Kevin Costello - Factorization Algebras

Note Title 5/25/2009

Deformation Quantization

Classical mechanics described by a Poisson algebra $(A, \{,\})$

$\rightsquigarrow$ replace A by an associative product $*$ on $A[[\hbar]]$ s.t. for $f, g \in A$

$$\{f,g\} = \lim_{\hbar \to 0} \frac{1}{\hbar} [f,g]$$

Today: want an analog in QFT

1. Describe classical + quantum algebraic structures
2. Show one can get classical structure from classical field theory
3. State a "quantization" theorem. Work in progress w/ Owen Gwilliam.

Quantum structure: factorization algebra.

Let M be a manifold ("spacetime")

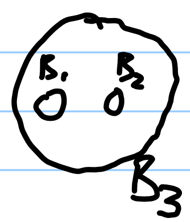
A factorization algebra on M can be described as follows: Let $B(M)$ = space of closed top. dim balls in M

$B_n(M)$ = configurations of n disjoint balls in M embedded in a larger ball

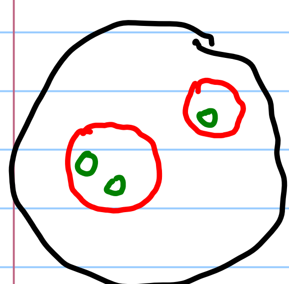
Structure: vector spaces for balls & maps corresponding to inclusions as in $B_n(M)$

— closely related to $E_{\dim}$ algebras.

A f. algebra is a vector bundle V on $B(M)$ with maps
$$B(M) \xleftarrow{p} B_n(M) \xrightarrow{q} B(M)^n$$
$$q^*(V^{\boxtimes n}) \longrightarrow p^*V$$
satisfying some evident compatibilities [& units..]



$$V(B_1) \otimes V(B_2) \longrightarrow V(B_3)$$



compatibility of two compositions from $V^{\text{small balls}} \longrightarrow V^{\text{large ball}}$.

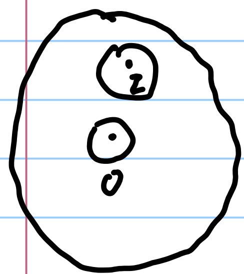
Close relative of E_n algebra ($\dim M = n$): replace V by a locally constant sheaf —

a <u>topological factorization algebra</u> is as above in category of locally constant sheaves.

<u>Theorem</u> An E_n algebra yields a topological factorization algebra on any framed M, $\dim M = n$

A factorization algebra on $\mathbb{C}$, where everything is holomorphic & invariant under $\mathrm{Aff}(\mathbb{C})$

$\rightsquigarrow V(\{|z|<1\}) =: W$ a vector space



$\Rightarrow$ get map $m_z : W \otimes W \longrightarrow W$ depending holomorphically on z in an annulus

$\rightsquigarrow$ vertex algebra:

$$m_z \sim \sum_{k \in \mathbb{Z}} \varphi_k z^k, \quad \varphi_k \in \text{(some completion of) } \mathrm{Hom}(W \otimes W, W)$$

- reminiscent of OPE.

__Claim__ Factorization algebras on M encode structure one expects from a QFT on M.

2d, holomorphic settings: fits with known pictures

1d: roughly associative algebra: observables of QM

In any dim construct a f. algebra using perturbative QFT.

F. algebras are a symmetric monoidal category.

__Def__ A classical F. algebra is a commutative algebra in this category

[Topology: $E_n \times E_\infty = E_\infty$: these are commutative algebras]

Suppose we have a classical field theory
eg M compact Riemannian mfld
Fields $= C^\infty(M, \mathbb{R})$

Action $S(\varphi) = \int \varphi \Delta \varphi + \varphi^3$

[elliptic quadratic term + higher order corrections]

Local nature of action $\rightsquigarrow$ Solutions to Euler Lagrange equations form a sheaf

EL on M : $\{ 2\Delta\varphi + 3\varphi^2 = 0 \}$

To any $B \subset M$, let
$\mathcal{O}(EL(B))$ = functions on $EL(B)$
$\longrightarrow$ defines a commutative factorization algebra:

$EL(B_3) \longrightarrow EL(B_1) \times EL(B_2)$
applying $\mathcal{O}$ gives a factorization algebra

[by being functions on all $EL(\text{ball})$, our "distributions" $\mathcal{O}(EL)$ are forced to have some compact support on our manifold]

Claim The E_∞ object in factorization algebras "wants" to become E_0, ie just a factorization algebra ...

deg of Poisson bracket	
1	E_0
0	E_1
-1	E_2
-2	E_3

Def The BV_0 operad is the operad over $\mathbb{R}[[\hbar]]$ generated by a commutative product, $*$ & a Poisson bracket of degree 1, $\{,\}$ s.t. $d* = \hbar\{\ \}$

So a BV_0 algebra has a differential, which is not a derivation, but failure to be so is governed by $\{\ \}$.

