V. Ginzburg - Calabi Yau algebras & Noncommutative Geometry

Note Title 11/27/2007

(w/ P. Etingof)

A assoc algebra.
Assume A has a finite projective resolution as A-bimodule (finite HH dim).

$\Rightarrow$ anti-dualizing complex
$A^! = R\mathrm{Hom}_{A\text{-}A}(A, A \otimes A)$

Motivation: X smooth variety $X \overset{i}{\hookrightarrow} X \times X$

$$\mathrm{Ext}^i(i_* \mathcal{O}_X, \mathcal{O}_{X \times X}) = \begin{cases} K_X^{-1} & \text{top deg} \\ 0 & \text{otherwise} \end{cases}$$

<u>Def</u> An algebra A as above is Calabi-Yau of dim d if
$A^! \cong A[-d]$

ie $\mathrm{Ext}^j_{A\text{-}A}(A, A \otimes A) = \begin{cases} A & j=d \\ 0 & j \neq d \end{cases}$ as A-bimodules

LHS is A-bimodule using <u>inside</u> bimodule structure on $A \otimes A$, which survives to Exts

Deformation Quantization

Suppose A is a deformation of $\mathbb{C}[X]$
X variety. When is A a CY_d?

- X must be a CY_d : zeroth order in def.
  $\mathrm{vol} \in \Omega^d(X)$
- What about to first order?
  $\mathbb{C}[X]$ carries Poisson bracket. What's the relation between the Poisson bivector & vol to make A CY to first order?

$\Longleftrightarrow$

Poisson structure is unimodular:
every hamiltonian vector field has 0 divergence,
$\operatorname{div} \xi_f = 0 \quad \forall f \in \mathcal{O}_X$

Another way to say this:

$$\mathrm{vol} \rightsquigarrow \quad \Lambda^p T_X \xrightarrow{\sim} \Lambda^{d-p} T_X^*$$

$$\text{bivector} \longmapsto \alpha \in \Lambda^{d-2} T_X^*$$

$$\text{unimodularity} \Longleftrightarrow d\alpha = 0$$

Today: $X = \mathbb{C}^3 \quad \mathrm{Vol} = dx \wedge dy \wedge dz$
Unimodular Poisson structure on $\mathbb{C}^3$
$\longleftrightarrow$ closed 1-form $d\varphi$, $\varphi \in \mathbb{C}[x,y,z]$

Corresponding Poisson bracket

$$\{f,g\}_\varphi = \frac{df\, dg\, d\varphi}{dx\, dy\, dz}$$

For f,g linear functions $\rightsquigarrow$

$$\{x,y\}_\varphi = \frac{\partial\varphi}{\partial z} \qquad \{y,z\}_\varphi = \frac{\partial\varphi}{\partial x} \quad \text{etc.}$$

$A_\varphi := \mathbb{C}[x,y,z], \{\}_\varphi$ Poisson algebra

- 0-locus of $\{\}_\varphi$ is critical locus of φ
- $\mathbb{C}[\varphi] \subset$ Poisson center of A_φ.
  & equality holds for generic φ

$B_\varphi = A_\varphi/(\varphi)$

Spec $B_\varphi \subset \mathbb{C}^3$ surface

If φ has isolated singularities $\Rightarrow$ this surface is symplectic outside a finite set of points (where $d\varphi = 0$).

Let's quantize A_φ & B_φ !

Want actual not formal quantization!
following Kontsevich (heuristically) must have
$\{\}$ vanishing at ∞.

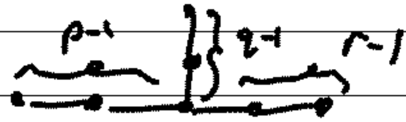
So we'll take S del Pezzo surface,
with elliptic curve at ∞ $\Rightarrow$
$S^{aff} = S \setminus \mathcal{E}$ elliptic curve
(anticanonical is ample & a section
of it is a Poisson bivector, with
0-locus precisely $\mathcal{E}$)
$\rightarrow$ S^{aff} symplectic (when smooth)

