Vienna Geometric Langlands

Note Title 1/8/2007

The geometric Langlands program provides the geometric harmonic analysis for moduli of bundles on curves.

<u>GL_1</u> C smooth projective curve/$\mathbb{C}$

$\mathrm{Pic}\, C = \mathrm{Bun}_{GL_1} C$ is an <u>abelian group</u> under $\otimes$.

Fourier transform / Pontrjagin duality:

- identify characters (e^{tx} for $\mathbb{R}$)
- Write universal character
  ($\hat{\mathbb{R}} = \{t\}$ e^{itx} fn on $\mathbb{R} \times \hat{\mathbb{R}}$)
- Use it as integral transform to identify $\mathrm{Fun}(\mathbb{R}) \cong \mathrm{Fun}(\hat{\mathbb{R}})$, i.e. write all functions as combinations of characters

Geometric harmonic analysis: substitute some kind of sheaves for functions, e.g. coherent sheaves (modules over $\mathcal{O}$) or <u>$\mathcal{D}$-modules</u>, modules over algebraic diff operators

- $e^{tx} \rightsquigarrow \mathcal{D}\cdot e^{tx} = \mathcal{D}/\mathcal{D}(\partial - t)$
  $\mathcal{D}$-modules capture functions by the equations they satisfy

- $\mathcal{D}$-modules $\Longleftrightarrow$ sheaves w/ flat connection
  $(\mathcal{E}, \nabla)$ ∇ gives action of T_X, extends to $\mathcal{D}$

- $\mathcal{D}$ deformed version of $\mathrm{Sym}\, T_X$ = functions on cotangent bundle ... $\mathcal{D}$-modules are quantized sheaves on cotangent bundle.

<u>Identity characters</u>: $x \in \mathrm{Jac}\,X = \mathrm{Pic}^0 X \longmapsto$ line L_x
s.t. $L_{x+y} = L_x \otimes L_y$, and vary algebraically in x

Such $L \Longleftrightarrow$ points in dual abelian variety which is again $\mathrm{Jac}\,X$.

Universal character: Poincaré sheaf
$$\begin{array}{ccc} & \mathcal{P} & \\ & \downarrow & \\ & \mathrm{Jac} \times \mathrm{Jac} & \\ \mathrm{Jac} \xleftarrow{\pi_1} & & \xrightarrow{\pi_2} \mathrm{Jac} \end{array}$$

Fourier-Mukai transform:
$$\mathcal{F} \longmapsto \pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{P})$$
$$D^b(\mathrm{Jac}) \xrightarrow{\sim} D^b(\mathrm{Jac})$$

Similar story if ask for L_x to vary <u>flatly</u>:
Now L form flat line bundles on Jac
Loc = flat line bundles on Jac, or on X: deforms $T^*\mathrm{Jac}$

$$\begin{array}{ccc} \mathcal{P}_{\mathcal{P}} & & \\ \downarrow & & Loc \\ \times & & \downarrow \\ Jac & & Jac \end{array}$$

Flat Fourier-Mukai (Laumon-Rothstein)

$$D^b(\mathcal{D}_{Jac}\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{O}_{Loc}\text{-mod})$$

deformed doublet F-M transform

Langlands: nonabelian harmonic analysis.

G reductive group, $Bun_G C$ = moduli of G-bundles
- no longer abelian, or a group!

What should replace group structure?

Two answers: • Hecke operators
• Hitchin's abelianization

<u>Hecke operators</u>: have many <u>correspondences</u> acting on $Bun_G C$. In case of GL_1:

$x \in C$, $n \in \mathbb{Z}$ ($= Rep\, GL_1$) have

$$\mathrm{Pic}\, X \xrightarrow{h_x} \mathrm{Pic}\, X, \qquad \text{generate divisors}$$
$$\mathcal{L} \longmapsto \mathcal{L}(nx) \qquad \Rightarrow \text{group structure}$$

All together: $Hecke_x = \{\mathcal{L}, \mathcal{M} : \mathcal{L}|_{X_x} \cong \mathcal{M}|_{X_x}\}$

$$\mathcal{L}\ \ \mathrm{Pic} \longleftarrow Hecke_x \longrightarrow \mathrm{Pic}\ \ \mathcal{M}$$

Geometric Satake theorem: the nonabelian version of these symmetries define for $x \in X$, $V \in \operatorname{Rep} {}^L G$ a functor

$$H_{V,x} : D_{Bun_G} \longrightarrow D_{Bun_G} \quad \left[\begin{array}{c}\text{vary flatly} \\ \text{in } x\end{array}\right]$$

(though not a map on spaces)

- These operators compose as $\otimes$ of reps (ie action of $\operatorname{Rep} {}^L G$) & commute for $x \neq y$.

