# David Nadler - Character Sheaves & TFT

Note Title 5/26/2009

(jt w/ D. Ben-Zvi) Northwestern TFT fest
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<u>Motivation</u>: Place representation theory of Lie groups in context of 3d TFT
(& relation to 4d TFT & geometric Langlands)

<u>Application</u>: duality for character sheaves

Outline: I. Review of character sheaves
II. TFT interpretation & duality
III. Relation to geometric Langlands

Ground rules: everything is derived

I. Fix $G$ complex reductive ($GL_n, SL_n, T, \ldots$)

$\frac{G}{G}$ adjoint quotient stack ($\rightsquigarrow$ study $G$ in conjugation equivariant way)

D-modules on $\frac{G}{G}$ ... ?

D-modules: $X$ smooth scheme / $\mathbb{C}$ (projective variety)

<u>Def</u> A D-module on $X$ is a quasicoherent sheaf $M$ on $X$ with a compatible action of ring of differential operators $D_X$

<u>Interpretations</u>: 1) $M$ is a quasicoherent sheaf with infinitesimal parallel transport

Examples: vector bundles w/ flat connection

2) $M$ is a "noncommutative module" on $T^*X$
... draw modules on $T^*X$ & ask which ones quantize ... roughly coisotropic submanifolds

3) $M$ is an A-brane on $T^*X$ (holomorphic)

<u>Def</u> $M$ is <u>holonomic</u> if its support on $T^*X$ ["singular support"] is Lagrangian [& fin. generated]

— Smallest $\mathcal{D}$-modules

<u>Theorem</u> $X, Y$ schemes over $Z$, smooth

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd} \qquad \mathcal{D}(X) \underset{\mathcal{D}(Z)}{\otimes} \mathcal{D}(Y) = \mathcal{D}(X \times_Z Y)$$
$$= \mathrm{Fun}_{\mathcal{D}(Z)}(\mathcal{D}(X), \mathcal{D}(Y))$$

- $\mathcal{D}$ modules on scheme can be made up nicely from local pieces.

<u>Exercise</u> $G$ reductive group$/\mathbb{C}$. Calculate
$\mathcal{D}(BG) = C_{-*}(G)$-mod   modules for chains on $G$
Justify!

<u>Back to $\frac{G}{G}$</u>  Character sheaves $Ch_G \subset \mathcal{D}(\frac{G}{G})$ full subcategory

... study NC geometry of $T^*G = G \times \mathfrak{g}^* \overset{\text{Killing}}{\simeq} G \times \mathfrak{g}$

$G \times \mathfrak{g} \supset G \times \mathcal{N}$ favorite picture

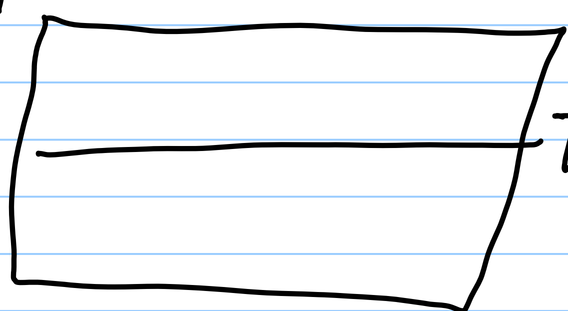
$G \times$ nilpotent cone

<u>Def</u> $Ch_G = \{ M \in D(\frac{G}{G}) : \text{sing support of } M \subset \mathcal{N} \}$

(Theorem of Mirković–Vilonen)

<u>Examples</u> 1. $G = T$ torus $\quad T \circlearrowleft_{adj} T$ trivial action

$$\frac{T}{T} = T \times BT$$



$T^*T$ : no nilpotents

$T$

$\Rightarrow$ support lies fully in 0-section

$\Rightarrow$ <u>local system</u> on $T$

$D$-mods module on $BT$ :

