Bertrand Toën - Secondary K-Theory

Note Title 5/18/2009

Secondary K-theory = "categorified" version of algebraic K-theory

- replace vector bundles by a categorical version

$K^{(2)}(k)$: secondary K-theory of the ring k

- contains "higher" invariants of $X = \operatorname{Spec} k$

eg • $H^1(X, \mathbb{G}_m) \simeq \operatorname{Pic} X \longrightarrow K_0(X)$

likewise $H^2_{et}(X, \mathbb{G}_m) \longrightarrow K_0^{(2)}(k)$

Another motivation: $K^{(2)}(k)$ seems close to a K-group of motives / Spec k in the noncommutative setting

(eg we will see the de Rham realization)

<u>Remark</u> The categorification process makes $K^{(2)}(k)$ more "discrete" than $K(k)$ ... relations with things like K-group of constructible sheaves.

<u>Definition of $K_0^{(2)}(k)$</u>

k commutative ring

$C(k)$ = category of complexes of k-modules

<u>Def</u> $dgcat/_k$ = category of $C(k)$-enriched categories

We will use dg cat/k to construct a symmetric monoidal category $H_0^{Mor}(\text{dgcat}/k)$ — version for us of the $\otimes$-category k-Mod.

$T \in$ dg cat/k

T-dg mod $=$ (left) dg modules $/T$
$= \mathrm{Hom}(T, C(k))$

Quasiisomorphisms in T-dg mod: morphisms which induce quasi isos in $C(k)$:

$f : E \to E' : \ \forall x \in T \quad E_x \longrightarrow E'_x$ is a quasiisomorphism

<u>Def</u> $D(T) = [\text{quasi isom}]^{-1}(T\text{-dgmod})$

<u>Def</u> $(f : T \longrightarrow T') \in$ dg cat_k is a Morita equivalence if
$f^* : D(T') \longrightarrow D(T)$ is an equivalence.

<u>Def</u> $H_0^{mor}(\text{dgcat}/k) := [\text{Morita equiv}]^{-1}(\text{dgcat}/k)$

<u>Theorem</u> $T, T' \in$ dgcat$/k$ (& T has flat homs$/k$)

$$\Rightarrow [T,T']_{H_0^{Mor}} \simeq \left\{ \begin{array}{l} E \in D(T \otimes T'^{op}) : \\ \forall x \in T \ \ E_x \in \underset{\text{compact}}{D_c}(T'^{op}) \end{array} \right\} \Big/ \sim$$

Remark D a triangulated category with ∞ sums, an object x is compact if $[x, \bigoplus_\alpha Y_\alpha] = \bigoplus_\alpha [x, Y_\alpha]$

Remark $Ho^{Mor}(dgcat)$ is the truncation of a natural 2-category [or really $(\infty,2)$-cat] with same objects

& $\underline{Hom}(T,T') = \left\{ \begin{array}{l} E \in D(T \otimes T'^{op}) \\ E_x \text{ compact } \forall x \end{array} \right\} \subset D(T \otimes T'^{op})$

$T, T' \in dgcat/k$

$T \underset{k}{\otimes} T' \in dgcat/k$:

$ob(T \underset{k}{\otimes} T') = ob\,T \times ob\,T'$,

morphisms are $\underset{k}{\otimes}$ of morphisms.

$\rightsquigarrow$ $dgcat/k$ is a closed symmetric monoidal cat,

$\rightsquigarrow$ induces a closed sym monoidal structure

$T \overset{\mathbb{L}}{\underset{k}{\otimes}} T'$ on $Ho^{Mor}(dgcat/k)$

Localisation T a dgcat $/k$

$[T]$: homotopy category, same objects

$[T](x,y) = H^0(T(x,y))$

$S \subset [T]$ subset of morphism.

<u>Def</u> A localisation of T along S is $T \xrightarrow{\ell} L_S T$ in $Ho^{Mor}(dg\,cat/k)$ s.t.

$$\forall T', \quad [L_S T, T'] \xhookrightarrow{\ell^*} [T, T']$$

& image consists of all morphisms $f : T \to T'$ with

$$\begin{tikzcd} {[f] : [T]} \arrow[r] \arrow[d, phantom, "\cup"] & {[T']} \\ S^{-1}[T] \arrow[ur] & \end{tikzcd}$$

<u>Prop</u> Localisations always exist.

