M. Khovanov - Matrix factorizations & link homology

Note Title 11/29/2007

(w. Lev Rozansky)

HOMFLY(PT) polynomial:

determined (up to scale) by

$$q^n P(L_+) - q^{-n} P(L_-) = (q-q^{-1}) P(L_0)$$

$n \in \mathbb{N}$: $P(L \sqcup \bigcirc) = [n] P(L)$

$$P(\bigcirc) = [n] = \frac{q^n - q^{-n}}{q-q^{-1}} = q^{n-1} + \dots + q^{1-n}$$

$\Rightarrow$ normalize $P(\emptyset) = 1$.

Related to reps of $U_q(sl_n)$ $n>0$

... Chern-Simons on links in S^3

colored by the n-dim rep of $U_q sl_n$

$P(L) \in \mathbb{Z}[q, q^{-1}]$

$n=0$: Alexander polynomial ($gl(1|1)$)

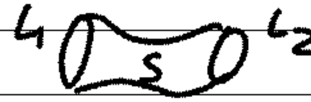
$n=1$: trivial

$n=2$: Jones

Upgrade $P(L)$ to a bigraded homology theory $H^{i,j}(L)$ fin dim $\mathbb{Q}$ vector spaces each i, j

$$\sum (-1)^i q^j \dim H^{i,j}(L) = P(L)$$

$n > 0$. ($n=0$: Oszvath-Szabo-Rasmussen knot Floer homology)

Should be functorial in link cobordisms up to isotopy: 

$S \subset S^3 \times [0,1]$

$\Rightarrow H(L_1) \xrightarrow{H(S)} H(L_2)$

Trivial link with k components $\Rightarrow$ 2d TFT

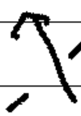
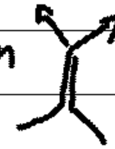
$\Rightarrow$ commutative Frobenius algebra with Poincaré polynomial $q^{1-n} + q^{3-n} + \dots + q^{n-1}$:

take $H(\bigcirc) = \mathbb{Q}[x]/x^n$ deg $x=2$

$1 \quad x \quad \dots \quad x^{n-1}$ basis

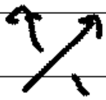
shift to balance around zero

... assume $i=0$ only: simplest possible bigraded Frobenius algebra with desired properties

 $= q^{1-n}$)($- \, q^{-n}$    double edges:

expand allowed diagrams: $\longrightarrow \iff$ fundamental rep V of sl_n, $= = \Lambda^2 V$

 : $V^{\otimes 2} \longrightarrow \Lambda^2 V \longrightarrow V^{\otimes 2}$, composite $\cdot [2]$

 $= q^{n-1}$)($- \, q^{n}$ 

$\rightsquigarrow$ get polynomial assigned to planar graphs with skein relation  $= [2]\,\|$,  $= [n]$

 $= [n-1]$  , & two more relations (version of Reidemeister moves)

For any such closed planar graph Γ, find that $P(\Gamma) \in \mathbb{N}[q, q^{-1}]$

... follows from state-sum formula.

— so can hope homology $H(\Gamma)$ can be only singly graded $\bigoplus_{j \in \mathbb{Z}} H^{0,j}(\Gamma)$

Matrix factorizations

$f \in \mathbb{Q}[x_1, \dots x_n] = R \quad f \neq 0$ .. should be sufficiently generic

Two free R-modules & maps

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^0, \quad d^2 = f$$

matrix factorization of f : if choose bases $R^k \xrightarrow[A]{} R^k \xrightarrow[B]{} R^k$

$A \cdot B = f \cdot \mathrm{Id}$ [find that M^0, M^1 have same rank since $f \neq 0$]

$\Longrightarrow$ modding out by homotopies get triangulated category of matrix factorizations.

