V. Toledano-Laredo : Stability conditions & Stokes factors

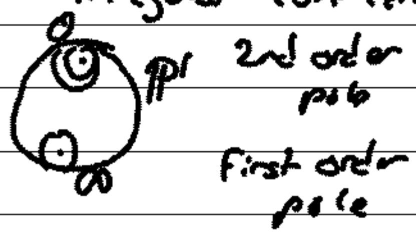
Note Title 11/30/2007

(with Tom Bridgeland)

Heroic work of Joyce:

try to "count" semistable coherent sheaves on a CY3 $\rightsquigarrow$ Donaldson-Thomas invariants

Gets wall-crossing formulas for Stab(Coh(X)) $\xrightarrow{\text{today}}$ Stokes phenomena for simplest irregular connections:



<u>Joyce's work</u>

$\mathcal{A}$ abelian category

$$K_0(\mathcal{A}) = \{ \sum_{M \in \mathcal{A}} n_M \cdot [M] \} / [B] = [A] + [C] \quad 0 \to A \to B \to C \to 0$$

$$\cup$$

cone $C(\mathcal{A}) = \{ \quad " \quad \cdot \; n_M \in \mathbb{N} \} / \cdots$

<u>Def</u> A stability condition on $\mathcal{A}$ is a homomorphism $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$

[central charge]

$$\begin{array}{ccc} K_0(\mathcal{A}) & \longrightarrow & \mathbb{C} \\ \vee & & \cup \\ C(\mathcal{A}) & \longrightarrow & \mathbb{H} \end{array}$$ upper half plane

[Today will stick to a fixed abelian category so rest of stability condition determined by central charge Z]

$\mathrm{Stab}\,\mathcal{A}$ = space of stability conditions

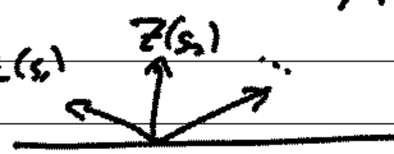
$= \{ Z \in \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{C}) : Z(C(\mathcal{A})) \subset \mathbb{H} \}$

<u>Main example</u>: A f.d. algebra / $\mathbb{C}$

$\mathcal{A} = \mathrm{Mod}(A)$ [from now on]

$K_0(\mathcal{A}) = \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$ $\quad S_i$ = simple A modules

$\mathrm{Stab}\,\mathcal{A} = \mathbb{H}^n$.



<u>Subcase</u>: Q quiver $= (V, E)$
vertices edges

Rep Q : $v \in V \mapsto M_v$ $\mathbb{C}$-vector space

$e \in E \mapsto \phi_e : M_{t(e)} \longrightarrow M_{h(e)}$

Simples : S_v $(v \in V)$: $M_u = \begin{cases} 0 & u \neq v \\ \mathbb{C} & u = v \end{cases}$

& all $\phi_e = 0$

If Q is a (finite) Dynkin quiver $\longleftrightarrow$ Φ finite root system $\subset \mathfrak{h}^*$

$K_0(\mathcal{A}) = \mathbb{Z}$-span of $\alpha_v = [S_v]$
= root lattice

$\mathrm{Stab}(\mathcal{A}) = \{ \varphi \in \mathfrak{h} : \mathrm{Im}\, \varphi(\alpha_v) > 0 \ \forall v \}$

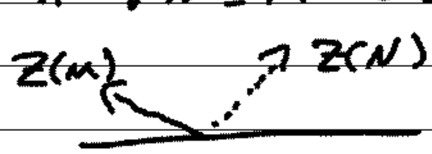
$= \mathfrak{h}_{\mathbb{R}} + i\,\mathcal{C}$ fundamental Weyl chamber

= complex fundamental Weyl chamber

Consider wall-crossings on $\mathrm{Stab}(\mathcal{A})$:
The set of semistable objects of a given class $\alpha \in K_0(\mathcal{A})$ will jump...

