Factorization & determination of periods

Note Title 11/26/2007

Deligne seminar at Bures 1984 : consider

<u>Problem</u> $X/\mathbb{C}$ smooth projective, M holonomic $\mathcal{D}$-module on X $\rightsquigarrow$ consider de Rham complex $dR(M) = M \xrightarrow{\nabla} M \otimes \omega_X$

$\boxed{\text{for } X \text{ a curve}}$

Global cohomology (in Zariski topology)

de Rham $R\Gamma_{dR}(X,M) = R\Gamma(X_{Zar}, dR^{alg}(M))$

$\downarrow\wr$ (in classical topology)

Betti: $R\Gamma_B(X,M) := R\Gamma(X_{cl}, dR^{hol}(M))$

Map $\downarrow$ is a quasiisomorphism $\rightsquigarrow$

can put rational structures in two complementary ways :

$k \subset \mathbb{C}$ $\rightsquigarrow$ de Rham rational structure:

X, M defined $/k \Longrightarrow R\Gamma_{dR}(X,M)/k$

or $k \subset \mathbb{C}$ $\rightsquigarrow$ Betti rational structure

constructible sheaf $dR^{hol}(M)/_k \rightsquigarrow R\Gamma_B(X,M)/k$

How are these related?

Easier question: look at determinants

$\det R\Gamma_{dR} \xrightarrow{\sim} \det R\Gamma_B$ isomorphism of lines

$\Rightarrow$ 2 k structures on same $\mathbb{C}$-line

$\Rightarrow$ complex number (defined mod k^*)

period $\in \mathbb{C}^*/k^*$

Deligne treated case of rank 1 bundles with connection by steepest descent method.

Bloch-Deligne-Esnault unpublished: higher rank version

Different vision of Deligne for periods:

ν non zero meromorphic 1-form on X

$\mathrm{div}(\nu)$ divisor of ν

P = locus of singularities of M

Want to assign to each $x \in X$ a graded super line ("epsilon factor")

$\mathcal{E}_{dR}(M,\nu)_x$, $\mathcal{E}_B(M,\nu)_x$

purely algebraic topological

+ graded superlines should be trivialized

for $x \in X \setminus \{\mathrm{div}\,\nu \cup P\}$

+ product formula:

$$\bigotimes_{x\in X} \mathcal{E}_{dR}(M,\nu)_x \xrightarrow{\sim} \det R\Gamma_{dR}(X,M)$$

& same for Betti.

+ identifications $\kappa_x : \mathcal{E}_{dR}(M,\nu)_x \xrightarrow{\sim} \mathcal{E}_B(M,\nu)_x$

so that $\bigotimes_{x\in X} \kappa_x$ gives the determinant of the period map $\det R\Gamma_{dR} \xrightarrow{\sim} \det R\Gamma_B$

Deligne constructed de Rham epsilon factors & suggested Morse theory approach to Betti factors.

At same time: holonomic quantum fields of Sato Kashiwara Kawai.

Today: write the holo. QFT approach to determinant lines.

(1) Factorization lines

$X/\mathbb{C}$ smooth curve, $P \subset X$ finite subset

$\mathcal{L}$ line bundle (will be ω_X in application)

<u>Def</u> An $\mathcal{L}$-twisted factorization line on (X,P) is a rule assigning for S a base scheme, $U \subset X \times S$ open (dense in each fiber) & ν meromorphic section of $\mathcal{L}$ on U with divisor $\operatorname{div} \nu \cup (P \times S) \cap U$ finite over S
[ie singularities don't go to ∞ in U]

$$(U/S, \nu) \longmapsto \mathcal{E}(U/S, \nu)$$

flat graded superline bundle on S satisfying

i. Compatible with base change

ii. Local nature wrt $P \cup \operatorname{div}(\nu)$
(identification of lines when we change U appropriately):
if $U' \subset U$ contains singular set $\Rightarrow$
$\mathcal{E}(U'/S, \nu) \xrightarrow{\sim} \mathcal{E}(U/S, \nu)$

iii. Factorization: if $P \cup \operatorname{div} \nu = \coprod D_\alpha$
disjoint union & $U_\alpha = U \setminus \bigcup_{\beta \neq \alpha} D_\beta$

$$\bigotimes_\alpha \mathcal{E}(U_\alpha/S, \nu) \xrightarrow{\sim} \mathcal{E}(U/S, \nu)$$

Remarks

- Such factorization lines have a local nature wrt X .... form a sheaf of Picard groupoids on X, $\mathcal{F}(X,P,\mathcal{L})$

- X proper $\Rightarrow$ $\mathcal{E}(X) = \mathcal{E}(X, \text{"}0\text{"})$: $\begin{bmatrix} U=X \\ \nu=0 \end{bmatrix}$
have well defined value at $\nu=0$: interpolates between $\mathcal{E}$ lines over space of ν's
... analytically get a constant line over the space of ν's. Role of properness!
we can't allow singularities to escape to ∞, so when we take $U=X$ need X proper.

