Dennis Gaitsgory - Global Geometric Langlands

Note Title 5/26/2009

Two historical approaches to constructing automorphic sheaves
- Whittaker models
- chiral algebras

Aim is to unify them.

$$\begin{array}{ccc} \mathcal{D}_c(\mathrm{Bun}_G X) & \longrightarrow & \bigotimes_{x \in X} \mathrm{Wh.} \mathcal{F}_G \\ \downarrow & & \\ \bigotimes_{x \in X} KL & & \end{array}$$

Plan:
I. Introduce categories
II. KL & localization
III. Whittaker categories & quantum Langlands
IV. The classical case

κ level : $\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{C}$

$\kappa + \frac{1}{2}\mathrm{Killing}|_{\mathfrak{h}}$: $\mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{C}$

$\Leftarrow$: form $\mathfrak{h}^\vee \otimes \mathfrak{h}^\vee \longrightarrow \mathbb{C}$

$\rightsquigarrow -\kappa^{-1} : \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \longrightarrow \mathbb{C}$

$D(\mathrm{Bun}_G)_\kappa$ sheaf of κ-twisted differ. on Bun_G

$\rightsquigarrow$ category $D\text{-mod}(\mathrm{Bun}_G)_{\kappa,!}$

... careful: stack of infinite type :

write $Bun_G = \bigcup U_i$ stacks of finite type

$\rightsquigarrow$ D-module is compatible system of D-modules on finite type pieces. – defined as inverse limit.

Drinfeld: this category is compactly generated! (nontrivial!)

<u>Def</u> $D\text{-mod}(Bun_G)_{X,*} = \left(D\text{-mod}(Bun_G)_{-X,!}\right)^{\vee}$

dual category

has natural functor to ! version, $\longrightarrow D\text{-mod}(Bun_G)_{X,!}$

far from an equivalence in general

but is an equivalence for X irrational.

$$KL(G)_X = (\hat{g}\text{-mod}_X)^{G[[t]]}$$

or for $x \in X$ $\quad KL(G)_{X,x} = (\hat{g}\text{-mod}_X)^{G(O_x)}$

– a <u>chiral category</u> on X

(D-module version of a E_2 category)

Have a version over configs of points

$KL(G)_X[Ran_X]$: specify object for every finite collection of points.

$$\begin{array}{c} KL(G)_{x,x} \\ \text{Loc} \big\downarrow \; \vdots \uparrow ? \\ \text{Dmod}(\text{Bun}_G)_{x,!} \end{array}$$

localization functor [analog of $\mathfrak{h}$-mod $\to$ Dmod(Y) for $H \circlearrowright Y$]

Here $G(K_x) \circlearrowright$ G-bundles with full level structure at x

$G(O_x)$-equivariant modules $\rightsquigarrow$ D-modules on Bun_G

Don't have a nice right adjoint to Loc

$C_1 \xrightarrow{F} C_2$ functor on compactly gen. categories

$\Rightarrow \quad C_1^\vee \xleftarrow{F^\vee} C_2^\vee$ dual functor

Dual functor to Loc:

$$\begin{array}{c} KL(G)_x \,[\text{Ran}_x] \\ \Gamma \big\uparrow \\ \text{Dmod}(\text{Bun}_G)_{x,*} \end{array}$$

Γ should be fully faithful

- equivalently Loc (Ran space version) should be a quotient....

