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Note Title 11/29/2007

Moore-Nekrasov-Shatashvili '97: Higgs bundle localization

From 4d perspective: $N=2^*$ theory
($N=4$ with masses for adjoints), topologically twisted (Donaldson twist)

Index $Tr(-1)^F e^{-\beta H}$: only vacuum contributes
$= Tr(-1)^F$ in SUSY theory

Perturb by special operators:
$Tr(-1)^F e^{-t_i O^i} = Tr\, e^{-t_i H^i}$: equivalent to a quantum mechanical system in a topological field theory: quantum mechanics of the vacuum.

Study theory on $\Sigma \times \mathbb{CP}^1$ $\Sigma = S^1 \times \mathbb{R}$
take $\mathbb{P}^1$ to zero size, get 2d QFT on Σ
$\rightsquigarrow$ 2d gauge theory:
Yang-Mills-Higgs theory, similar to 2d YM

This manifold has $b_2^+ = 1$: have

wall crossing phenomena: limit with $\mathbb{P}^1$ big
e.g. not accessible from limit with
$\mathbb{P}^1$ small, pass walls in between ...

Hitchin equations: $F_{z\bar{z}}(A) + [\phi_z, \phi_{\bar{z}}] = 0$

$\nabla_z(A)\,\phi_{\bar{z}} = 0$

A, ϕ: 1-forms in $\mathfrak{g}$

$\varphi_0, \varphi_\pm$ 0-forms in $\mathfrak{g}$

[we'll discuss only G = unitary group]

Lagrangian $\mathcal{L} = \mathrm{Tr}\, \varphi_0 (F(A) - \phi\wedge\phi)$

$+ \mathrm{Tr}\, \varphi_+ \nabla_z(A)\phi_{\bar{z}} +$

$+ \mathrm{Tr}\, \varphi_- \nabla_{\bar{z}}(A)\phi_z \quad - \underbrace{C\phi_z\phi_{\bar{z}}}_{\text{gives mass to } \phi}$

+ fermions

[without ϕ get 2d topological YM...]

BRST transformations determining fermions:

$$\delta A = \psi_A \qquad \delta \phi_z = \psi_z$$

$$\delta \psi_A = \nabla_A \varphi_0 \qquad \delta \psi_{\substack{z \\ \bar{z}}} = [\varphi_0, \phi_{\substack{z \\ \bar{z}}}] \pm c \phi_{\substack{z \\ \bar{z}}}$$

$\delta^2 = 0$ up to action of Lie algebra & circle action

$$\delta^2 = \mathcal{L}_{\varphi_0} + c\mathcal{L}_V$$

$\mathcal{L}_V$ = circle action rotating Higgs field

- computes intersection theory on moduli of Higgs bundles.

As $c \to \infty$ φ decouples (acquires ∞ mass)

$\Rightarrow$ 2d YM:

The operators $(\delta \mathcal{O}^i = 0)$ are just traces $\mathrm{tr}\, \varphi_0^k$. Can add these to Lagrangian

ans: $$\int e^{-\int \mathcal{L} + t_i \mathcal{O}^i} = \sum_R d_R^{2-2g} e^{-t_i C_i(R)}$$

C_i = Casimirs of representation R

As $c \longrightarrow 0$: flat connections for complex group, bad limit since reps are infinite dimensional.

$c \neq 0$: partition function calculated by abelianizing & then localizing:
on abelian fields (A, φ_0) $\varphi_0 = (\lambda_1, \ldots, \lambda_n)$ diagonal
get $$\int \sum \frac{\partial I(\lambda)}{\partial \lambda_i} F^i + \frac{\partial^2 I(\lambda)}{\partial \lambda_i \partial \lambda_j} \psi^i \wedge \psi^j$$

$$+ \int T(\lambda) \sqrt{g}\, R^{(2)}$$

... & compute what are $I(\lambda)$, $T(\lambda)$:

$$I(\lambda) = \sum_i \frac{1}{2} \lambda_i^2 - 2\pi \lambda_i \lambda_j$$

$$+ \sum_{ij}^{N} \int_0^{\lambda_i - \lambda_j} \arctan \frac{\lambda}{c}\, d\lambda$$: Yang's counting function

Let $\alpha_k(\lambda) = \frac{\partial I}{\partial \lambda_k}$

$\rightsquigarrow$ $I(\lambda)$ localizes to

$$P(\lambda) = \sum_{n_k} e^{i\alpha_k(\lambda) n_k} = (\quad)\cdot\delta(\lambda_i - \lambda_i^*)$$

$\Rightarrow$ Bethe ansatz for NLS

$$e^{2\pi\lambda_j} \prod_{i\neq j} \frac{\lambda_i - \lambda_j + i\sigma}{\lambda_i - \lambda_j - \sigma} = \mathbb{1}$$

$$Z_g(t) = \sum_\lambda D_\lambda^{2-2g} \; e^{-\sum t_k P_{2k}(\lambda)}$$

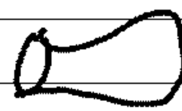
$$D(\lambda) = \mu^{\frac{1}{2}}(\lambda) \prod_{i<j} (\lambda_i - \lambda_j) \left(c^2 + (\lambda_i - \lambda_j)\right)^2$$

sqrt of Hessian $\quad$ Vandermonde $\quad$ shifted Vandermonde

As $c \to \infty$ get sum of reps of unitary group (2d YM)

As $c \to 0$ get sum over discrete series reps of complex group, of renormalized dimensions .....

c ←--→ equivariant parameter

2d YM:  disc partition function is character of irrep

What will we get at $c=0$ for these partition functions? function of abelian gauge fields (ie symmetric functions on torus)

$\psi_\lambda(x_1, \dots x_N)$ symmetric & invariant under $x_i \mapsto x_i + 1$

All local operators are same as in 2d YM

Phase space: T^*H/W $\quad$ H = real Cartan

At $c=0$ have extra operators, & need to look at <u>complexified</u> Cartan.