2. Basic observation

Description of $\mathcal{F}(X,P,\mathcal{L})$ in the analytic setting.

For $n\in\mathbb{Z}$ let $F_X^{(n)}$ be the $\mathbb{G}_m$-torsor over X corresponding to $\mathcal{L}\otimes\omega_X^{\otimes n}$

— carries principal parts of sections ν!

Given ν with n-th order pole at $x \in X$ can consider the top polar part of ν at x

$\rightsquigarrow$ lives in torsor $F_X^{(n)} \ni \nu_x$

Space of ν's with given leading term form a contractible space & our lines were flat in families

$\Rightarrow$ so we get a local system on $F_X^{(n)}$ ...

So every factorization line $\mathcal{E}$ yields a local system of graded superlines on each $F_{X \setminus P}^{(n)}$ & $F_P^{(n)}$

[Let $\mathcal{P}(Y)$ on a space Y = Picard groupoid of graded superlines on Y.]

<u>Def</u> $\phi(X, P, 1) \subset \mathcal{P}(F_P^{(-1)}) \times \mathcal{P}(F_{X \setminus P}^{(-1)})$

subcategory of pairs $(\underset{G_P}{\in}, \underset{G_{X \setminus P}}{\in})$

satisfying extra condition.

G_{X-P} • the monodromy on the circle $F_x^{(-1)}$ $x \in X-P$
of G_{X-P} equals $(-1)^{\deg G_{X-P}}$.

G_P • the monodromy on the circle $F_x^{(-1)}$ $x \in P$
is $(-1)^{\deg G_P} \cdot \mu(G_{X-P}$ around $x)$

where

we take any trivialization of $F^{(-1)}$ over the disc at x (! / homotopy)

.... take in this trivialization
the monodromy around x of G_{X-P}
call it $\mu(G_{X-P}$ around $x)$.

<u>Theorem</u> $\mathcal{F}(X,P,\mathcal{L}) \xrightarrow{\sim} \phi(X,P,\mathcal{L})$
equivalence of Picard groupoids
.... reduce everything to first order poles
of v! (study confluence of poles ...)

Our setting: $\mathcal{L} = \omega_X$.
then $F^{(-1)}$ is canonically trivialized
by fixing condition "$-\frac{df}{f}$" :

form with residue -1.

Now local system on $F^{(-1)}_{X \setminus P}$ is the same as a local system on $X \setminus P$ & a number = monodromy on fiber. But we've fixed this number.

Likewise on $F_P^{(-1)}$ we've determined the monodromy completely:

$$\phi(X, P, \omega) = \left\{ \begin{smallmatrix} \text{super line} \\ \text{on } X \setminus P \end{smallmatrix} \right\} \times \left\{ \begin{smallmatrix} \text{super line} \\ \text{on } P \end{smallmatrix} \right\}$$

$$= \gamma(P) \times \gamma(X \setminus P)$$

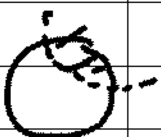
3. Now construct $\mathcal{E}_{dR}$ & $\mathcal{E}_B$ as factorization lines.

$\mathcal{E}_{dR}(M, \nabla)_x$: $M \otimes F_x$ ($F_x = K_x$ Laurent series at x)

$\nu^{-1} \nabla$

$\nu^{-1}\nabla$ Fredholm endomorphism of $M \otimes F_x$

$\rightsquigarrow$ polarized determinant line: fix $\mathcal{O}_x$ lattice, apply operator, take index.

$\mathcal{E}_B(M,\nu)_x$: look at real part of ν
$\rightsquigarrow$ defines a class of locally closed

subspaces of X (Morse theory):
Cover ball by such subsets.

Can compute det of cohomology on the constructible set which is the complement of these opens: defined using stratification ...

Now when our differential forms have only $\frac{df}{f}$ poles $\rightsquigarrow$ Betti cohomology is trivialized on $X - D$.

de Rham version: connection has regular singularities, preserves lots of lattices, define regularized determinant using gamma function ... reduce irregular to regular ...

Now need to prove product formula! do it by global methods - move curve, Drinfeld etc. But in global setting this is just a question of equality of numbers (independent of diff. form)

Use Theorem of Goldman on Teichmüller group action on moduli of local systems: any function invariant under the action depends only on the singularities of the local system & satisfies factorization — use degeneration of curves into pairs of pants.

⇒ find that our numbers are products of numbers depending on singularities separately.