Def κ is positive / negative / critical / irrational if κ restricted to any simple factor satisfies $\kappa + \frac{1}{2}\text{Killing} = c \cdot \text{Killing}$

with $c \in \mathbb{Q}^{>0}$ $\quad \mathbb{Q}^{<0}$ $\quad 0$ or irrational

Lemma Let κ be positive or irrational. Then

a. Loc sends compacts to compacts

$\rightsquigarrow$ nice right adjoint $\Gamma : \text{Dmod}_! \to KL[\text{Ran}]$

b. Γ is fully faithful

Lemma' Let κ be negative or irrational.

a. Γ admits a left adjoint $\text{Loc} : KL \to \text{Dmod}_*$ sending compacts to compacts

b. Γ is fully faithful embedding

Whittaker category $\quad Gr_{G,x} = G(K_x)/G(O_x)$

$\text{Whit}(G)_\kappa \subset \text{Dmod}(Gr_{G,x})_\kappa$

objects equivariant for $N(K_x)$ against a nondegenerate character $N(K_x) \longrightarrow \mathbb{G}_a$.

$$\begin{tikzcd} Gr_{G,x} \arrow[d] & \mathrm{Dmod}(Gr_G)_K \arrow[d, "\pi_*"] & \supset & \mathrm{Whit}(G)_K \arrow[dll, "\mathrm{Poinc}"] \\ Bun_G & \mathrm{Dmod}(Bun_G)_{K,*} & & \end{tikzcd}$$

$$\pi^! : \mathrm{Dmod}(Bun_G)_{x,!} \xrightarrow[\mathrm{coeff}]{} \mathrm{Whit}(G)_K$$

Whittaker coefficients / Poincaré series.

Would like coeff to be fully faithful, but it's not.....

$$\begin{tikzcd} \mathrm{Whit}(G)_K\,[\mathrm{Ran}] \arrow[r, "\mathrm{Poinc}"] & \mathrm{Dmod}(Bun_G)_{K,*} \\ & \mathrm{Dmod}(Bun_G)_{K,!} \arrow[ul, "\mathrm{coeff}"] \end{tikzcd}$$

<u>Define</u> $\mathrm{Dmod}(Bun_G)^{\mathrm{nondegen}}_{x,!} = \mathrm{Dmod}_! / \mathrm{Kernel}(\mathrm{coeff})$

<u>Conjecture</u> The resulting functor

$$\mathrm{Dmod}^{\mathrm{nondegen}} \longrightarrow \mathrm{Whit}(G)_x\,[\mathrm{Ran}]$$

is fully faithful

Def $\mathrm{Dmod}^{\mathrm{nondeg}}_{\kappa,x} = \langle \mathrm{Im}(\mathrm{Poinc}_!) \rangle$

Dual conjecture: $\mathrm{Poinc} : \mathrm{Whit}[\mathrm{Ran}] \longrightarrow \mathrm{Dmod}^{\mathrm{nondeg}}$ is fully faithful

Theorem (modulo a local conjecture) Assume κ positive or irrational
$\exists$ a functor $\Psi^{\kappa,-1/\kappa}_{G,G^\vee}$

$$
\begin{tikzcd}
\mathrm{Whit}(G)_\kappa \arrow[rrr, "\sim"] & & & \mathrm{KL}(G^\vee)_{-\frac{1}{\kappa}}[\mathrm{Ran}] \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{\kappa,!} \arrow[u, "\mathrm{coeff}"] \arrow[r] & \mathrm{D}^{\mathrm{nondeg}}_{G,\kappa,!} \arrow[ul] \arrow[r, "\simeq"] & \mathrm{D}^{\mathrm{nondeg}}_{G^\vee,-\frac{1}{\kappa},*} \arrow[r] & \mathrm{Dmod}(\mathrm{Bun}_{G^\vee})_{-\frac{1}{\kappa},*} \arrow[u, "\Gamma"] \\
\mathrm{KL}(G)_\kappa[\mathrm{Ran}] \arrow[u, "\mathrm{Loc}"] \arrow[rrr, "\sim", "\mathrm{Lurie}"'] & & & \mathrm{Whit}(G^\vee)_{-\frac{1}{\kappa}}[\mathrm{Ran}] \arrow[u, "\mathrm{Poinc}"'] \arrow[ul]
\end{tikzcd}
$$

equivalence of chiral categories

[Middle equivalence would follow from above conjecture]