For $M \in \mathcal{A}$ $\Rightarrow$ phase $\phi(M) = \arg(Z(M))$

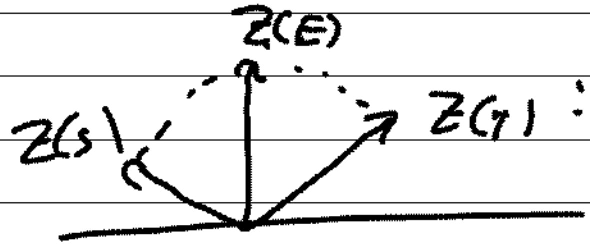
Def M is (semi)stable if $\forall N \subseteq M$ one has
$\phi(N) \underset{(=)}{<} \phi(M)$



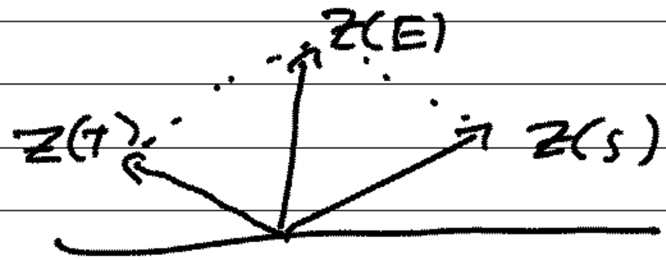
Simplest example of wall crossing:

$\mathcal{A}$ has two simples S, T & a nonsplit extension $0 \to S \to E \to T \to 0$

Z is determined by $Z(S)$, $Z(T)$



: E not semistable



: E stable

Jumping locus for objects of class

$$[E] = [S] + [T]$$

is $Z \in \operatorname{Stab} \mathcal{A} : \dfrac{Z(S)}{Z(T)} \in \mathbb{R}_{>0}$

Joyce's first idea:

Encode wall crossings into discontinuities of functions on $\operatorname{Stab} \mathcal{A}$ with values in the Hall algebra of $\mathcal{A}$.

(Schofield, Lusztig, Riedtmann, Kapranov-Vasserot)
... constructible function version of Hall algebra
... flavor close to Harvey-Moore algebra
of BPS states.

X complex variety
$f : X \longrightarrow \mathbb{C}$ is constructible if
$f = \sum_{i=1}^{n} c_i 1_{Z_i}$ $\quad Z_i \subseteq X$ locally closed
1_Z : characteristic function

Integration: $\int_X^{\chi} f = \sum_{i=1}^{n} c_i \chi(Z_i)$

χ = compactly supported Euler characteristic

If X = moduli stack of representations of A
$\overset{''}{M_d} = Rep_d / Gl_d$ $\quad$ (Rep_d = A-mod structure on $\mathbb{C}^d$)

f constructible on M_d := Gl_d invariant constructible f on Rep_d (ie ignore motivic invariants of automorphism groups)

$$H(\mathcal{A}) = \bigoplus_{d \geq 0} H_d(\mathcal{A}) = \text{Const. functions on } \coprod_d \mathcal{M}_d$$

$$f * g(M) = \int_{0 \subseteq N \subseteq M} f(N) \cdot g(M/N) = p_{2*}(p_1^* f \, p_3^* g)$$

$$\begin{tikzcd}
 & \{N \subset M\} \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] & \\
\{N\} & \{M\} & \{M/N\}
\end{tikzcd}$$

e.g. $$f_1 * \cdots * f_n (M) = \int_{0 = M_0 \subset M_1 \subset \cdots \subset M_n = M} f_1(M_1/M_0) \cdots f_n(M_n/M_{n-1})$$

Identity : δ_0 delta function on 0 element.

<u>Huge</u> algebra : contains δ-functions on all objects / isom. ....

Composition algebra $C(\mathcal{A}) \subset H(\mathcal{A})$:
algebra generated by χ_α = char. function of objects of class $\alpha \in K_0(\mathcal{A})$.

