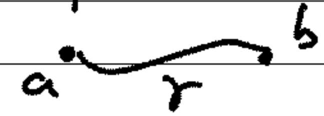
M. Kapranov - NC differential operators and Tannakian duality for nonflat connections

Note Title 11/30/2007

<u>Gauge-string correspondence</u>

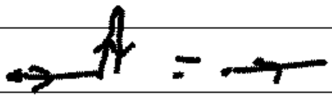
bundle w/ connection (E,∇) $\dashrightarrow$ path



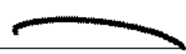
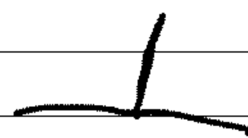
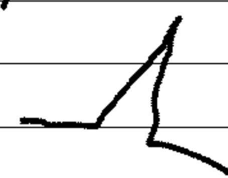
$\Rightarrow$ holonomy $H_\gamma : E_a \longrightarrow E_b$
invariant under reparametrization of path
- defined on unparametrized paths ...
"ill-behaved" space.

Category $Bun_\nabla(X)$, $\otimes$ tensor category of C^∞ bundles w/ connection.
Fiber at a gives functor $Bun_\nabla(X) \xrightarrow{\Phi_a} Vect$
which is a tensor functor
("fiber functor") & $H_\gamma : \Phi_a \xrightarrow{\sim} \Phi_b$
isomorphism of tensor functors.
$\rightsquigarrow$ this in fact determines γ
(over all Bun_∇)

H_γ Holonomy is also invariant under cancellations

 "zig zag symmetry"

Paths/reparam & zigzag is very ill behaved

e.g.  $=$  $\overset{\text{close}}{\sim}$ 

Still, rather attractive POV!

<u>Naive idea</u> #1 Generalized path := isomorphism of $\otimes$ functors $\Phi_a \xrightarrow{\sim} \Phi_b$

This destroys continuity of paths γ...
– need instead stronger condition such as Lipschitz (Young 1930s)... too analytic!

<u>Naive idea</u> #2 Use connections to understand space of paths infinitesimally

$\Pi_X(a,b) =$ piecewise smooth paths $a \to b$
modulo reparametrization
& cancellation

forms a groupoid Π_X (objects are points, morphisms $= \Pi_X(a,b)$)

Let's try to understand it infinitesimally.

Lie group(oid) $\longrightarrow$ Lie algebr(oid)

$$\begin{tikzcd} X \arrow[r, hook, "e"] & \Pi_X \arrow[l, shift left, "s"] \arrow[l, shift left=3, bend left, "t"] \end{tikzcd} \qquad \text{want } e^* T_{\Pi_X/X}.$$

Lie algebroid: $\mathfrak{g}$ vector bundle on X with a bracket on sections & a linear morphism of bundles $\mathfrak{g} \xrightarrow[\alpha]{} T_X$ preserving the bracket + property $[f\cdot X, Y] = f\cdot[X,Y] + \mathrm{Lie}_{\alpha(X)}(f)\cdot Y$

$\mathring{\mathfrak{g}} = \operatorname{Ker} \alpha$ is then a bundle of Lie algebras (ie fiber-by-fiber bracket)

Free Lie algebroid: adjoint to functor "forget"

$$\begin{tikzcd} \{\text{Lie Alg}/X\} \arrow[r, "\text{forget}"] & \{\text{bundles } E \xrightarrow{\alpha} T_X\} \arrow[l, bend left, "FL"] \end{tikzcd}$$

Def The path Lie algebroid of X is $\mathcal{P}_X =$ $FL(T_X \xrightarrow{id} T_X)$ (X C^∞ or smooth/$\mathbb{C}$)

Claim $\mathcal{P}_X$ is the algebraic analogue of $\mathrm{Lie}(\Pi_X)$

Reason: {Representations of $\mathcal{P}_X$} = Bun_∇

Sections of $\mathcal{P}_X$ =: noncommutative vector fields

- derivations of ring category $\mathrm{Bun}_\nabla, \otimes$

or rather of the forgetful functor

$\mathrm{Bun}_\nabla \longrightarrow \mathrm{Bun}$ vector bundles

Such a derivation $\mathcal{D}$ is a collection

$\{\mathcal{D}_{E,\nabla} : E \longrightarrow E\}$ for all E, ∇, natural in E

& $\mathcal{D}_{E\otimes E', \nabla\otimes\nabla'} = \mathcal{D}_{E,\nabla}\otimes 1 + 1 \otimes \mathcal{D}_{E',\nabla'}$

Examples 1. v usual vector field $(\mathcal{D}_v)_{E,\nabla} = \nabla_v$

2. ξ bivector field $\Rightarrow \Phi_\xi$ acting by curvature

$(\Phi_\xi)_{E,\nabla} = (F_\nabla \lrcorner \xi) : E \longrightarrow E$

$[\mathcal{D}_v, \mathcal{D}_{v'}]_{\mathcal{P}} = \mathcal{D}_{[v,v']_{Lie}} + \Phi_{v\wedge v'}$.