<u>Examples of dg categories</u>

- $C_{per}(k) \ni dg\,cat/k$ perfect complexes

$$L(k) := L_{quasi\ isom}\, C_{per}(k) \in Ho^{Mor}(dg\,cat/k)$$

- X scheme$/k$

quasi isom $\subset Perf(X)$ = dg cat of perfect complexes

$$L_{pe}(X) := L_{quasi\ isom}\, Perf(X).$$

$$[L_{pe}(X)] = D_{per}(X), \quad L_{pe}(T) : D_c(T)$$ compact objs.

$$L_{pe}(X)(E,E') \simeq \underline{\mathbb{R}\mathrm{Hom}}(E,E')$$

Analogy:

$Ho^{Mor}(dg\,Cat/k)$	k-Mod
$\overset{\mathbb{L}}{\otimes}_k$	$\otimes_k$
$\otimes$-closed	$\otimes$-closed
saturated dg cats = dualizable objects	Proj. modules of finite type = dualizable objects in k-mod
$T \hookrightarrow T' \to T/T'$ fully faithful	short exact sequences

$$K_0^{(2)}(k) = \frac{\mathbb{Z}[\text{saturated dg cat}]}{T' = T + T/T'}$$

Example X smooth proper $/k$ $\Rightarrow$ $L_{pe}(X)$ saturated

$\Rightarrow [L_{pe}(X)] \in K_0^{(2)}(k)$

$$K_0(Var/_k) \xrightarrow{\text{Bondal-Larsen-Lunts}} K_0^{(2)}(k)$$

$\shortparallel$

$\mathbb{Z}[\text{varieties}/k] \Big/ X = Y + U \quad Y \subset X$ closed; $U = X \setminus Y$

Next lecture: a few more examples.

Ultimate goal: try to understand Chern character

Lecture II

$H_0^{Mor}(\text{dgcat}/k, \overset{L}{\underset{k}{\otimes}})$: $[T,T'] = \left\{ \begin{array}{l} E \in D(T \otimes T'^{op}) \\ E_x \text{ compact } \forall x \end{array} \right\}$

$\cap$

$H_0^{ct}(\text{dgcat}/k, \overset{L}{\underset{k}{\otimes}})$: same objects
$[T,T'] = D(T \otimes T'^{op})$

rigid $\otimes$ category: every object has a dual in here.

In H_0^{Mor} dualizable objects are <u>saturated</u> dg categories

<u>Prop</u> $T \in H_0^{Mor}(\text{dgcat}/k)$ is saturated

$\Longleftrightarrow$ T is equivalent to B, a dg alg$/k$

s.t. B is compact $\in D(k)$ ("proper")

& in $D(B \otimes B^{op})$ ("smooth").

<u>Example</u> $\alpha \in H^2_{et}(X, \mathbb{G}_m)$ $X = \operatorname{Spec} k$

$\rightsquigarrow R_\alpha$ Azumaya k-algebra, smooth & proper

$\longrightarrow \operatorname{Lpe}(R_\alpha)$ saturated

$\Rightarrow$ map $H^2_{et}(X, \mathbb{G}_m) \longrightarrow K_0^{(2)}(k)$

More generally: A k-algebra, projective of finite type $/k$

$\Rightarrow L$ field $k \to L$,

$A \otimes_k L$ has finite homological dim,

$\rightsquigarrow \operatorname{Lpe}(A)$ saturated.

Basic properties

- $k \longmapsto K_0^{(2)}(k)$ is a functor in k (via $\otimes_k k'$ on dg categories)
- $K_0^{(2)}(k) = \pi_0(K^{(2)}(k))$ secondary K-theory spectrum ... in fact commut. ring spectrum
- $k = \operatorname{colim} k_\alpha$ filtered colimit $\Rightarrow$ $K^{(2)}(k) = \operatorname{colim} K^{(2)}(k_\alpha)$ (continuity)

We want to extend $K^{(2)}(f)$ to schemes & stacks
& describe $ch : K^{(2)}(X) \to ?$
using the cobordism hypothesis [de Rham realization]

Segal categories

$(1,\infty)$-categories: ∞-categories where
n-morphisms are invertible $n>1$
$= \infty$-categories whose Homs are ∞-groupoids

∞-groupoids $\longleftrightarrow$ homotopy types
$= SSet$

Naive definition: a $(1,\infty)$-category is
a $SSet$-enriched category

... too naive: get bad notions of categories of functors

Def A Segal category is a functor

$$A : \Delta^{op} \longrightarrow SSet \quad \text{s.t.}$$

- $A(0)$ is a discrete simplicial set (i.e. set)
- $\forall n \quad A(n) \longrightarrow A_1 \times_{A_0} \cdots \times_{A_0} A_1$
  is a weak equivalence of simplicial sets.