Let $\pi \in K_S^{-1}$ Poisson bivector.
have a log symplectic structure:
forms with log poles at ∞

$$\Omega^1_S(\log \mathcal{E}) \xrightarrow{\pi} T_{S,\mathcal{E}}$$

$T_{S,\mathcal{E}}$ vector fields tangent to $\mathcal{E}$

del Pezzo's $\longleftrightarrow$ E_n type Dynkin diagrams



$\tilde{E}_k$ $k=6,7,8$

($*$ -shaped, with legs of length
$p-1$, $q-1$, $r-1$)

Corresponding del Pezzo:

$\varphi = \tau xyz + P(x) + Q(y) + R(z) = 0$

P poly of deg p $= \frac{x^p}{p}$ + lower order

$Q = \frac{y^q}{q} + \dots$

$R = \frac{z^r}{r} + \dots$

Most degenerate case: no lower order terms, get simple elliptic singularity

The rest are deformations of this homogeneous one

Milnor number of this degeneration $= p+q+r-1$

Problem: quantize these surfaces in $\mathbb{C}^3$
i.e quantize B_φ for φ as above.

Ansatz for solution (motivated by CY story):

family of algebras $A^{t,c}_{P,Q,R}$

$$A^{t,c}_{P,Q,R} = \frac{\mathbb{C}\langle x,y,z\rangle}{\left(\left\{\begin{array}{l} xy - tyx - c\frac{dR}{dz}, \\ yz - tzy - c\frac{dP}{dx}, \\ zx - txz - c\frac{dQ}{dy} \end{array}\right\}\right)}$$ 2-sided ideal

family with same number of parameters

Grading on variables (making the elliptic singularity quasi homogeneous)

deg	x	y	z
E6	1	1	1
E7	1	1	2
E_8	1	2	3

This makes $\deg \varphi = \deg x + \deg y + \deg z$ homogeneous.

Our $A^{t,c}$ is thus a filtered algebra.

<u>Theorem</u> For generic [outside countable collection of divisors]
t, c we have
1. A is CY3
2. $\mathrm{gr}\, A \cong A^{t,c}_{P_0, Q_0, R_0}$ P_0, Q_0, R_0 leading terms (homogeneous part)

& Hilbert series
$h(\mathrm{gr}\, A) = h(\mathbb{C}[x,y,z])$ with degrees as deg
(ie family is flat).

3. The classical limit $c \longrightarrow 0$, $t = \tau c + 1 \longrightarrow 1$
(τ fixed) is a flat deformation degenerating
to A_φ (ie $\frac{fg - gf}{c} \rightsquigarrow \{\}_\varphi$)

4. $Z(A) = \mathbb{C}[\Phi]$ polynomial algebra
in one variable. $\deg \Phi = \deg \varphi$
[Φ is unique up to $c_1 \Phi + c_2$.]

<u>Comments</u> • In homogeneous case
$A^{t,c}_{P_0 Q_0 R_0}$ is graded. In case of
E_6 this is the standard Sklyanin algebra

• Φ is very complicated, see Etingof's webpage

- The family $A^{t,c}_{P,Q,R}$ is a versal family of CY algebras

In general: deformations of A as algebra are controlled by $HH^2(A,A)$

<u>Theorem</u> If A is a CY of dimension d $\Rightarrow$
NC BV operator $\Delta : HH^{\bullet}(A,A) \longrightarrow HH^{\bullet-1}(A,A)$

Claim: CY deformations are controlled by
$\operatorname{Ker} \Delta : HH^2 \longrightarrow HH^1$.

'Versality': maps from family to this CY def. space is an isomorphism (for generic t,c)

Define $B(\Phi) = A^{t,c}_{PQR} / ((\Phi))$ two-sided ideal
flat deformation of $B_\varphi = A_\varphi / (\varphi)$

We're interested in $D^bCoh(\operatorname{Spec} B(\Phi))$
Understand theoretically a large piece of this.

Let R = Rees algebra of A,
contains homogeneous version of Φ which we'll denote Ψ

For any NC graded algebra R & $\varphi \in R^d$ homogeneous element in center $\rightsquigarrow$ triangulated category of matrix factorizations

$$M_+ \underset{g'}{\overset{g}{\rightleftarrows}} M_-$$

one preserves degree, other shifts by d

so that composition either way is φ.

$$gg' = g'g = \varphi.$$

Eisenbud, Orlov this category $\mathcal{M}$ sits inside derived category of coherent sheaves ... would be same if it were CY, but here have understandable discrepancy.
(labelled by pts of elliptic curve)