Jacob Lurie - Topological Geometric Langlands

Note Title 5/26/2009

(w/ D. Gaitsgory)

G algebraic group /ℂ, $G^\vee$ Langlands dual

X algebraic curve /ℂ

$Bun_G X$: stack of G-bundles

$Loc_{G^\vee} X$: derived stack of local systems

• "Geometric Langlands" : Conjecture

$$QCoh(Loc_G(X)) \simeq Dmod(Bun_{G^\vee} X)$$

[roughly speaking!!]

How do these categories depend on X?
not topological invariants... but on LHS
closely related to a topological invariant

$$Loc_G(X) \simeq BG^X = \{\pi_1 X \longrightarrow G\}/\sim$$

equivalence of underlying analytic spaces
... so these two are analogous...

Physics: Kapustin-Witten - geometric Langlands
comes from an equivalence of 4d TQFTs
"$N=4$ super Yang Mills in the GL twist"
for G & $G^\vee$.

	topological / Betti: regime	algebraic / deRham regime	physics
	$QCoh(BG^X)$	$QCoh(Loc_G X)$ $\wr$ $Dmod(Bun_{G^\vee} X)$	equivalence of 4d TQFTs
	$q = e^{2\pi i c}$	twisting parameter $c \in \mathbb{C}$	c or $E = \mathbb{C}/(1,c)$
TQFT?	Yes, for a boring $\mathcal{C}$	No	Maybe, but will need a very interesting target $\mathcal{C}$
	$q \longleftrightarrow e^{4\pi i/\log q}$????	$c \longleftrightarrow -\frac{1}{c}$	

<u>Quantum version</u> $Dmod(Bun_G) \simeq QCoh(Loc_{G^\vee})$

On $Bun_G X$ there's a canonical line bundle $\det^c$.

"Quantum geometric Langlands"

$$Dmod_c(Bun_G) \simeq Dmod_{-\frac{1}{c}}(Bun_{G^\vee})$$

$\mathrm{Bord}_n =$ Obj = oriented 0-manifolds

(∞, n) category   Mor = bordisms of 0-manifolds

$\vdots$

n-Mor : n-manifolds with corners

$(n+1)$ mor : diffeos

$(n+2)$ mor : isotopies

. . .

Def An (extended) TQFT is a $\otimes$-functor

$$\mathrm{Bord}_n \longrightarrow \mathcal{C}$$

Def If M is an m-manifold $(m \leq n)$, an n-framing of M is a trivialization

$$T_M \oplus \mathbb{R}^{n-m} \simeq \mathbb{R}^n$$

Bord_n^{fr} : use n-framed manifolds rather than oriented.

Theorem $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \simeq$ n-dualizable objects of $\mathcal{C}$

$$\underset{O(n)}{\circlearrowright} \qquad Z \longmapsto Z(*)$$

$\Rightarrow$ action of $O(n)$ on collection of n-dualizable objects of $\mathcal{C}$

[case $n=1$: $\mathbb{Z}/2$ action takes a dualizable object to its dual]

Theorem $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \simeq \left(\begin{matrix}n\text{-dualizable} \\ \text{objects in } \mathcal{C}\end{matrix}\right)^{SO(n)}$

Example of a 2-category:

Alg: Ob = $\mathbb{C}$-algebras

Mor = bimodules

2-Mor = maps of bimodules

Every object of Alg is one-dualizable

($A \longmapsto A^{op}$ opposite algebra)

2-dualizable $\Longleftrightarrow$ A semisimple !

dgAlg ... an $(\infty, 2)$ category

ob = dg algebras

Mor = dg bimodules

. . . .

higher morphisms = chain homotopies

2-dualizable $\Longleftrightarrow$ A smooth & proper (in sense of Kontsevich)

To increase "category level", replace dg Vect by dg categories

& also add an additional associative product:

categories w/ two associative products

$\rightsquigarrow$ braided monoidal category.

$\Rightarrow$ $(\infty, 4)$ category Braid

Ob: braided monoidal dg categories

Mor = monoidal categories in bi-modules

2 Mor : dg cats

3-Mor = functor

4-Mor = nat. transformations

. . .

Example of an object: $\mathrm{Rep}_q G$

braided monoidal (dg-) category

- Every object of Braid is 2-dualizable
- $\mathrm{Rep}_q G$ is 3-dualizable [$\Leftarrow$ rigidity] but not 4-dualizable.

What about $O(3)$ action?

$\mathrm{Rep}_q G$ is an SO_3 fixed point in Braid

($\Leftarrow$ $\underline{\text{ribbon}}$ category)

So get an oriented 3d TFT $\mathrm{Bord}_3 \to \mathrm{Braid}$.

$$Z(\Sigma) = \text{"} \bigotimes_{x \in \Sigma} \mathrm{Rep}_q G \text{"}$$ configs of points labelled by $\mathrm{Rep}_q G$.

defined using tensor product

example: $q=1$ $\mathrm{Rep}\, G$ is symmetric

$$\bigotimes_{x \in \Sigma} \mathrm{Rep}\, G \simeq QC(BG^{\Sigma})$$

$Z(S^2)$ = higher Drinfeld center of $\mathrm{Rep}_q G$

S^2 as algebraic curve:

- $S^2 \simeq \mathbb{P}^1$, study $\mathcal{D}(\mathrm{Bun}_G \mathbb{P}^1)$
- $S^2 = D^2 \amalg_{\dot{D}^2} D^2$ formal disc with double point at origin

… Bun_G here is $G(\mathbb{C}[[t]]) \backslash G(\mathbb{C}((t))) / G(\mathbb{C}[[t]])$

$= G(\mathbb{C}[[t]]) \backslash \mathrm{Gr}_G$ affine Grassmannian

So we're studying Satake/ spherical Hecke category $D(Gr_G)^{G[[t]]}$: monoidal category which acts on $D(Bun_G X)$ for any $x \in X$

Geometric Satake Isomorphism (Mirkovic-Vilonen)

$$D(Gr_G)^{G[[t]]} \simeq Rep\, G^\vee \text{ as abelian } \otimes \text{ category}$$

$$\simeq Z_{E_2}(Rep\, G^\vee) \text{ as derived categories}$$

[$D(Bun_G \mathbb{P}^1)$ roughly free module of rank 1]

For generic choice of c, analog of Satake $Z_{E_2}(Rep_q G)$ is very small: trivial for c generic.

What is algebro geometric analog of $Rep_q G$? Answer: $D(Gr_G)$

$D(Gr_G)$ is "like a braided monoidal category" ... use Beilinson-Drinfeld Grassmannian

$Gr_G(x,y)$ = G-bundles trivialized off x,y

How is $D_c(\mathrm{Gr}_G)$ like $\mathrm{Rep}_q G$?

Can simplify in two ways:

1 — consider $G[[t]]$-equivariant D-modules

<u>KL</u> $\rightsquigarrow$ reps of affine Lie algebra

$\xleftrightarrow[\text{Lusztig}]{\text{Kazhdan-}}$ $\mathrm{Rep}_q G$

local unramified Langlands correspondence

2 — consider Whittaker objects

<u>Whit</u> $\rightsquigarrow$ $\mathrm{Rep}_{q'} G^\vee$

So now try to reconstruct what we assign to a surface via integration

$$\begin{tikzcd} \bigotimes_{x\in X} \mathrm{Whit} & & \bigotimes_{x\in X} \mathrm{KL} \\ & D_c(\mathrm{Bun}_G X) \arrow[ul] \arrow[ur] & \text{fully faithful} \end{tikzcd}$$

is this fully faithful?

would give geometric Langlands... something like this should be true....