Remark An S-category A gives a Segal category

$A : \Delta^{op} \to SSet, \quad n \longmapsto \coprod_{(a_0,\dots a_n)} A(a_0,a_1) \times \dots \times A(a_{n-1},a_n)$

$\Rightarrow$ S-cat $\hookrightarrow$ Segal Cat $\underset{\text{full}}{\subset}$ bisimplicial sets

A Segal category A comes from an S-category iff $A_n \longrightarrow A_1 \times_{A_0} \dots \times_{A_0} A_1$ are isomorphisms.

Def A an ∞-category, $[A]$ = category with same objects & $[A](x,y) = \pi_0 A(x,y)$

... homotopy category or 1-truncation

- $f: A \to B$ is fully faithful if $A(x,y) \longrightarrow B(fx, fy)$

essentially surjective if $[A] \to [B]$ is so.

Main Theorem (Hirschowitz-Simpson / Pellissier / Bergner)

There exists a model structure on

$$\{ F: \Delta^{op} \longrightarrow SSet : F_0 \text{ discrete} \}$$

s.t. • Weak equiv $\cap$ {Segal cat} are fully faithful ess. surj. functors

- cofibrations = monomorphisms
- fibrant objects are Segal categories (with Reedy fibrancy condition)
- It is a $\otimes$-model cat for the direct product.

- $S\text{-Cat} \longrightarrow$ Pre Segal categories "is" a Quillen equivalence

$\Rightarrow \quad Ho(S\text{-Cat}) \simeq Ho(\infty\text{-cat})$

Consequence: $\mathbb{R}Hom(A,B) = Hom(A, RB)$

RB = fibrant replacement for B

Fibrant Segal categories give a category enriched (in strict sense) over ∞-categories $\rightsquigarrow$ examples of a $(2,\infty)$-category

<u>Localization</u> $\quad S \subset [A]$

$A \rightsquigarrow L_S A$ satisfies universal property of inverting morphisms in S.

eg $A = M$ a model category, S = Weak eqs

$\Rightarrow \quad L_W M = M^{\text{fib, cofib}}$ as ∞-categories

(for M nice enough)

$L_W M(x,y) = Map_{M^{f,c}}(x,y)$

- A any category, $\mathbb{R}Hom(A, L_W M) \simeq L_W(M^A)$

$\simeq \quad (M^A)^{\text{fib, cofib}}$

Part III

$(\mathcal{C}, w)$ category w/ weak equivalences
$\rightsquigarrow L_w\mathcal{C} \in \infty$-cat

Example $(\mathrm{Perf}(k), q\text{-isom}) \rightsquigarrow L_{pe}(k) \in$ dgcat
$\qquad\qquad\qquad\qquad\qquad\qquad L\mathrm{Perf}(k) \in \infty$-cat

$(\mathrm{dgcat}/k, \text{Morita eq}) \rightsquigarrow \mathbb{D}_g^{Mor}(k) \in \infty$-cat
- ∞ category of dg categories

$[L\mathrm{Perf}(k)] \simeq D_{perf}(k)$
$[\mathbb{D}_g^{Mor}(k)] \simeq H_0^{Mor}(\mathrm{dgcat}/k)$

$\pi_i(L\mathrm{Perf}(k)(E,F), \bullet) \simeq \mathrm{Ext}^{-i}_{D(k)}(E,F)$

$T, T' \in \mathbb{D}_g^{Mor}(k)$,
$\pi_1(\mathbb{D}_g^{Mor}(T,T'), E) \simeq \mathrm{Aut}\, E$
$E \in D(T \otimes T')$ bimodule

$\pi_i(\mathbb{D}_g^{Mor}(T,T'), E) \simeq \mathrm{Ext}^{1-i}_{D(T\otimes T')}(E,E)$

$\Rightarrow \pi_i(\mathbb{D}_g^{Mor}(T,T)) \simeq HH^{1-i}(T)$ Hochschild cohomology
$:= \mathrm{Ext}^{1-i}(T,T)$ ext of bimodule

So existence of negative Hochschild cohomology of T is related to higher homotopy groups – ie obstructs T living in a 2-category or 3-category or... inside the ambient ∞-category.

