Sasha Beilinson - Connections on Punctured Disc

Note Title 5/18/2007

NC geometry - as language for bad quotients — such as moduli of connections on disc.

G group, conn. $\operatorname{Spec} F$ punctured disc

$F = k((t))$

G-bundles with connections on $\operatorname{Spec} F$: orbits on space of connections on trivial G-bundle $\Longleftrightarrow$ $\omega(F) \otimes \mathfrak{g} = C$

$\circlearrowleft$

$G(F)$ gauge group

$\mathrm{Conn} = C/G(F)$: unpleasant space, not alg. stack

- looks naively like 0-dimensional space

Tangent at $(\mathcal{F}, \nabla)$ is $H^1_{dR}(\operatorname{Spec} F, \mathfrak{g}_{\mathcal{F}})$

or rather full cohomology $R\Gamma_{DR}(\mathfrak{g}_{\mathcal{F}})[1]$

Euler char $= 0$, determinant line canonically trivialized $\rightsquigarrow$ 0-dim

What are $\mathcal{O}$-modules on Conn?

$\mathcal{C}$, $G(F)$ are ind-schemes $\rightsquigarrow$
have notion of families of points
(R-points for R a k-algebra)

$G(F)(R) := G(F \hat{\otimes} R)$ completed tensor
$= G(R((t)))$ inductive limit
of family of schemes + closed embeddings

$\mathcal{C}$: ∞-dim vector space

in fact $\mathcal{C}$, $G(F)$ are ind-affine:
functions are projective limits of comm. algebras $\longleftrightarrow$ topological algebras

$$\mathcal{R} = \varprojlim \mathcal{R}/I_\alpha$$

On an ind-affine scheme can consider
$\mathcal{R}$-modules: topologically discrete $\mathcal{R}$-modules
(annihilators of vectors are open ideals)

$\mathcal{R}/I_\alpha \twoheadrightarrow \mathcal{R}/I_\beta$ — need property

$$\boxed{\begin{array}{l} I_\beta/I_\alpha \text{ is finitely generated} \\ \Longleftrightarrow \underline{\text{reasonable}} \text{ ind-schemes.} \end{array}}$$

Usual notion of $G(F)$-equivariant $\mathcal{O}_C$-modules are essentially constant on orbits ⇝
do not lie in our category: they're not discrete, don't give abelian category...
each section's support should be a true finite-dim subscheme ⟷ can't be equivariant for ∞-dim group

Like BRST reduction: don't take annihilated subspace of all currents, must do peculiar procedure of adding fermions.

- add fermions
- peculiar notion of derived cat of $\mathcal{O}$-modules

First example where there are no such difficulties to see $\mathcal{O}$-mod on Conn are interesting:

Ex. $G = \mathbb{G}_m$, $C = \omega(F)$
$G(F) = F^*$ (these are the k-points.)

$\sum a_i t^i dt \in C$: a_i indep functions if fix order of pole set Spec(polynomials in ∞ many variables), take union of these.

$G(F)$ as ind-scheme: product of 4 pieces

- $1 + tk[[t]] = \prod \mathbb{G}_a$
- $\mathbb{Z} = \{t^{\mathbb{Z}}\} = \mathbb{Z}$
- $k^* = \mathbb{G}_m$
- $\{1 + a_{-1}t^{-1} + \dots + a_{-n}t^{-n}\} = \coprod \widehat{\mathbb{G}_a}$
  $\forall\ a_i$ nilpotent

Contou-Carrère : $G(F)$ is canonically Cartier self-duality

Space of orbits in this abelian case is a group... Picard stack

$$\mathrm{Cone}\left(F^\times \xrightarrow{d\log} \omega(F)\right) = \mathrm{Conn}$$: stack is the core Picard stack

$d\log : 1 + tk[[t]] \xrightarrow{\sim} \omega(\mathcal{O}_F)$

So can rewrite as

$$\mathrm{Cone}\left(F^*/_{1+tk[[t]]} \longrightarrow \omega(F)/\omega(\mathcal{O}_F) \right)$$

$\mathbb{G}_m$ = kernel of dlog, let's forget these $\mathbb{G}_m$ autos of every connection.

Image of dlog : dlog : $t \longmapsto \frac{dt}{t}$
acting by translation

$\Rightarrow k \frac{dt}{t} \simeq \mathbb{G}_a \circlearrowright \frac{dt}{t}$ acting by translation

So get $\mathbb{G}_a/\mathbb{Z}$.

$\underline{11}\ \hat{\mathbb{G}}_a$ maps isomorphically to formal completions of connection coeffs

$$\Rightarrow \mathrm{Con} \simeq \mathbb{G}_a/\mathbb{Z} \times \underline{11}\ \mathbb{G}_a/\hat{\mathbb{G}}_a$$
(mod $\mathbb{G}_m$)

So $\mathcal{O}$-modules on second factor are $\mathcal{D}$-modules on ∞ many variables, & $\mathbb{Z}$-equiv. modules on G_a.

