David Nadler - Langlands-Vogan-Mirror Duality

Note Title 5/18/2007

(w/ D. Ben-Zvi)

$$\begin{tikzcd} LG \arrow[r, leftrightarrow, "\text{Langlands}"] & \text{"loop geometry" of } G^\vee \\ G \arrow[u, leftrightarrow, "S^1\text{-fixed pts}"] \arrow[r, leftrightarrow] & G^\vee \arrow[u, leftrightarrow, "S^1 \text{ fixed pts}"'] \end{tikzcd}$$

<u>Vogan's mystery/challenge</u>

Automorphic/A side (topology)	Spectral / B-side (algebra)
G reductive $/\mathbb{C}$	$G^\vee$ dual $/\mathbb{C}$
$\theta \circlearrowright G$ conjugation	$\eta \circlearrowright G^\vee$ dual involution
Example: $SL_2\mathbb{C}$, θ = usual conjugation $\rightsquigarrow SL_2\mathbb{R}$	$PGL_2\mathbb{C}$, η = trivial involution
$1 \to G \to G_\theta \to \mathbb{Z}/2 \to 1$	$1 \to G^\vee \to G^L \to \mathbb{Z}/2 \to 1$
$\Theta = \{g \in G_\theta \setminus G : g^2 = 1\}/G$ set of conjugations	$I = \{g \in G^L \setminus G^\vee : g^2 = 1\}/G^\vee$ set of involutions

<u>Vogan duality</u>

fixed gr   fixed group

$$\coprod_{\theta\in\Theta} G^{\theta}_{\mathbb{R}} \backslash G/B \qquad \Big| \qquad \coprod_{\eta\in I} K_{\eta} \backslash G^{\vee}/B^{\vee}$$

$D_c($ " $)$ <-------> D-mod (")

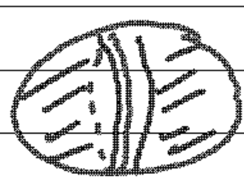
duality on level of K-groups

Θ = 1 elt $(1,\theta)$

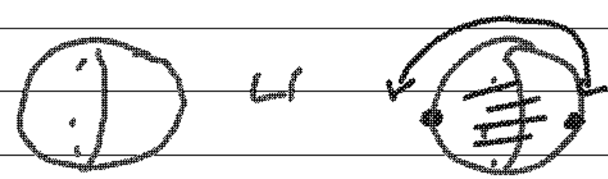
$G_{\mathbb{R},\theta} = SL_2\mathbb{R}$

I = 2 elts $\underset{\eta_0}{(1,\eta)}, \underset{\eta_1}{(\sqrt{-1},\eta)}$

$K_{\eta_0} = PGL_2\mathbb{C}$ $K_{\eta_1} = O_2\mathbb{C}$



3 orbits



$SL_2\mathbb{R} \circlearrowright \mathbb{CP}^1$

$PGL_2\mathbb{C} \circlearrowright \mathbb{CP}^1$ $O_2\mathbb{C} \circlearrowright \mathbb{CP}^1$

Calculate irred local systems: 1 orbit   2 orbits

4 irred $SL_2\mathbb{R}$-local systems   1 irreducible   3 irreducibles

$4 = 1+3$

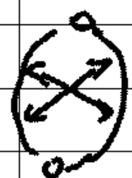
Soergel conjecture: lift Vogan duality to categories.

Goal: Find a loop version of Vogan duality.

A side

$LG = \{g: \mathbb{C}^* \to G\}^\circ$
$\quad = G(\mathbb{C}[t,t^{-1}])$

$L_{\mathbb{R}}G = \{g \mid g(-\bar{z}^{-1}) = \Theta(g(z))\}^\circ$

$\mathbb{C}^*$  $\xrightarrow{\text{antipodal}} G \circlearrowleft_{\Theta}$

Geometry of $L_{\mathbb{R}}G \backslash \boxed{LG/I}$ affine flags

$I = \{g: \mathbb{C} \to G : g(0) \in B\}^\circ$

- take $D_c(L_{\mathbb{R}}G \backslash LG/I)$

B side

Need loop version of
$\bigsqcup_{\ell \in I} K_\ell \backslash G^\vee / B^\vee$

$\rightsquigarrow$ take $S^1 \xrightarrow{\text{maps}} G^L \backslash G^\vee / B^\vee$
important piece of loop space