(Knörrer-Buchweitz, Landau-Ginzburg models)

 $\longleftrightarrow \mathbb{Q}[x]/x^n$. Get from matrix factorizations:

$$Y \longrightarrow X \qquad f = x^{n+1} - y^{n+1} = (x-y)(x^n - \dots - y^n)$$

$$R = \mathbb{Q}[x, y]$$

$\Rightarrow$ matrix factorization

$$R \xrightarrow{\sum x^i y^{n-i}} R \xrightarrow{x-y} R$$

Close up arc ie set $x = y \Rightarrow$

$$M(\alpha) = \left\{ \mathbb{Q}[x] \xrightarrow{(n+1)x^n} \mathbb{Q}[x] \xrightarrow{0} \mathbb{Q}[x] \right\}$$

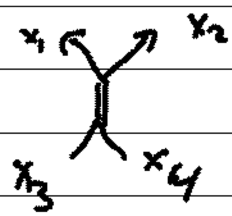
$$H^*(\text{''}) = \quad 0 \qquad \mathbb{Q}[x]/x^n \qquad 0$$

$x \xrightarrow{\alpha} y$ plays role of identity functor:

given factorization

$$\xrightarrow{z} (M) \xrightarrow{y} \otimes \; y \xrightarrow{\alpha} x$$

$g = y^{n+1} + h(z) \Rightarrow M \otimes M(\alpha) \cong M$

Given graph  $f = x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$

Find $f = (x_1 + x_2 - x_3 - x_4)\, u_1 + (x_1 x_2 - x_3 x_4)\, u_2$ $\quad u_1, u_2$ polynomials

Since two parts of f ($+$ & $-$)
are of form $g(x_1+x_2, x_1x_2)$ &
$g(x_3+x_4, x_3x_4)$ so lies in
above ideal.

$R = \mathbb{Q}[x_1, \dots x_4]$

tensor of complexes
$$\left\{\begin{array}{c} R \xrightarrow{x_1+x_2-x_3-x_4} R \xrightarrow{u_1} R \\ \otimes_R \\ R \xrightarrow{x_1x_2-x_3x_4} R \xrightarrow{u_2} R \end{array}\right. = M(e)$$

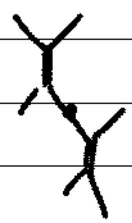
e = double edge

$D^2 = f$.. This is a cyclic version of
the Koszul complex: R generated by
$a_1, \dots a_m \Rightarrow$ take $\bigotimes_{i=1}^{n} (0 \to R \xrightarrow{a_i} R \to 0)$
& if a_i regular sequence this is exact.

We're doing a cyclic version: given (a_i, b_i)
take $\bigotimes_{i=1}^{m} (R \xrightarrow[a_i]{} R \xrightarrow[b_i]{} R)$ of $f = \sum a_i b_i$

To a graph Γ assign $\underbrace{\bigotimes_e M(e)}_{\text{double edges}} \underbrace{\bigotimes_\alpha M(\alpha)}_{\text{usual edges}}$

- variables correspond to markings on the graph.

e.g.  tensor $M(e) \underset{\text{shared variable}}{\otimes} M(e)$

$$D^2 = \sum_e f_e = 0 : \text{ 2-periodic complex!}$$

Give every x degree 2 $\Rightarrow$ $\deg d^2 = 2(n+1)$

$\deg d = n+1$

$\rightsquigarrow$ have $\mathbb{Z}$-grading on the terms.

<u>Theorem</u> • $\forall \Gamma$: $H^1(\Gamma)$ or $H^0(\Gamma)$ vanishes

(analog of regular sequence)

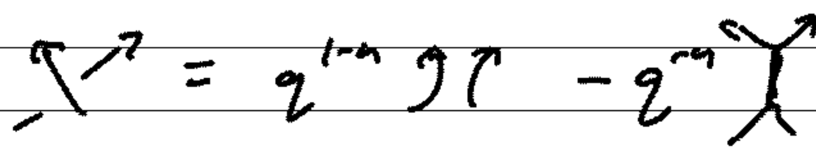
$H(\Gamma) :=$ nonzero cohomology.

• graded dim $H(\Gamma) = P(\Gamma)$.

categorification of the graph polynomial.

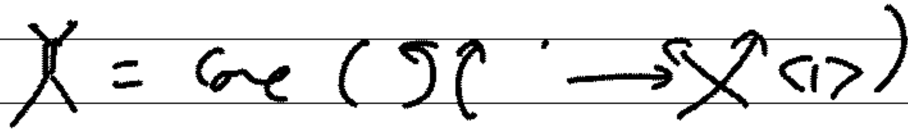
Proof - lift skein relations to matrix factorizations.