$c = \infty$: λ is highest weight, λ_i integers

$\psi_\lambda(x_1, \dots x_N)$ $\quad \lambda_1 \geq \dots \geq \lambda_N$

$c \neq \infty$ λ_i not integers but rather solutions of Bethe ansatz: but Yang shows these are exactly labelled still by partitions $p_1 \geq \dots \geq p_N$ of integers.

$c = \infty$ $H_i \psi_\lambda(x) = p_i(\lambda) \psi_\lambda(x)$ eigenfunctions

$$H_i = \sum \left(\frac{\partial}{\partial x_i}\right)^i$$

$c \neq \infty$ Yang-Mills-Higgs:

$$H_i \psi_\lambda(x) = p_i(\lambda) \psi_\lambda(x)$$

where H_i are deformed by c

-- discrete spectrum & multiplicity (still!

$$H_2 = \sum_1^N \left(\frac{\partial}{\partial x_i}\right)^2 + c \sum_j^N \delta(x_i - x_j)$$

second Hamiltonian ...

Dunkl operator $D_i = -i \frac{\partial}{\partial x_i} + \frac{c}{2} \sum_{j \neq i}^N \left(\delta(x_i - x_j) + 1\right) s_{ij}$

S_{ij} = switch i,j. Then $H_1 = \sum_i D_i^k$.

$$Z(t) = \sum D_\lambda^{2-2g}\, e^{-t_i C_i(\lambda)}$$

quantum integrable system, solved by Cherednik algebra

H's are center of deg. affine Hecke algebra.

Back to 4 dimensions

Gerasimov - Shatashvili, Nekrasov - Shatashvili.

$N=2^*$ theory, adjoint scalar with mass m

Low energy theory is abelian theory,

(Donagi - Witten) know low energy Lagrangian, prepotential etc, get integral over u plane...

Let's solve the theory in 4d & then compactify:

Donaldson theory for $b_2^+=1$;

On $\Sigma \times \mathbb{CP}^1$ (radius R_1, R_2) have walls

whenever R_1/R_2 rational!!

Seiberg-Witten prepotential

$\mathcal{F}(a_e, \tau; m)$ $\qquad \tau = \frac{i}{g^2} + \theta$ complex coupling

$\mathcal{F} = \mathcal{F}_{pert} + \mathcal{F}_{inst}$ perturbative + instanton

Bethe ansatz equation get corrections in τ
Perturbative part gives solution of Moore, Nekrasov, S. — but now gets corrected by instantons.

$$\frac{\partial}{\partial a_i} \frac{\partial \mathcal{F}}{\partial m} + \oint \frac{\partial U_2}{\partial x_i} = n_i$$

full Bethe ansatz equation

U_2 : first zero observable: $U_2 = \frac{\partial \mathcal{F}}{\partial \tau}$
$\mathcal{F}$ = prepotential

2d Yang-Mills-Higgs + $\sum a_n q^n$ instanton corrections
$q = e^{2\pi i \tau}$, all a_n known since prepotential is known. $\rightsquigarrow$
deformation of 2d system!

2d YM has another generalization, to G/G WZW: has parameter giving k <u>level</u> of Kac-Moody, $k \to \infty$ get YM.

$$Z = \sum_{\substack{\text{irreps} \\ \text{of quantum} \\ \text{group}}} \dim V_\mu^{2-2g} e^{(\dots)}$$

$$q = e^{2\pi i/k+h^\vee}$$

In a limit get complexified group G^c/G^c ...

Introduce this new parameter to Cherednik algebra ⟿ double affine Hecke?

$\{\lambda\}$ solutions for Bethe ansatz for XXZ chain with spin $\to \infty$

$$Z_g(t) = \sum D_\lambda^{2-2g} e^{-t_i ch_i(g)}$$ ← characters of quantum group

... quantizing complex Chern-Simons for $\Sigma \times S^1$ roughly, corrected with operator = generator of circle action

$t = \exp c \to 1$ get Yang Mills Higgs

$t \to \infty$ get Verlinde formula

$4d\ N=2^*$ $\xrightarrow{\Sigma \times \mathbb{P}^1}$ YMH

5d theory $\xrightarrow{\mathbb{CP}^1}$ this theory

Yangian (XXX)	YMH	(Hall-Littlewood)
XXZ	G/G	(Macdonald)
XYZ	6 dim theory: <u>elliptic version</u>	

each of these models are the quantum mechanics of the vacuum.

DAHA: Dunkls + permutation

Hamiltonians of our theory.

Raising & lowering operators: change the particle number.. (= rank of gauge group)

These equations are basically the WDVV equations of our TFT.