$St^L = \{(\delta \in G^L - G^\vee, B) : \delta^2 \in B^\vee \text{ (unipotent)}\}/G^\vee$

Take $D_{coh}(St^L)$
"real" version of Steinberg

<u>Conjecture</u> (Langlands/mirror for real groups)

$$D_c(L_{\mathbb{R}}G \backslash LG/I) \xleftrightarrow{\text{duality}} D_{coh}(St^L)$$

Picture

$$\begin{tikzcd} LG \arrow[r, dash, "\text{Langlands}"] \arrow[d, dash] & \text{``loop group'' of } G^\vee \arrow[d, dash] \\ G \arrow[r, dash, "\text{Soergel's conjecture}"'] & G^\vee \end{tikzcd}$$

S^1 fixed points

Evidence

1. Complex groups: two presentations of affine Hecke algebra

$D_c(I_\infty \backslash LG / I_0)$ | $D_{coh}(St^\vee)$

opposite Iwahoris

$$St^\vee = \left\{ \begin{array}{c} g \in G^\vee, B_1, B_2 \in \mathcal{B}^\vee \\ g \in B_1, B_2 \\ g \text{ unipotent} \end{array} \right\} \Big/ G^\vee$$

On K-groups: Kazhdan-Lusztig

Categories: Bezrukavnikov (& friends)

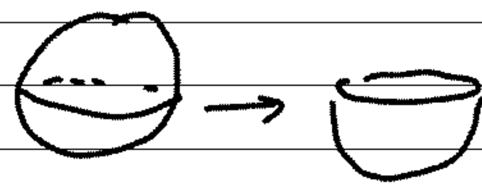
⇒ Any question in complex case should have a chance!

Example: Soergel's conjecture in complex case understood - Beilinson-Ginzburg-Soergel

→ this now becomes canonical

Theorem Conjecture is true on Grothendieck groups
"affine Vogan duality"

Proof – Geometric version of Langlands base change
($\mathbb{C} \rightsquigarrow \mathbb{R}$)

$\mathbb{P}^1$  $\mathbb{RP}^2$

Reinterpret our objects in terms of these curves:

	A side		B side
complex	$D_c(\mathrm{Bun}_G(\mathbb{P}^1)_{0,\infty})$	$\xleftrightarrow[\text{Langlands}]{\text{complex}}$	$D_{coh}(\mathrm{Conn}_{G^\vee}(\mathbb{P}^1)_{0,\infty})$
	$\updownarrow$ Base change		$\updownarrow$
real	$D_c(\mathrm{Bun}_{G,\theta}(\mathbb{RP}^2)_{0=\infty})$	$\xleftrightarrow[\text{real?}]{}$	$D_{coh}(\mathrm{Conn}_{G^\vee,\eta}(\mathbb{RP}^2)_{0=\infty})$

What is base change?

B side : Tannakian story ("Hecke descent")

$\mathcal{F}$ should have following structure:

$x \in \mathbb{P}^1 \quad V \in \mathrm{Rep}\, G^\vee$

$$V \underset{x}{*} \mathcal{F} \xrightarrow{\sim} \eta(V)_{\mathrm{antipode}(x)} * \mathcal{F}$$

Prop $\mathcal{F} + \varphi \longleftrightarrow \mathbb{RP}^2$- B brane

Mystery - does base change work on A side?
Does a $\mathbb{P}^1$ A-brane + Hecke descent
$\to \mathbb{RP}^2$ A-brane ?
no natural map on underlying spaces!

But if G is abelian there's a norm map
Norm: $\mathrm{Bun}_G \, \mathbb{P}^1 \longrightarrow \mathrm{Bun}_{G,\theta} \, \mathbb{RP}^2$.

Hecke descent is just descent for norm map.

$\mathbb{P}^1$ + $\mathbb{RP}^2$: all bundles have reductions
to abelian bundles
$\Longrightarrow$ can prove theorem on K-groups