${}^L G$: defined by geometry of modifications: reductive group with dual root data
($GL_n \circlearrowleft$, $SL_n \longleftrightarrow PGL_n$, $Sp_n \longleftrightarrow SO_{2n+1}$
$Spin_{2n} \longleftrightarrow SO_{2n}/\mu_2$ ......)

So can try to diagonalize all these commuting operators! ie make all the $H_{V,x}$ into multiplication operators
Natural parameter space for "eigenvalues":
$Loc_{{}^L G} C$ = space of flat ${}^L G$ bundles E

$D(Loc_{{}^L G}, \mathcal{O})$ carries multiplication operators $W_{V,x}$:
$x \in X$ $V \in \operatorname{Rep} {}^L G \Longrightarrow$

$$\begin{array}{lccc} & \mathcal{W}_{V,x} & \supset & (E)_V|_x \\ \text{vector bundle} & \downarrow & & \downarrow \\ & Loc_{^LG} & \ni & E \end{array}$$

: E parameterises an "eigenvalue"
$x, V \longmapsto$ Vector space

<u>Geometric Langlands conjecture</u>

$$\begin{array}{ccc} D^b(Bun_G, \mathcal{D}) & \xrightarrow{\sim} & D^b(Loc_{^LG}, \mathcal{O}) \\ \circlearrowright & & \circlearrowright \\ \mathcal{H}_{V,x} & & \otimes \mathcal{W}_{V,x} \end{array}$$

<u>ie</u> $\mathcal{A}_E \longleftarrow\!\!\!\shortmid\; E \in Loc,\ \mathcal{O}_E$ skyscraper

Hecke eigensheaf

Deformed version of T^*Bun_G $\longleftrightarrow$ Deformed version of $T^*Bun_{^LG}$

<u>Abelianization</u>

dual abelian group stacks

$$\begin{tikzcd} T^*Bun_G \arrow[drr, "H"'] & \supset \mathcal{A} \arrow[dr] & & \mathcal{A}^\vee \arrow[dl] \subset & T^*Bun_{^LG} \arrow[dll] \\ & & Hitch^{reg} \subset Hitch & & \end{tikzcd}$$

- analog of locus $G^{rss} \subset G$ of elements whose centralizer is a maximal torus of G.

GL_n: $T^*Bun_{GL_n}$ = vector bundles $\mathcal{E}$ + liftings of $\mathcal{E}$ to sheaf on T^*C

$\cup$

$\left\{ \text{line bundles on smooth curves } \begin{array}{c} \tilde{C} \subset T^*C \\ \searrow_n \ \downarrow \\ C \end{array} \right\}$

Hitch = (linear series of) curves in T^*C

Fibers of H over $\tilde{C}$ is $\mathrm{Pic}\,\tilde{C}$

Donagi - Pantev: $D^b(\mathcal{A}, \mathcal{O}) \simeq D^b(\mathcal{A}^\vee, \mathcal{O})$

"classical limit" of geometric Langlands conjecture (on open locus $\mathcal{A}$)

.... Hecke symmetries reduce to group structure on $\mathcal{A}$.

<u>TFT</u>: T^*Bun_G is a Calabi-Yau

$\rightsquigarrow$ defines topological field theories (A-, B-models of string theory).

In fact $D(\mathrm{Bun}_G, \mathcal{D})$, $D(\mathrm{Loc}_{{}^LG}, \mathcal{O})$, $D(T^*\mathrm{Bun}_G, \mathcal{O})$

CY categories (NC CY)

& Costello: NC CY $\Longleftrightarrow$ 2d (oc) TFT

So we have assigned $C \longrightarrow$ 2d TFT:

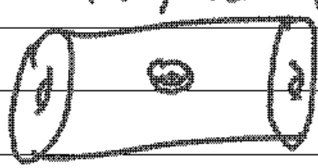
really should be part of 4d TFT

associated to $M^4 = C \times \Sigma$.

Hecke operators fit in naturally:

Hecke operators at x $\cong$ NC CY assigned to $C = S^2$

act on theory for C via



Kapustin-Witten: indeed have 4d TFT

($N=4$ SYM in GL twist) producing this

geometry, & geometric Langlands equivalence

comes from Montonen-Olive/electric-magnetic/S-duality

identifying theories for G, ${}^LG$.