Critical level

$$\kappa = -\tfrac{1}{2}\kappa_{\text{crit}}, \qquad -\tfrac{1}{\kappa} = \infty$$

$$\begin{tikzcd}
\mathrm{Dmod}(\mathrm{Bun}_{G^\vee})_{!,\infty} \arrow[rr, "\simeq"] \arrow[dr, "\sim"'] & & \mathrm{Dmod}(\mathrm{Bun}_{G^\vee})_{*,\infty} \arrow[dl, "\sim"] \\
& \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee}) &
\end{tikzcd}$$

[∞ behaves like an irrational level]

Identify everything at ∞:

$$KL(G^\vee)_\infty = \mathrm{Rep}\, G^\vee$$

$$\mathrm{Whit}(G^\vee)_\infty = \mathrm{QCoh}(\mathrm{Op}_{G^\vee,x})$$

opers on punctured disc, unramified as local systems

$$\mathrm{Loc} : \mathrm{Rep}\, G^\vee_x \longrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{G^\vee})$$

$$\shortparallel$$

$$p_x^* \qquad p_x : \mathrm{LocSys}_{G^\vee} \longrightarrow BG^\vee$$

$$\begin{tikzcd}
& \mathrm{Op}_{G^\vee}(X \setminus x) \arrow[dl] \arrow[dr] & \\
\mathrm{LocSys}_{G^\vee} & & \mathrm{Op}_{G^\vee,x}
\end{tikzcd}$$

<u>Conjecture</u> The functor $\mathrm{coeff}: QC(\mathrm{LocSys}_{G^\vee}) \to QC(\mathrm{Op}_{G^\vee})[R_{an}]$ is fully faithful

– ie at ∞ entire category is non degenerate

Equivalently $\mathrm{Poinc}: QCoh(\mathrm{Op}_{G^\vee})[R_{an}] \to QC(\mathrm{Loc}_{G^\vee})$ is a localization (quotienting)

$$\begin{tikzcd}
KL(G)_{crit} \arrow[rrr, "\sim"] \arrow[d, "\mathrm{Loc}"] & & & QC(\mathrm{Op}_{G^\vee})[R_{an}] \arrow[d, "\mathrm{Poinc}"] \\
\mathrm{Dmod}(\mathrm{Bun}_G)_! \arrow[r, two heads] \arrow[d, "\Gamma"] & \mathrm{Dmod}(\mathrm{Bun}_G)_!^{\mathrm{nondeg}} & & QC(\mathrm{Loc}_{G^\vee}) \arrow[ll, "\sim"'] \arrow[d, "\Gamma"] \\
\mathrm{Whit}(G) \arrow[rrr, "\sim"] & & & \mathrm{Rep}\, G^\vee [R_{an}]
\end{tikzcd}$$

$$\begin{tikzcd}
KL(G)_{crit} \arrow[rrr, "\simeq"] & & & QC(\mathrm{Op}_{G^\vee})[R_{an}] \\
\mathrm{Dmod}(\mathrm{Bun}_G) \arrow[u, "\Gamma"] & \mathrm{Dmod}(\mathrm{Bun}_G)_{*}^{\mathrm{nondeg}} \arrow[l, hook'] & & QC(\mathrm{Loc}_{G^\vee}) \arrow[ll, "\sim"'] \arrow[u, "\mathrm{coeff}"] \\
 & \mathrm{Whit}(G)[R_{an}] \arrow[u] & & \mathrm{Rep}\, G^\vee [R_{an}] \arrow[u, "\mathrm{Loc}"]
\end{tikzcd}$$

How to describe full categories of $\mathcal{D}$-modules, not just nondegenerate part?

Loc_G is a derived stack — so could redefine QC of it:

different variants: $QC(Y)^! = Ind(Coh\, Y)$

$\downarrow$ localization

$QC(Y)$

Ind(coherent) too big...

look for some intermediate version?