$\delta_\alpha(Z) :=$ Char fn of Z-semistable objects of class α

$\delta_\alpha(Z) \in C(\mathcal{A})$: consequence of Harder-Narasimhan & Reineke inversion:

Harder-Narasimhan $\Rightarrow$

$$\chi_\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \dots + \alpha_n = \alpha} \delta_{\alpha_1}(Z) * \dots * \delta_{\alpha_n}(Z)$$

& can invert this (Reineke) to get explicit formula for $\delta_\alpha(Z)$ in terms of χ_α.

<u>Joyce's second idea</u> Replace $C(\mathcal{A})$ by a smaller Lie algebra.

Coproduct $\Delta : C(\mathcal{A}) \to C(\mathcal{A}) \otimes C(\mathcal{A})$

$$\Delta f(M, N) = f(M \oplus N)$$

<u>Exercise</u> 1. f is primitive ($\Delta f = f \otimes 1 + 1 \otimes f$) iff f is supported on indecomposables

2. $\Delta f = f \otimes f$ (grouplike) $\iff f = \delta_0$ ["connected"]

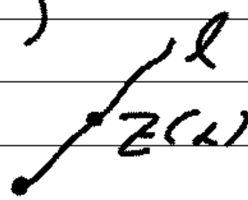
$\Rightarrow$ (Milnor-Moore)

$$\mathcal{C}(\mathcal{A}) = \mathcal{U}\,\mathfrak{n}(\mathcal{A})$$

Lie alg. $\mathfrak{n}(\mathcal{A}) = \{ f \in \mathcal{C}(\mathcal{A}) : \text{supported on indecomposables} \}$

e.g. $\mathcal{A} = \mathrm{Rep}\, Q$, Q Dynkin quiver

$\mathfrak{n}(\mathcal{A}) = \mathfrak{n}_+$ (Ringel)



Fix a ray l in $\mathbb{C}$

$S_l = 1 +$ char fn of semistables of phase in l $\in \overline{\mathcal{C}(\mathcal{A})}$ completion wrt grading

$$= 1 + \sum_{Z(\alpha) \in l} \delta_\alpha$$

$\Delta S_l = S_l \otimes S_l$ grouplike

$\Rightarrow$ can take $\log S_l =: \sum_{Z(\alpha) \in l} \varepsilon_\alpha$ graded components

with $\varepsilon_\alpha \in \mathfrak{n}(\mathcal{A})$

<u>Joyce's 3rd idea</u> : $\varepsilon_\alpha \rightsquigarrow$ generating function on $\mathrm{Stab}\,\mathcal{A}$ which is continuous, in fact holomorphic!

Theorem (Joyce)

Let $f_\alpha = \sum_{\substack{n \geq 1 \\ \alpha_1 + \dots + \alpha_n = \alpha}} J_n(Z(\alpha_1) \dots Z(\alpha_n)) \, \mathcal{E}_{\alpha_1} * \dots * \mathcal{E}_{\alpha_n}$

Then $\exists!$ function J_n s.t.

1. $J_1 = 1$ J_n satisfy growth conditions
2. f_α is continuous & holomorphic

$$\mathrm{Stab}\, \mathcal{A} \longrightarrow C(\mathcal{A})$$

These functions satisfy analog of WDVV:

$$d f_\alpha = \sum_{\beta + \gamma = \alpha} [f_\beta, f_\gamma] \frac{d \log \beta}{d \log \gamma}$$

(thinking of β, γ as linear functions on $\mathrm{Stab}\, \mathcal{A}$

Observation (Bridgeland, Toledano-Laredo) [BLT]

Pretend . $\alpha \longleftrightarrow$ roots of a simple Lie algebra $\mathfrak{g}$

- f_α takes value in $\mathfrak{g}_\alpha$
- $Z \in \mathfrak{h}_{\mathbb{C}}^{\mathrm{reg}}$

Thm The above equation is the isomonodromy equations (Jimbo-Miwa-Ueno GL_n, Boalch G general))

for connection 

Stokes factors & matrices:

G algebraic group, $\mathfrak{g} = \mathrm{Lie}\, G$

$\cup \qquad\qquad\qquad\qquad \cup$

H torus (not nec. maximal) $\quad \mathfrak{h} \ni Z$.

