M. Liu - Instanton Counting on toric surfaces

Note Title 11/29/2007

Nakajima's talk explained a relation

SU_2 instanton counting on $\mathbb{R}^4 = \mathbb{C}^2$
(K-theoretic version)

$\Updownarrow$

GW invariants of certain toric CY3s.

S projective smooth toric surface
K_S = total space of the canonical line bundle
For above application want
$S = \mathbb{F}_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$ Hirzebruch surface

We'll discuss GW invariants of K_S for
S a projective smooth toric surface $/ \mathbb{C}$
& the HOMFLY polynomial of the Hopf link
— arises in Chern-Simons theory on S^3.

<u>HOMFLY polynomial of Hopf link</u>

$W_{R_1, R_2}(q, \lambda)$ R_1, R_2 irreps of $U(N)$
$(N \gg 0)$
$\Longleftrightarrow$
q, λ numbers Young diagrams or partitions

Leading terms in $\lambda \rightsquigarrow$ function

$W_{\mu\nu}(q)$ $\quad \mu, \nu$ partitions

$$= (-1)^{|\mu|+|\nu|} q^{\frac{\kappa_\mu}{2} + \frac{\kappa_\nu}{2}} \sum_\eta S_{\mu/\eta}(q^{-\rho}) S_{\nu/\eta}(q^{-\rho})$$

$|\mu| = \sum \mu_i$, $l(\mu) =$ number of $\mu_i = l$

$\mu = \mu_1 \geq \cdots \geq \mu_l$

$\kappa_\mu = \sum \mu_i(\mu_i - 2i + 1)$

$S_{\mu/\eta}$ skew Schur function, we're plugging in $(q^{\frac{1}{2}}, q^{3/2}, q^{5/2}, \ldots\ldots)$

GW invariants of K_S:

Given $g \geq 0$ integer, $0 \neq \beta \in H_2(S, \mathbb{Z}) = H_2(K_S, \mathbb{Z})$

$\overline{\mathcal{M}}_{g,0}(K_S, \beta)$ genus g degree β stable maps into K_S

- not necessarily compact,

but will be compact e.g. for S Fano & $\beta \neq 0$ $\Rightarrow$ can define

$$N_{g,\beta}^{K_S} = \int_{[\overline{\mathcal{M}}_{g,0}(K_S,\beta)]^{vir}} 1 \in \mathbb{Q}$$

$[\ ]^{vir} \in H_0(\overline{M}_{g,0}(K_S,\beta);\mathbb{Q})$,

$1 \in H^0(\overline{M}_{g,0}(K_S,\beta);\mathbb{Q})$.

Compute by virtual localization for
$T=(\mathbb{C}^*)^2 \curvearrowright K_S$
$\curvearrowright S$ $\rightsquigarrow$ (Graber-Pandharipande virtual localization)

$$N_{g,\beta} = \int_{[\overline{M}_{g,0}(K_S,\beta)^T]^{vir}} \frac{1}{e_T(N^{vir})} \in \mathbb{Q}\left(\frac{u_2}{u_1}\right)$$

- makes sense even when the moduli space is noncompact since the T fixed points are always compact $\Rightarrow$ gives definition of $N_{g,\beta}$

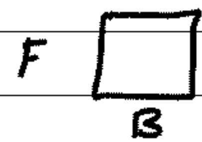
$H^*_T(pt) = \mathbb{Q}[u_1,u_2]$

$$Z^{K_S}_{GW}(\lambda;Q) := \exp\Big(\sum_{\substack{\beta\in H_2(S,\mathbb{Z})\\ \neq 0}} \sum_{g\geq 0} \lambda^{2g-2} N^{K_S}_{g,\beta}\, Q^\beta\Big)$$

Aganagic-Mariño-Vafa: propose explicit expression for Z (using large N duality) via $W_{\mu\nu}(q)$

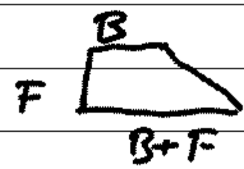
Example $S = \mathbb{F}_m$

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$

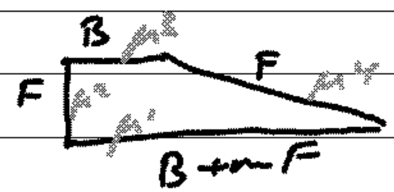


moment polytope pictures

$\mathbb{F}_1$



$\mathbb{F}_m$



$q = e^{i\lambda}$

$$Z^{K_{\mathbb{F}_m}}_{GW}(\lambda, Q_B, Q_F)$$

$$= \sum_{\mu^1\mu^2\mu^3\mu^4} W_{\mu^1\mu^4}(q)\, W_{\mu^4\mu^3}(q)\, W_{\mu^3\mu^2}(q)\, W_{\mu^2\mu^1}(q)$$

$$\cdot\ (-1)^{m|\mu^4| + m|\mu^3|}\, q^{m\kappa_\mu(|\mu^1| - |\mu^3|)}$$

$$\cdot\ Q^{|\mu^1| + |\mu^3|}\, Q_F^{|\mu^2| + |\mu^4| + m|\mu^1|}$$

Want to prove this mathematically!