To lift to links: recall



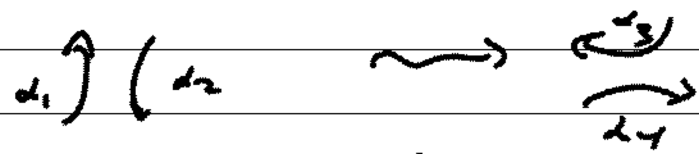
So need a map of the associated factorizations to take cone...

But



$(\langle 2 \rangle = [1])$

comes from simplest cobordism

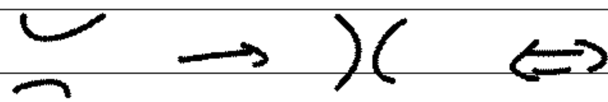


$M(\alpha_1)\otimes M(\alpha_2) \longrightarrow M(\alpha_3)\otimes M(\alpha_4)$

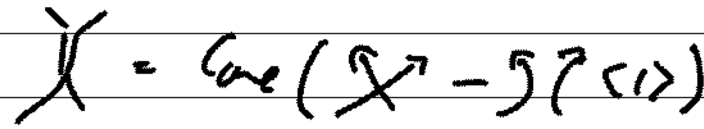
... comes from 0,1,2 dim TFT

pt $\longmapsto$ matrix factorizations of $f = \sum \pm x_i^{n+1}$

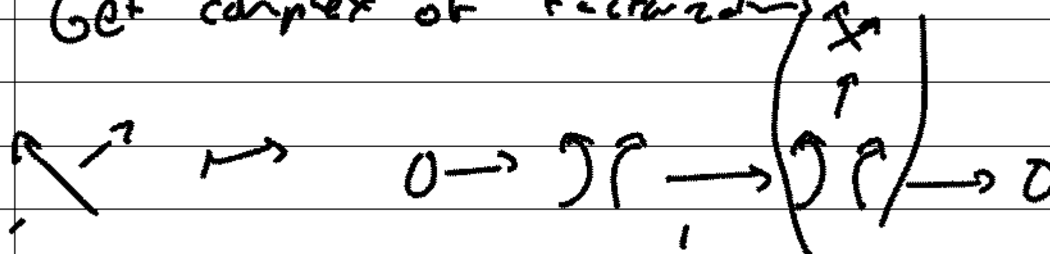
& above is part of this structure.



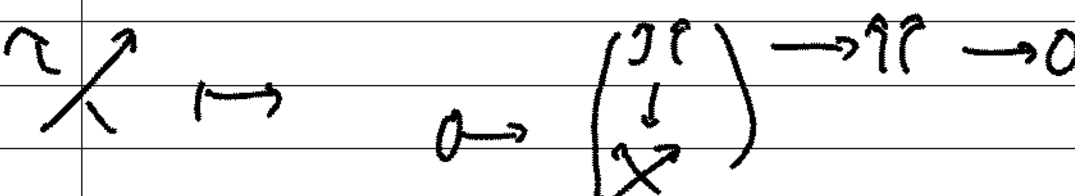
$$\begin{tikzcd} R \arrow[r, "a"] \arrow[d, "1"] & R \arrow[r, "bc"] \arrow[d, "b"] & R \arrow[d, "1"] \\ R \arrow[r, "ab"'] & R \arrow[r, "c"'] & R \end{tikzcd}$$

Also 

Get complex of factorizations



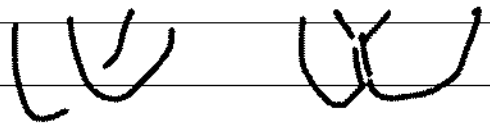
objects of HMF_f triangulated category, above lives in complexes of objects in HMF_f ...



Tensor together to get a matrix factorization of a potential depending on the ends of the tangle & a <u>complex</u> for a link

-- all homologies live only in single degrees.

Can avoid matrix factorizations by working just with the subcategory of pictures



Sussan: use parabolic-singular blocks for sl_N all N.

Can use Fukaya categories on quiver varieties, coherent sheaves on affine Grassmannian . . . --
4 or 5 conjectural approaches.