Northwestern Loop Spaces

Note Title 5/16/2007

w. D. Nadler. Project: relate local geometric Langlands program to complex & real local Langlands programs.

Local geometric Langlands concerns geometry & rep theory (broadly construed) of LG.
while local Langlands "at ∞" concerns representations of G & real forms $G_{\mathbb{R}}$. Langlands parameters for both given in terms of geometry of dual group $G^{\vee}$.

Relation of LG to G: S^1 fixed points.
$\rightsquigarrow$ (following Witten etc) use equivariant localization

M compact space with S^1 action
$\rightsquigarrow$ equivariant cohomology:
$$H^*_{S^1}(M) = H^*([M/S^1])$$

cohomology of "correct quotient":
$$(M \times E)/S^1 \longrightarrow E/S^1 = BS^1 = \mathbb{CP}^\infty$$
E contractible space with free S^1 action

Moreover $H^*_{S^1}(M) \Leftrightarrow H^*(M^{S^1})$:

periodic cyclic homology
$$H^*_{S^1}(M) \underset{H^*(BS^1)}{\otimes} \mathbb{C}[u,u^{-1}] \simeq H^*(M^{S^1}, \mathbb{C}[u,u^{-1}])$$

Categorified version:

$$D^b_{c,S^1}(M) \underset{H^*(BS^1)}{\otimes} \mathbb{C}[u,u^{-1}] \simeq D^b_c(M^{S^1}) \otimes \mathbb{C}[u,u^{-1}]$$

(follows from GKM)

On Langlands parameter side, we're interested e.g. in $G^\vee$-equiv. coherent sheaves on Springer & Steinberg varieties: $\tilde{G} = \{(g, B) \in G^\vee \times \mathcal{B}^\vee : g \in B\}$

$$St^\vee = \tilde{G} \times_{G^\vee} \tilde{G} = \{(g, B_1, B_2) : g \in B_1 \cap B_2\}$$

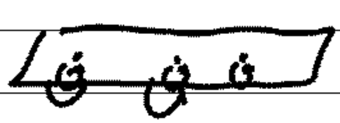
Doesn't look like a loop space...

What could the analog of S^1-equivariant localization be here?

Derived loop spaces

Would like to make sense of $\mathcal{L}X = Map(S^1, X)$ in world of algebraic geometry .... X a scheme or stack.

First approximation: X a stack, e.g. $X = Z/\Gamma$

 : space where points have automorphisms

$S^1 \longrightarrow X$: too quick to connect different points of X (not homotopy invariant)

but can zip around an automorphism

$$\Rightarrow \quad \mathcal{I}X = \{(x,g) \;\; x \in X \;\; g \in \mathrm{Aut}\, X\} / \mathrm{Stab}\, g$$

inertia stack

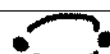
One motivation:

$$\begin{tikzcd} \text{rings} \arrow[r, "\text{schemes}"] \arrow[dr, "\text{stacks}"'] & \text{sets} \arrow[d] \\ & \text{spaces} \end{tikzcd}$$

really just see 1-truncated spaces : $\coprod K(\Gamma, 1)$'s

S^1 also defines such a functor, look at homotopy classes of maps $\Rightarrow \mathcal{I}X$

e.g. $X = BG$ $\quad \mathcal{I}X = G/G$ adjoint quotient

Another attempt: $S^1 =$ 

Map $S^1 \to X$ a smooth scheme:

two points in X, & they're equal, & they're equal.

ie $\quad \mathcal{L}X = X \times_{X \times X} X$

How to interpret well this nontransverse intersection? $\mathcal{O}_{\mathcal{L}X} \overset{?}{=} \mathcal{O}_X \underset{\mathcal{O}_{X\times X}}{\otimes} \mathcal{O}_X$

should be derived: $\mathcal{O}_{\mathcal{L}X} = \mathcal{O}_X \overset{\mathbb{L}}{\underset{\mathcal{O}_{X\times X}}{\otimes}} \mathcal{O}_X$

= Hochschild homology sheaf.

Calculate by Koszul complex $\Rightarrow$

(HKR) $\mathcal{O}_{\mathcal{L}X} = \mathrm{Sym}\, \Omega[1]$

$$\mathcal{L}X = T_X[-1] \quad \text{odd tangent bundle}$$

– usual answer from physics or supergeometry

(Replace S^1 by Spec
$H^*(S^1) = \mathbb{C}[\eta]$ $|\eta|=1$ $\eta^2=0$: <u>odd</u> algebra
... works eg in supergeometry: <u>odd line</u> $\mathbb{A}^1[1]$
$\mathrm{Map}(\mathbb{A}^1[1], X) = T_X[-1]$ odd tangent bundle)

Where does this answer live? DAG

$$\begin{array}{ccc} \text{rings} & \longrightarrow & \text{Set} \\ \downarrow & & \downarrow \\ \text{sRings} & \longrightarrow & \text{sSet} \end{array}$$

For general (smooth) stacks $\mathcal{L}X$ is combination of odd tangents & inertia:

$$\widehat{\mathbb{T}_X[-1]} = \mathbb{H}X \hookrightarrow \mathcal{L}X$$

small loops $\qquad$ loops

$$\hat{\tilde{G}}/G \hookrightarrow G/G$$

$S^1 \circlearrowright \mathcal{L}X$. On $\widehat{\mathbb{T}_X[-1]}$: the derivative of this action is the action of $H_*(S^1)$

.. odd vector field, the de Rham differential

$\longleftrightarrow$ cyclic structure of HH gives Connes differential

Categorify: $HH \rightsquigarrow \mathcal{C}oh\, \mathcal{L}X$

$HP \rightsquigarrow \mathcal{C}oh(\mathcal{L}X)^{S^1, loc}$

Theorem $D(\mathbb{H}X, \mathcal{O})^{S^1} \otimes_{BS^1} \mathbb{C}[u, u^{-1}]$

$$\simeq D(X, \mathcal{D}) \otimes \mathbb{C}[u, u^{-1}]$$

(unlocalized version $\rightsquigarrow$ $\mathcal{R}(\mathcal{D})$-modules)

Application: $X = {}_{B^\vee}\backslash G^\vee / B^\vee = {}_{G^\vee}\backslash G^\vee/B^\vee \times G^\vee/B^\vee$

$$\mathcal{L}X = \widetilde{St}^\vee / G^\vee \longrightarrow H^\vee \times H^\vee$$

<u>Corollary</u> $D(\mathcal{B}^\vee/B^\vee, \mathcal{D}) \otimes \mathbb{C}[u,u^{-1}]$

$$\simeq D(\widehat{St^\vee/G^\vee}|_{e\times e}, \mathcal{O})^{S^1, loc}$$

ie (tame) Langlands parameters for LG &

" " " $G_{\mathbb{C}}$ related

by same process – S^1 localization –

as the representations.