(∞-cat, equiv) ⇝ ∞-Cat, the $(1,\infty)$-cat. of $(1,\infty)$ categories (really underlying a $(2,\infty)$ category)

Adjoints & limits $\quad f: A \to B$ in ∞-cat

Def f has a right adjoint if

$$\exists\, g: B \longrightarrow RA \hookleftarrow A$$ fibrant model for A

$$+\ u \in \underline{\mathrm{Hom}}(A, RA)(i, gf)$$

s.t. $\forall a \in A,\ b \in B$

$$\begin{tikzcd} B(f(a),b) \arrow[r] \arrow[dr, "\text{weak equiv}"'] & RA(gf(a), g(b)) \arrow[d] \\ & RA(i(a), g(b)) \end{tikzcd}$$

⇝ notion of lim & colim in a given ∞-cat

A ∈ ∞-cat $\quad$ I ∈ cat

$$\begin{tikzcd} A \arrow[r, "\text{const}"'] & R\mathrm{Hom}(I, A) \arrow[l, bend right, "\lim = \text{right adjoint}"'] \arrow[l, bend left, "\mathrm{colim} = \text{left adj}"] \end{tikzcd}$$

$\mathbb{D}_g^{\mathrm{Mor}}(X)$, $\mathbb{D}_g^{ct}(X)$, $\mathbb{D}_g^{\mathrm{Sat}}(X)$ for

X any scheme or stack: defined as limit over affines

$$\mathbb{D}_g^{\mathrm{Mor}}(X) = \lim_{(\mathrm{Spec}\, k \to X)} \mathbb{D}_g^{\mathrm{Mor}}(k) \in \infty\text{-cat}$$

$$= \Gamma(X, \underline{\mathbb{D}_g}^{\mathrm{Mor}})$$

eg $\mathrm{End}_{\mathbb{D}_g^{ct}(X)}(\mathbb{1}) = LQCoh(X)$ ..

Idea of construction of Chern character:

$T \in \mathbb{D}_g^{ct}(X)$: T = sheaf of dg categories$/X$

Consider the "loop space" $\mathcal{L}X \xrightarrow{\pi} X$
"$\mathrm{Hom}(S^1, X)$"

$\pi^* T$ comes equipped with a natural "monodromy" autoequivalence $m : \pi^* T \circlearrowleft$

- monodromy along the loop ...

$\mathbb{D}_g^{cf}(Y)$ is a rigid $\otimes$-category

$\Rightarrow \mathrm{Tr}(m) \in \mathbb{D}_g^{cf}(\mathcal{L}X)(\mathbb{1},\mathbb{1})$

$= LQcoh(\mathcal{L}X)$

<u>Key result</u> $\mathrm{Tr}(m)$ is S^1-invariant

- follows from the cobordism hypothesis in dimension one.

<u>Remark</u> Can replace $\mathbb{D}_g^{cf}(-)$ by any functor "Schemes" $\xrightarrow{\mathcal{A}}$ rigid monoidal ∞-cat

$\Rightarrow \mathcal{A}(X) \xrightarrow[ch]{} \mathrm{End}_{\mathcal{A}(\mathcal{L}X)}(\mathbb{1})^{S^1}$

eg $\mathcal{A} = LPerf \Rightarrow$ RHS is negative cyclic homology of X, & this is the Chern character.

<u>Remark</u> $Ch(T) \in D_{qcoh}^{S^1}(\mathcal{L}X)$

" $\simeq$ " $D(\mathcal{D}_X\text{-mod})$ de Rham realization

Theorem (Ben-Zvi/Nadler)

$\otimes$ ∞-categories : [symmetric everywhere]

a "commutative monoidal object in ∞-cats"

... eg a commutative object in Set
is a Γ-object in Set

Γ = category of pointed finite sets

$\Gamma \xrightarrow[M]{} Set$ satisfying Segal conditions

$$\begin{cases} M_0 = * \\ M_n \xrightarrow{\sim} M_1^n . \end{cases}$$

$\rightsquigarrow$ A $\otimes$-∞ cat is a functor

$A : \Gamma \longrightarrow \infty\text{-cat}$

s.t.
$$\begin{cases} A_0 \xrightarrow{\sim} * \\ A_m \xrightarrow{\sim} A_1 \times \dots \times A_1 \end{cases}$$
equivalences of ∞-categories

$\otimes$ ∞ cats form an ∞-category

$\underline{\infty\text{-cat}^{\otimes}}$

<u>Remark</u> $\otimes$-∞-cat can be obtained by localizations of $\otimes$-cats

<u>Example</u> dgcat/k, $\otimes_k$, Morita

$\rightsquigarrow$ get monoidal ∞-category of dgcats.