J. Maldacena - Integrability in N=4 SYM

Note Title 11/27/2007

'70s : Large N limit of SU(N) gauge theory simplifies in a certain sense.

Lagrangian $S = \frac{1}{g^2} \int d^4x \, Tr[F_{\mu\nu} F^{\mu\nu}]$

⇒ we're considering the limit where

$g \to 0$ & $g^2 N = \lambda$ fixed

⟿ theory simplifies & looks like a string theory: see only planar Feynman diagrams  → discretized surfaces.

This is a 2d system (living on string worldsheet).

2d systems often exhibit integrability:

Integrability: useful for solving discrete spin chains

$\uparrow \downarrow \uparrow \uparrow$

or in 1+1 dim QFT, scattering of solitons.

Integrability for us will mean a large number of conserved charges, allowing us to solve the system

Integrability appears in N=4 SYM for SU(N), large N limit - single parameter λ.

$\mathcal{N}=4$ SYM
$SU(N)$ $\longleftrightarrow$ Strings on $AdS_5 \times S_5$

ie in this particular case we know <u>which</u> string theory the large N limit corresponds to

Parameter matching : $\lambda^{\frac{1}{4}} = R/l_s$
where $R = R_{AdS} = R_S$ radius of curvature of AdS = radius of sphere.

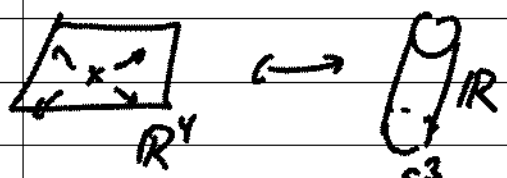
(AdS_5: Lorentzian version of $\mathbb{H}_5$)

Two regimes: $\lambda \ll 1$ YM is weakly coupled, use standard perturbative techniques.

$\lambda \gg 1$ string moving on space w/ very large radius of curvature $\rightsquigarrow$ weakly coupled 2d worldsheet theory

Want to interpolate between these regimes!
What do we want to compute?
- dimensions of operators in this CFT
- correlation functions (eventually...)

Scaling dimensions of operators.



Energy of state on S^3
$\longleftrightarrow$ anomalous dim. of operator on $\mathbb{R}^4$

Fields in $N=4$: A_μ gauge fields,

$SO_6 \circlearrowright$ $\varphi_1, \dots \varphi_6$ scalars, $\psi \dots$ fermions.

Let $Z = \varphi_1 + i\varphi_2$ complex scalar, with charge $J = J_{12} = 1$.

The operators we're considering are of form
$\mathrm{Tr}[Z^J](0)$

SUSY $\Rightarrow$ dimension is independent of coupling (this is a BPS operator)
— so find same kind of operator for large λ, in the string theory.

$\leftrightarrow J_{34}$ Let $W = \varphi_3 + i\varphi_4$ $\Longrightarrow$ new operators

$\mathrm{Tr}\,[Z[W,Z]]$: dimension $\Delta = 3 + \# \cdot \lambda$

— becomes $\gtrsim \lambda^{\frac{1}{4}}$ for large λ
(ie Δ depends on coupling).

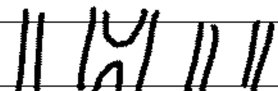
In the planar limit only <u>single trace</u> operators will contribute. Can write any such in form

$$\mathrm{Tr}\,[\,Z\,Y\,F\,D^{m}\phi\,D^{n}Z\dots\,]$$

ie trace of combination of fields & derivatives
— depends on order of operators up to cyclic permutation.

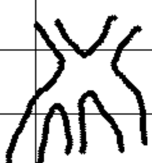
Only operators with same dimensions mix: diagonalize wrt dilatation operator D (acts in block form) — only fin.
Many operators with given classical dimension: Operator D is a matrix with fin dim block form.

Planarity restricts interactions to be among near neighbours: to n^{th} order in λ see only distance n interactions

e.g.  $\longleftrightarrow$ first order in λ interaction

$\rightsquigarrow$ get spin chain where at each site have ∞ dim space of states: can put any of the fields & their derivatives.

Number of sites is <u>not</u> conserved under interactions: e.g. could take



But consider a special class of operators where Hamiltonian acts in a simpler way.
We'll consider only fields made out of W, Z: e.g

$\mathrm{Tr}\,[ZWZZW]$ Relation $\Delta = J_{12} + J_{45}$ holds classically, quantumly $\Delta = \Delta(\lambda) \geq J_{12} + J_{45}$

$SU(2)$ symmetry rotating Z, W (part of $SO6$ symmetry of $N=4$)

$\Rightarrow$ an $SU(2)$ spin chain

$$\underset{Z}{\uparrow}\ \underset{W}{\downarrow}\ \underset{Z}{\uparrow}\underset{Z}{\uparrow}\ \underset{W}{\downarrow}$$

$\mathcal{H} = \Delta - J_{12} - J_{34}$ SU(2) invariant matrix. To leading order in λ can only mix nearest neighbors $\Rightarrow$ must be of form

$$\mathcal{H} = * \lambda \left(\sum_{i}^{k} \vec{\sigma}_i \vec{\sigma}_{i+1} + \text{const} \right)$$

$$= * \lambda \left(\sum_{i}^{k} P_{i,i+1} + \text{const} \right)$$

$P_{i,i+1} = \diagup\!\!\!\diagdown$.

$*$ = some fixed coefficient that can be determined

Existence of BPS states of form

$\sum_{l} \mathrm{Tr}\left[Z^l W Z^{k-l}\right]$ fixes the constant above to be -1 : $\lambda(\Sigma P - 1)$

– relates by SU(2) to BPS operator $\mathrm{Tr}\left[Z^J\right]$ so we know its dimension independent of λ.