This formula for $Z^{K_S}_{GW}$ is a key ingredient in the proof of the GW – DT correspondence for K_S

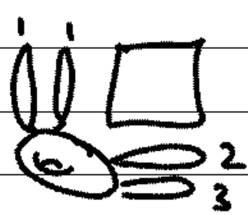
--- conjectured for any projective 3-fold by MNOP. Proved for smooth toric 3-fold MOOP. Local curves BP, OP

Aganagic Mariño Vafa algorithm $\Rightarrow N_{g,\mu}^{K_S}$ independent of u_1, u_2

<u>Step 1</u> Localization

$$N_{g,\mu}^{K_S} = \int_{[\overline{M}^T]^{vir}} \frac{1}{e_T(N^{vir})} \in \mathbb{Q}\left(\frac{u_2}{u_1}\right)$$

What are the T fixed points?



$$\in \overline{\mathcal{M}}_{2,0}(K_{\mathbb{F}_0}, 5B+2F)^T$$

$$\updownarrow$$

$$\overline{\mathcal{M}}_{g,n}$$

fixed points are (up to a finite group) unions of products of $\overline{\mathcal{M}}_{g,n}$.

Contributions: 2-partition Hodge integrals

$g \in \mathbb{Z}_{\geq 0}$ μ^+, μ^- partitions $\ell^{\pm} = \ell(\mu^{\pm})$

[$\tau = \frac{u_2}{u_1}$ formal variable]

$$G_{g,\mu^+,\mu^-}(\tau) = \frac{-(\sqrt{-1})^{l_+ + l_-}}{|\mathrm{Aut}\,\mu^+|\,|\mathrm{Aut}\,\mu^-|} \cdot$$

$$\prod_{i=1}^{l_+} \frac{\prod_{a=1}^{\mu_i^+ - 1}\left(\frac{\mu_i^+}{2} + a\right)}{(\mu_i^+ - 1)!} \prod_{j=1}^{l_-} \frac{\left(\frac{\mu_j^-}{2} + a\right)}{(\mu_j^- - 1)!} \left(\tau(\tau+1)\right)^{l^+ + l^- - 1}$$

$$\cdot \int_{\overline{\mathcal{M}}_{g,l^+ + l^-}} \frac{\Lambda_g^\vee(1)\,\Lambda_g^\vee(\tau)\,\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l_+}(1 - \mu_i^+ \psi_i) \prod_{j=1}^{l_-} \tau(\tau - \mu_j \psi_{l^+ + j})}$$

where $\psi_i = c_1(L_i)$ ψ classes

$\lambda_j = c_j(E)$ Hodge bundle

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \dots + (-1)^g \lambda_g.$$

$\rightsquigarrow$ Hodge integrals $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_1^{j_1} \cdots \lambda_g^{j_g} \in \mathbb{Q}$

Generating series for them:

$$G_{\mu^+,\mu^-}(\lambda,\tau) = \sum \lambda^{2g-2+l^+ + l^-} G_{g,\mu^+,\mu^-}(\tau)$$

By exponentiation create a disconnected version of this sum. Can write $Z_{GW}^{K_3}$ in terms of $G_{\mu^+\mu^-}(\lambda, \tau)$.

So need to relate $G_{\mu\mu}$ to HOMFLY

<u>Theorem</u> (Liu Liu Zhou)

(*) $G_{\mu^+\mu^-}(\lambda, \overline{\tau}) = R_{\mu^+\mu^-}(\lambda, \tau)$

where
$$R_{\mu^+\mu^-}(\lambda, \tau) = \sum_{|\nu^\pm| = |\mu^\pm|} \frac{\chi_{\nu^+}(\mu^+)}{z_{\mu^+}} \frac{\chi_{\nu^-}(\mu^-)}{z_{\mu^-}}$$
$$\cdot\ q^{\frac{k_{\nu^+}}{2}\tau + \frac{k_{\nu^-}}{2}\tau - 1} \cdot W_{\nu^+\nu^-}(q)$$

& for a partition μ, $z_\mu = \mu_1 \cdots \mu_\ell \cdot |\mathrm{Aut}\,\mu|$

& $\chi_\nu(\mu)$ = character associated to ν evaluated at conjugacy class μ.

Special case: $\tau = -1$

$G_{g,\mu^+\mu^-}(-1) = 0$ unless $l_+ + l_- - 1 = 0$

$G_{g,(d),\emptyset}(-1) = G_{g,\emptyset,(d)}(-1) = \sqrt{-1}(-1)^d d^{2g-2} b_g$

& $b_g = \begin{cases} 1 & g=0 \\ \int \lambda_g \psi^{2g-2} & g>0 \end{cases}$

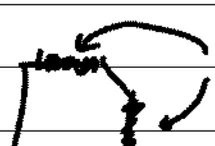
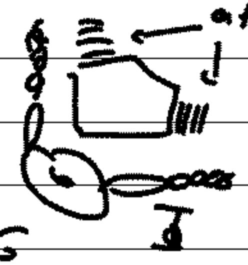
---> get the theorem from results of Faber-Pandharipande

Relate different cases using disconnected double Hurwitz numbers Φ

For R have a convolution equation determining it:

$R(\tau) = e^{A(\tau-\tau_0)} R(\tau_0) e^{A(\frac{1}{\tau}-\frac{1}{\tau_0})}$.

- so prove same for G! use relative GW invariants $K_{\mu^+\mu^-}(\lambda)$ of $X = Bl_{0\infty,\infty 0}\,\mathbb{P}^1\times\mathbb{P}^1$

 relate to these divisors Φ 

all the $\mathbb{P}^1$'s

& localization $\Rightarrow$ $K \longleftrightarrow G\cdot\Phi \ldots$