Nonabelian case much harder: rather than $K^*/\mathcal{O}^*$ need to "quotient" by affine Grassmannian

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X reasonable ind scheme, G group ind scheme

What are $\mathcal{O}$-modules on X/G?

in two steps... spell it out in derived setting.

First what is "right" derived category of $\mathcal{O}$-modules on X? (so as to be able to make descent ... need to understand pull back functors: how e.g. to pull back from a point in topological module world?? - keeping discreteness)

$X \to pt$ What is $\mathcal{O}_X \leftrightarrow p^* \mathcal{O}_{pt}$?

How about ω_X instead - "right" version?

if $X = \varinjlim X_\alpha$ X_α of finite type $\Rightarrow$

$i_{\alpha\beta} : X_\alpha \hookrightarrow X_\beta$ closed

$\Rightarrow \omega_{X_\alpha} \hookrightarrow \omega_{X_\beta}$ (Cousin complexes embed)

So can take inductive limit, get nice Cousin complex. But many times this complex is acyclic: eg $X = \mathbb{A}^\infty$ union of ∞ many: $\omega_{\mathbb{A}^n}$ is n-forms in deg $-n$, when take limit cohomology vanishes.

So can't see $\omega_{\mathbb{A}^\infty}$ in usual derived category

$\Longrightarrow$ modify derived category so as to see it.

Formally: construction of ind objects in dg category $\supset$ original dg category as compact objects.

So let's consider coherent $\mathcal{O}$-modules! each supported on some X_α & coherent on it. $\Rightarrow \mathrm{Coh}\, X$.

Keller. $\underrightarrow{Coh}(X) \subset$ Complexes of $\mathcal{O}$-modules

full dg subcategory of complexes $(M^\bullet, d)$

so that can decompose $M^\bullet = \bigoplus_{n \geq 0} M^\bullet_{\alpha_n, n}$

with each $M^\bullet_{\alpha_n, n}$ coherent,

& differential $d(M^\bullet_{\alpha_n, n}) \subset \bigoplus_{m<n} M^\bullet_m$

— filtration, & d vanishes on gradeds.

$$\underrightarrow{Coh}(X) \supset \text{bounded complexes of coherents} \cup \text{acyclic complexes}.$$

[Want 1. filtration preserved by d, d vanishes on grad'd

2. filtration splits without d

3. each graded is $\bigoplus$ of coherents]

Take right orthogonal complement to acyclic complexes in $\underrightarrow{\mathrm{Coh}}(X)$, & localization of this will be the correct derived cat.

$$\Big(C^b\big(\mathrm{Coh}(X)\big)^{acycl}\Big)^{\perp} \subset \underrightarrow{\mathrm{Coh}}(X)$$
$$\| $$
$$\{ M : \forall N \in C^b(\mathrm{Coh})^{acycl} \quad \mathrm{Hom}(N, M) = 0 \}$$

This category maps to $D(\mathcal{O}\text{-mod})$ but this functor kills many objects:

eg dualizing object lives here & is nonzero:

$$i_\alpha^! \, \omega_X = \omega_{X_\alpha} \neq 0$$

<u>Step 2</u> Twisted $\mathcal{O}$-modules:

One can twist $\mathcal{O}$-modules by an $\mathcal{O}^*$-gerbe.

Here we need determinant gerbe of tangent bundle — consider only formally smooth ind schemes.

$\rightsquigarrow$ have Ω_X, Θ_X (co)tangent bundles:

Take vector bundles

(vector space object of same nature as X)

To such a Tate vector bundle $\rightsquigarrow$ determinantal gerbe (Deligne, Brylinski, Kapranov...)
$\longleftrightarrow$ notions of lattices in Tate vector space, get groupoid of relative determinant $\longrightarrow$ gerbe on X.

The corresponding twisted $\mathcal{O}$-modules can be described as modules for $Cliff(\mathcal{O} \oplus \Omega)$
— modules with continuous action of creation / annihilation operators

Over a point such Clifford modules $\longleftrightarrow$ vector spaces, but over a family get exactly $\mathcal{O}$-modules twisted by det gerbe

This Cliff is naturally $\mathbb{Z}$-graded.
Det lines are $\mathbb{Z}$-graded lines,
Here get $\mathbb{Z}$-torsor on Nisnevich topology of X

Shenkbrit rarely consider $\mathbb{Z}$-graded complexes, but complexes with gradings in dimensional $\mathbb{Z}$-torsor of tangent bundle.

These objects have good theory of pullbacks
$\Longrightarrow$ can consider equivariant objects

Why is all this needed for pullback?

In fin. dim have two pullbacks $f^*, f^!$
In ind setting $f^!$ nice, in pro setting f^* nice.

Clifford modules helps escape this contradiction.

In smooth setting difference of $f^*, f^!$ captured by $\omega_{X/Y}[\dim]$.

Clifford modules have <u>canonical</u> pullback, only one of them! If use one of two standard trivializations of the $\mathbb{Z}$-torsor get f^* and $f^!$.