Let's diagonalize $\mathcal{H}$: problem invariant under discrete translations, diagonalize in momentum basis

$$\sum_{l} \mathrm{Tr}\left[Z^l W Z^{k-l}\right] e^{ipl} \quad \boxed{=0}$$

Cyclicity of trace $\Rightarrow$ invariant under overall translation $\Rightarrow$ total momentum $= 0$.

But have states of form

$$\sum_l \mathrm{Tr}\left[Z^{l_1} W Z^{l_2-l_1} W Z^{l_3-l_2} \cdots\right] e^{i(p_1 l_1 + p_2 l_2 \cdots)}$$

where $\sum p_i = 0$ (mod 2π)

i.e. $\sum_l [Z Z Z Z \overset{l}{W} \overset{p}{\rightarrow} Z Z Z \ldots] e^{ipl}$

W particle moving with momentum p & spin l

$$H \cdot |p\rangle = -\lambda(e^{ip} + e^{-ip} - 2)|p\rangle$$
$$= \lambda \, 4 \sin^2 p/2$$

$= \mathcal{E}(p) =$ energy of particle with momentum p.

- dispersion relation

Can continue this analysis to higher order in λ using translation symmetry of the problem' dispersion relation $\mathcal{E}(p)$ must be 2π periodic in p.

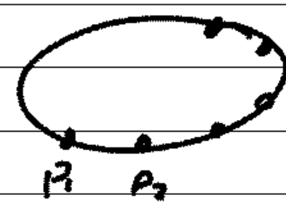
$$Z Z Z Z Z \overset{\xrightarrow{p_1}}{W} Z Z Z Z Z Z \overset{\overset{p_2}{\leftrightarrow}}{W} Z Z Z Z Z$$

when far apart find $\mathcal{E} = \mathcal{E}(p_1) + \mathcal{E}(p_2)$

but gets corrected when the W's are close by.

$$\psi \sim \begin{cases} e^{ip_1 l_1 + ip_2 l_2} & l_1 < l_2 \\ S(p_1,p_2)\, e^{ip_1 l_1 + ip_2 l_2} & l_1 > l_2 \end{cases}$$

Finite spin chain
of length L



Satisfies Bethe equation expressing periodicity
of wave function

$$1 = e^{i p_i L} \prod_{j \neq i} S(p_j, p_i)$$

S = phase factor above ... here we're
using integrability in strong ways,
to determine n-particle scattering from
2-particle scattering!

Problems in higher order:

- Do not know the Hamiltonian
- Conjecture integrability.

evidence: full 1-loop Hamiltonian is integrable,
some higher loop calculations integrable.
Large λ get classical σ-model which

is indeed integrable. Then hope it's integrable to all orders!

- Need dispersion relation $\mathcal{E}(p)$ & 2-particle S-matrix, that's all we really need.

Consider now limit of an infinite chain. What are the symmetries of the problem?

- $SU(2,2|4)$ conformal group of $N=4$ SYM

- subgroup preserving lowest excitations $PSU(2|2)^2 \ltimes \mathbb{R}$

- Supercharges $[Q,Q] = Q(u)(1-e^{-ip})e^{ix}$
$=: X = X_1 + iX_2 \in \mathbb{C}$
[on momentum p state]

$\rightsquigarrow$ find two extra central charges
$PSU(2|2)^2 \ltimes \mathbb{R} \ltimes \mathbb{R}^2$
See evidence for Hopf algebra structure.....

Another simplification: can describe all states as bound states of simplest low energy BPS representation, made of 8 bosons & 8 fermions

~ symmetries completely constrain the dispersion relation, $E(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 p/2}$

Scattering: have 16 states for each particle

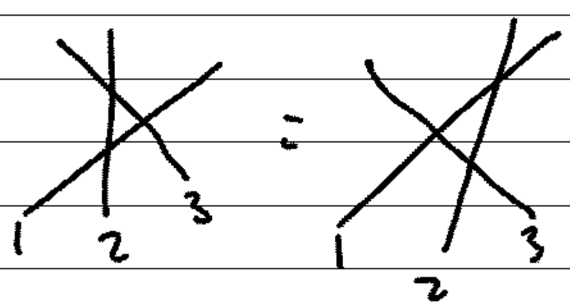
before $\quad \underset{A=1\ldots16}{\xrightarrow{p_1}} \qquad \underset{B=1\ldots16}{\xleftarrow{p_2}} \qquad \Rightarrow \; S_{AB}^{\;\;CD}$

after $\quad \underset{C}{\xleftarrow{p_2}} \qquad \underset{D}{\xrightarrow{p_1}} \qquad$ scattering matrix

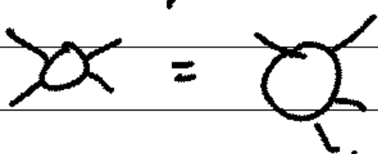
$$S_{AB}^{\;\;CD} = M_{AB}^{\;\;CD}\, S_0(p_1, p_2, \lambda)$$

M given matrix which is <u>known</u>!

— satisfies Yang-Baxter / star-triangle!



- Relativistic theory: use crossing symmetry –
outgoing particle with negative energy
= incoming particle with positive energy
... analytic continuation of amplitudes:



⤳ implies constraints on phase factor S_0
– enough to guess a solution.

This is an assumption: crossing symmetry
holds in general for <u>continuous</u> not
<u>discrete</u> systems...