— closely related as $\mathbb{Z}/2$-graded to framed little discs operad, but different with $\mathbb{Z}$ grading.

Invert $\hbar \rightsquigarrow$ this is contractible, $BV_0 \simeq E_0$

$\hbar = 0$: $BV/\hbar$ = comm algebras w/ bracket of degree 1.

- i.e. interpolates between E_∞ & E_0.

So algebra with deg 1 Poisson bracket "really wants to become" E_0.

<u>Claim</u> derived space of solutions of EL has Poisson bracket of degree 1:

Let X be a manifold, $f \in \mathcal{O}(X)$

$\Rightarrow$ the derived critical scheme $h\mathrm{Crit}(f)$ has a $\{\}$ of degree 1, so wants to become E_0.

$h\mathrm{Crit}(f)$ = Koszul resolution of $df = 0$

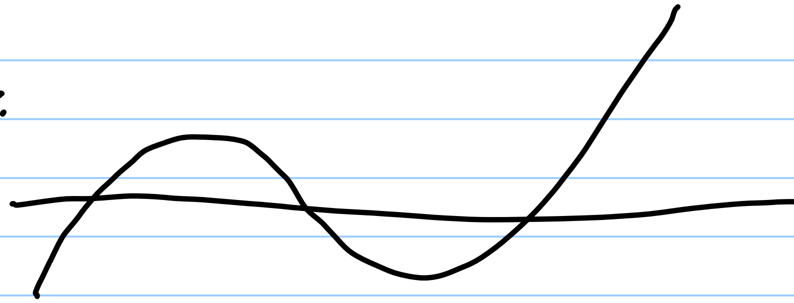
$\mathcal{O}(h\mathrm{Crit}(f))$ is the dga

$$\cdots \to \Lambda^2 TX \xrightarrow{\vee df} TX \xrightarrow{\vee df} \mathcal{O}(X)$$

= derived intersection of Γ_{df} $\cap$ 0-section inside T^*X.

- functions on here are polyvector fields, carrying Schouten bracket, deg 1 Poisson bracket.

Picture: T^*X



derived intersection = path space $\Gamma_f \rightsquigarrow$ 0-section
& we're transgressing the Poisson bracket on T^*X.

Apply this to the Euler-Lagrange situation:
- look at derived space of solutions

= critical points of action, so should acquire Poisson bracket.

e.g. for free theory, derived critical locus of $\int \varphi \Delta \varphi$ is the linear object

(2-term complex) $C^\infty(M) \xrightarrow{\Delta} C^\infty(M)$

(sheaf of complexes on M)

Take $\mathcal{O}$ get dg comm. factorization algebra.

Action with a cubic term $\rightsquigarrow$ nonlinear piece to differential

derived EL sols

For $B \subset M$, $\mathcal{O}(hEL(B))$ looks like

$$\prod \mathrm{Hom}_{\text{cont.}}\left(\mathrm{Sym}^n C^\infty(B) \otimes \Lambda^k C^\infty(B), \mathbb{R}\right)$$

... some space of distributions on B^{n+k},

with a differential coming from the action

Carries deg 1 $\{\}$, comes from pairing two copies of functions [?...]

$\rightsquigarrow$ wants to become E_0,

If X has a measure, ie a trivialization of ω_X, we get a BV operator $\Delta : \Lambda^\bullet TX \circlearrowleft$ s.t. $d + \hbar\Delta$ gives a BV_0 algebra.

[In our ∞-dimensional situations, this is the main goal, to find perturbatively such structures... . . . . . .]

Theorem The commutative factorization algebra associated to EL equations of a classical field theory can be quantized into a factorization algebra in interesting situations, e.g.

- Scalar field theories
- Yang-Mills on $\mathbb{R}^4$

[Note: false over $\mathbb{Q}$! essentially transcendental]