$Ch_T$ = local systems on $T \otimes C_{-*}(T)$-mod

$\cup$

$Ch_T^u$ unipotent character sheaves: generated by constant local system

$= C^*(T) \otimes C_{-*}(T)$-module

2. Construct our favorite character sheaf (unipotent) on $G$: the Springer sheaf

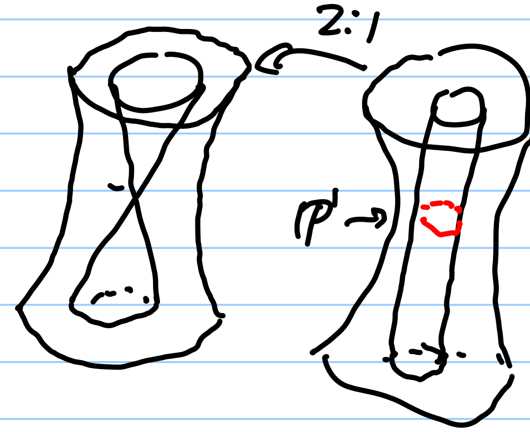
$$\frac{\tilde{G}}{G} \xrightarrow{\mu} \frac{G}{G}$$

Grothendieck–Springer resolution

$\tilde{G}$ = pairs $\{g \in B\} \subset G \times \{\text{flag variety}\}$

$SL_2$: nbhd of identity

Springer sheaf $= \mu_* \mathbb{C}_{\tilde{G}/G}$ is a character sheaf




Lusztig's original definition/construction:

$$B\backslash G/B \xleftarrow{\delta} \frac{G}{B} \xrightarrow{p} \frac{G}{G}$$

$$D(B\backslash G/B) \xrightarrow{p_*\delta^!} D\left(\frac{G}{G}\right)$$

fin many iso classes of objects $\longleftrightarrow W$
all D-modules are holonomic

All character sheaves arise by correspondence above

Goal: understand character sheaves categorically in terms of $D(B\backslash G/B)$ = Hecke category $H_G$

1. - monoidal dg category

B G B

$B\backslash G/B$ = G bundles on interval with B reductions

2. $H_G$ is made of holonomic D-modules

3. Kazhdan-Lusztig duality: composition of Verdier duality with flipping of interval $g \mapsto g^{-1}$

Role of $H_G$ in representation theory: natural symmetries of $\mathfrak{g}$-modules

<u>Theorem</u> There is a (partial, ie 0,1,2) 3d TFT $\longleftrightarrow$ categorified 2d TFT $\mathcal{X}_G$ (the character theory) s.t.

$\mathcal{X}_G(\bullet) =$ 2-category of $H_G$-modules (eg $D(Y/B)$ for any $Y$ with $G$-action ... includes many interesting categories of representations)

$\mathcal{X}_G(S^1) =$ unipotent character sheaves

$\mathcal{X}_G(S^2) = H_T^*(pt)$

<u>Implications</u>

- gives internal characterization of $Ch_G$ in terms of $H_G$:

$$Ch_G = \left\{ \begin{array}{l} Z \text{ Drinfeld center} \\ \shortparallel \\ THH^* \\ HH_* \end{array} \right\} (H_G)$$

$\Rightarrow$ character sheaves carry rich operadic structure

<u>Langlands duality</u> $\Rightarrow$ relation between field theories associated to $G$ & $G^\vee$

$\chi_G$ unipotent character theory for $G$ associated to $H_G = D(B\backslash G/B)$

$\chi_{G^\vee}^{mon}$ character theory assoc to $H_{G^\vee}^{mon} \subset D(U\backslash G^\vee/U)$

$$H_G \simeq H_{G^\vee}^{mon} \quad \text{2-periodically : Beilinson - Ginzburg - Soergel}$$

$\Rightarrow$ equivalence of (2-periodic) character theories

$$\chi_G \simeq \chi_{G^\vee}^{mon}$$

<u>Cor</u> $Ch_G$ & $Ch_{G^\vee}$ are equivalent (2-periodically)

<u>Example</u> $G = T$, $G^\vee = T^\vee$

$$C^*(T) \otimes C_*(T)\text{-mod} \quad \text{vs} \quad C^*(T^\vee) \otimes C_{-*}(T^\vee)\text{-mod}$$

$\mathbb{Z}/2$ periodically