Properties of $\mathcal{P}$:

1. Has a filtration by bracket order
$T_X = \mathcal{P}_X^{\leq 1} \subset \mathcal{P}_X^{\leq 2} \subset \ldots$ & gr $\mathcal{P}$ = free Lie algebra on T_X over $\mathcal{O}_X$
(bundle of fiberwise free Lie algebras)

2. $\mathring{\mathcal{P}}_X = \ker \mathcal{P}_X \longrightarrow T_X$ is a bundle of Lie algebras
with induced filtration $\underbrace{\{\Phi_\xi\}}_{} = \Lambda^2 T_X \twoheadrightarrow \mathring{\mathcal{P}}_X^{\leq 2} \subset \mathring{\mathcal{P}}_X^{\leq 3} \subset \ldots$

with gr $\mathring{\mathcal{P}}_X$ = free Lie algebra of T_X over $\mathcal{O}_X$
= $[\mathrm{gr}\,\mathcal{P}, \mathrm{gr}\,\mathcal{P}]$.

3. $\mathcal{P}(X,x)$ = fiber at x of $\mathring{\mathcal{P}}_X$ =: fundamental Lie algebra of x
= Lie $\Omega(X,x)$ unparametrized loops

$X/\mathbb{C}$ complex algebraic
Choice of (formal) coordinate system $x_1, \ldots x_n$ near x identifies $\mathcal{P}(X,x)$ with $[FL(\mathbb{C}^n), FL(\mathbb{C}^n)]$
- identification comes from choice of coordinate
$\Rightarrow$ Aut $\mathbb{C}[[x_1, \ldots x_n]]$ acts by automorphisms.

It is free as an abstract Lie algebra.

4. $H^1_{Lie}(P(X,x), \mathbb{C}) = \hat{\Omega}^{2,cl}_X$ closed 2-form on formal nbhd

5. $P(X,x)$ is a receptacle for higher covariant derivatives of curvature
$F_\nabla \in \Omega^2_X \otimes \mathrm{End}\, E$ ... this space doesn't carry a connection unless we put a Riemannian connection on X

∇F wants to lie in kernel of
$\mathrm{Alt}: \Omega^1 \otimes \Omega^2 \otimes \mathrm{End}\, E \longrightarrow \Omega^3 \otimes \mathrm{End}\, E$
which is $FL_3(\Omega^1_X) \otimes \mathrm{End}\, E$ --
really lives in our fundamental Lie algebra instead.

6. Finite dim representations of $P(X,x)$ are equivalent (as $\otimes$ category) to $\mathrm{Bun}_\nabla(\hat{D}_x)$ formal disc at x.

$P(X,x)$ free $\Longrightarrow$ $\mathrm{Bun}_\nabla(\hat{D}_x)$ is described by arbitrary assignments to generators which are $H_1^{Lie}(P(X,x), \mathbb{C})$

$\Omega^{2,cl}(\mathbb{C}^n) = \bigoplus_k$ (Young diagram with k boxes in the first row) as GL_n rep

$\Rightarrow$ get generators $[z_{i_1}, [z_{i_2}, \dots [z_{i_{p-1}}, z_{i_p}] \dots]$
$i_1 \geq \dots \geq i_{p-1} < i_p$

$\rightsquigarrow$ Taylor formula for connections:

7. (E, ∇) $\mathfrak{g}$-connection on $\hat{D}_n$ is uniquely determined by $[\nabla_{i_1}, [\nabla_{i_2}, \dots [\nabla_{i_{p-2}}, F_{i_{p-1}, i_p}] \dots] \in \mathfrak{g}$ arbitrary. up to conjugation

For any Lie algebroid there is an enveloping algebra $U(\mathfrak{g})$ eg $U(T_X) = \mathcal{D}_X$ sheaf of diff ops

$U(P_X) =: \mathbb{D}_X$ NC differential operators — act as natural differential operators

Ex. (X, g) Riemannian. $(E, \nabla) \Rightarrow \Delta_E : E \to E$ Laplace operator.

In coordinates $x_1, \dots x_n, D_1, \dots D_n$ $[D_i, x_j] = \delta_{ij}$
$D_1, \dots, D_n$ form a free algebra among themselves:

$\mathcal{D}_X$ has filtration with graded the tensor algebra on T_X.

Formal integration of $\mathcal{T}_X$:

any Lie algebra integrates to $e^{\mathfrak{g}}$ formal group

$$e^{\mathfrak{g}}(\Lambda) = (U(\mathfrak{g}) \otimes \Lambda)_{\text{grouplike}} \quad \Lambda \text{ nilpotent}$$

$\mathfrak{g}$ Lie algebroid $\rightsquigarrow e^{\mathfrak{g}} = \hat{G} \rightrightarrows X$

formal groupoid

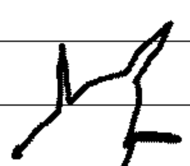
Apply to $\mathcal{T}_X$: $\hat{\Pi}_X = e^{\mathcal{T}_X}$ groupoid of formal unparametrized paths:

"formal nbhd of $X \xhookrightarrow{\Delta} \Pi_X$"

Classifying space of Π_X $\{B_n \Pi_X\}$ simplicial set

$$a_0 \xrightarrow{\gamma_1} a_1 \xrightarrow{\gamma_n} \cdots a_n$$

Simplices here are maps from a tree to X



$\mathbb{R}$ analog of moduli of stable maps: space of unparametrized curves

In fact can match formal neighborhoods
in Kontsevich space to $B_n \widehat{\Pi}_X$

Can also consider "NC $\mathcal{D}$-modules"

e.g local cohomologies of a vector bundle
with connection along a subvariety)