E. Frenkel – Geometric Endoscopy & Mirror Symmetry

Note Title 11/28/2007

(w/ E. Witten)

C smooth projective curve / $\mathbb{C}$ irreducible
G [reductive or] simple Lie group

${}^LG$ · Langlands dual group

$\mathcal{M}_H(G)$ = moduli space of stable Higgs G-bundles / C
$= \left\{ (E, \varphi) : \begin{array}{l} E \to C \text{ principal } G\text{-bundle} \\ \varphi \in H^0(C, \mathfrak{g}_E \otimes K_C) \end{array} \right\}_0$
$\mathfrak{g}_E$-valued 1-form

Hitchin map:

$\mathcal{M}_H(G) \longrightarrow \mathbb{B} = \oplus H^0(C, K_C^{\otimes(d_i+1)})$

d_i+1 $(i=1,\dots l = \text{rk}\,\mathfrak{g})$ orders of Casimirs (generators of invariant polynomials on $\mathfrak{g}$)

$G = SL_n$ this is just characteristic polynomial map.

The set $\{d_i\}$, hence $\mathbb{B}$, are same for G & for ${}^LG$:

$$\begin{tikzcd} \mathcal{M}_H(G) \arrow[dr] & & \mathcal{M}_H({}^LG) \arrow[dl] \\ & \mathbb{B} & \end{tikzcd}$$

These are two dual fibrations in the sense that generic fibers are smooth tori that are dual to each other. (One side is moduli of flat line bundles on other).
... must take with grain of salt: gerbes, disconnected, torsors

These two moduli spaces are mirror dual
Homological Mirror Symmetry for them $\rightsquigarrow$ Geometric Langlands program

Both $\mathcal{M}_H(G)$, $\mathcal{M}_H({}^LG)$ they are hyperkähler.
Complex structure I : moduli of Higgs bundles
" " J : " " local systems

A-model in symplectic structure K ω_K $\longleftrightarrow$ B-model in cx. structure J

A-branes on $\mathcal{M}_H(G)$ $\longleftrightarrow$ B-branes on $\mathcal{M}_H({}^LG)$

= pairs (L, ∇) L Lagrangian, ∇ flat unitary line bundle, + coisotropic ones....

- derived category of coherent sheaves (in Betti picture)

This equivalence is meant to intertwine
't Hooft operators & Wilson operators:

Wilson operators :
$$\begin{array}{c} E_{univ} \\ \downarrow \\ \mathcal{M}_H({}^LG)\times C \end{array} \implies$$

$$\begin{array}{c} V_{univ,x} \\ \downarrow \\ \mathcal{M}_H({}^LG) \end{array}$$ universal vector bundle assoc. to $V \in \mathrm{Rep}\,{}^LG$, $x \in C$

Tensoring with it is the Wilson operator $W_{V,x} * \mathcal{B} = V_{univ,x} \otimes \mathcal{B}$

$\implies W_{V,x} * \mathcal{O}_\mathcal{E} = \mathcal{O}_\mathcal{E} \otimes V_{univ,x}|_\mathcal{E}$:

applying to skyscraper at $\mathcal{E} \in \mathcal{M}_H(\mathcal{B})$
just multiplies $\mathcal{O}_\mathcal{E}$ by a vector space
-- eigenbrane of Wilson operators

Dual of $\mathcal{O}_\mathcal{E}$ for $\mathcal{E}$ in a smooth Hitchin fiber:

$$\begin{array}{ccccc} F_b \subset \mathcal{M}_H(G) & & & & \mathcal{M}_H({}^LG) \supset {}^LF_b \ni \mathcal{E} \\ & \searrow & & \swarrow & \\ & & \mathcal{B} \ni b & & \end{array}$$

Fiber ${}^L F_b$ = flat unitary line bundle on F_b

So $\mathcal{E}$ gives $\begin{matrix} \text{flat bundle} \\ \downarrow \\ F_b \end{matrix}$ $\in$ A-branes

which is the dual to the B-brane $\mathcal{O}_\mathcal{E}$.

... according to SYZ prescription

Kapustin-Witten: these are indeed eigenbranes of the 't Hooft operators

Geometric Langlands correspondence:

$\mathcal{E}$ ${}^L G$ local system $\longmapsto$ $\mathcal{F}_\mathcal{E}$ D-module on Bun_G moduli stack of G-bundles

which is a Hecke eigensheaf with eigenvalue $\mathcal{E}$:

Hecke operators $H_{V,x}$ $\{ V \in \mathrm{Rep}\, {}^L G \quad x \in C \}$

act by multiplication by $(\mathcal{E}_x)_V$.

What happens at singular fibers, in particular at singular pt?

mildest case: orbifold singularities: local systems with Γ a finite groups of automorphisms (assume for simplicity ${}^LG$ is of adjoint type, so has trivial center).

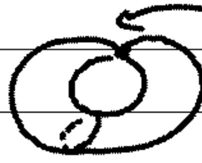
Example: $G = SL_2$ $\quad {}^LG = PGL_2 = SO_3$

$$\begin{array}{c} \\ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & \pm 1 \end{array}\right) \end{array} \begin{array}{c} SO_3 \\ \cup \\ = O_2 \end{array}$$

O_2 has a center $\mathbb{Z}/2$

Let $\mathcal{E}$ be an O_2 local system, thought of as SO_3 local system. A generic such $\mathcal{E}$ will have $\mathrm{Aut}\,\mathcal{E} = \mathbb{Z}/2 \rightsquigarrow \mathbb{Z}/2$ orbifold point.

$g=1$ $\quad {}^LG = SO_3$ single point of ramification, get a Hitchin fiber



$\mathcal{E}$ $\mathbb{Z}/2$ orbifold point on a nodal $\mathbb{P}^1$.

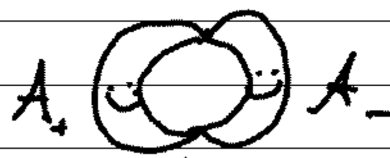
B-branes supported at $\mathcal{E}$ are equivalent to $\mathrm{Rep}\,\Gamma \rightsquigarrow 2$ irreducible branes,

$\mathcal{B}_+ \longleftrightarrow$ trivial rep $\quad \mathcal{B}_- \longleftrightarrow$ sign rep

(actually gets twisted by a gerbe, can't distinguish $\mathcal{B}_\pm$)

$\mathcal{B}_+ \oplus \mathcal{B}_-$ is an eigenbrane

Describe this by a limiting procedure:
generic eigenbrane on nearby fiber degenerates
to $\mathcal{B}_+ \oplus \mathcal{B}_-$

Dual picture: 

two irred components

these points are smooth
in the ambient space

So eigenbrane on fiber decomposes into
the two fractional branes $A_\pm$.

– even though we can't explicitly describe
the eigen D-modules, we can "touch"
the A-branes – draw their pictures!

— so expect $\mathcal{F}_\mathcal{E}$ eigenbrane $= \mathcal{F}_{\mathcal{E}+} \oplus \mathcal{F}_{\mathcal{E}-}$
decomposes into a direct sum.
each piece satisfies a fractional Hecke property

So expect for endoscopic local systems same
property holds $\rightsquigarrow$ check on level of
functions.