Ex 1. G reductive, H maximal torus

2. N = prounipotent group associated to $\mathfrak{n}(A)$. $G = N \rtimes H$, $H = \mathrm{Hom}(K(A), \mathbb{C}^*)$

($\mathfrak{n}(A)$ graded by indecomposables)

Let $\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right)dt$ connection

on $P = G \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

Let Φ = root system of G rel H $\subset \mathfrak{h}^*$

<u>Good solutions</u> • If $[Z, f] = 0 \Longrightarrow$

take $\phi = e^{-Z/t}\, t^{f}$

• $\phi = F(t)\, e^{-Z/t}\, t^{[f]}$

$[f]$ = projection onto centralizer of Z

Would like F holomorphic & $F(0) = 1$: does not exist...

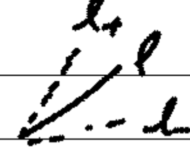
But could have for F well defined on a half-plane...

<u>Def</u> ℓ is a Stokes line of ∇ if it's of the form $\mathbb{R}_{>0} Z(\alpha)$ $\alpha \in \Phi$

<u>Theorem</u> (Balser-J.-Lutz, Boalch, BLY)
If ℓ is <u>not</u> a Stokes line, $\exists!$ ϕ_ℓ solution with F defined on halfplane defined by ℓ.

Stokes factors: $\phi_\ell = \phi_{\ell'} \cdot S_\Sigma$ $(S_\Sigma \in G)$

Def ℓ a Stokes line $\Rightarrow$ perturb



$$\phi_- = \phi_+ \cdot S_\ell \quad \text{where } S_\ell = \exp \mathcal{E}$$

with $\mathcal{E} \in \bigoplus_{Z(\alpha)\in \ell} \mathfrak{g}_\alpha$

Stokes map: $f \in \mathfrak{g} \oplus \mathfrak{g}^h \xmapsto{S} \mathcal{E} \in \mathfrak{g} \oplus \mathfrak{g}^h$:

collect logs of Stokes factors associated to all Stokes lines.

Theorem (BLT) 1. The map S is analytic & given by

$$\mathcal{E}_\alpha = \sum_{n\geq 1} \sum_{\substack{\alpha_1+\dots+\alpha_n \\ =\alpha}} M_n(Z(\alpha_1)\dots Z(\alpha_n))\, f_{\alpha_1}\cdots f_{\alpha_n}$$

where $M_n(z_1,\dots,z_n) = \int_{[0,s_n]} \frac{dt}{t-s_n}\circ\dots\circ\frac{dt}{t-s_1}$

$s_i = z_1+\dots+z_i$

2. S^{-1} has Taylor series

$$f_\alpha = \sum_{n \geq 1} \sum_{\underline{\alpha_1 + \dots + \alpha_n} = \alpha} J_n(Z(\alpha_1) \dots Z(\alpha_n)) \cdot \varepsilon_{\alpha_1} \dots \varepsilon_{\alpha_n}$$

J's are Joyce's functions

f_α = residues of connection: don't jump,

ε_α are Stokes data, jump!

3. The isomonodromy equation for ∇ rel to H are exactly

$$d f_\alpha = \sum_{\beta + \gamma = \alpha} [f_\beta, f_\gamma] \frac{d \log \beta}{d \log \gamma}$$

Kontsevich-Soibelman: full triangulated category for a CY3.

Study $\mathcal{L}(X) = \bigoplus_{\alpha \in K(\mathrm{Coh}\, X)} c^\alpha$

with bracket $[c^\alpha, c^\beta] = \chi(\alpha, \beta) c^{\alpha + \beta}$

above picture: have $\mathfrak{n}(\mathrm{Coh}\,X)$ or maybe $\mathfrak{g}(D(\mathrm{Coh}\,X))$ – should have Lie algebra homomorphism

$$\mathfrak{g}(X) \longrightarrow \mathcal{L}(X)$$

$$\mathcal{E}_\alpha(Z) \longmapsto c^\alpha \cdot DT(\alpha, Z) \quad \text{Donaldson-Thomas}$$