Overview

Joint work with David Nadler (Northwestern)
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- Motivations from representation theory:
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- New organizing principle for representations of Lie groups via topological field theory
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  - See also arXiv:0805.0157 (with J. Francis) and arXiv:0904.1247.
Loop Spaces

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- Fixed points are constant loops $X \subset LX$. 
Loop Spaces in Physics

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- Rotation of loops key to E. Witten’s work on Atiyah-Singer index theorem and elliptic genus.
- The main tool here is equivariant localization.
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- **Definition**: Equivariant cohomology $H^*_{S^1}(X)$ is ordinary cohomology of the quotient $(M \times S^\infty)/S^1$. 
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- **Theorem:** The equivariant cohomology $H^*_{S^1}(M)$, after inverting $u$, coincides with the cohomology of the fixed points $H^*(M^{S^1})$ tensored by $\mathbb{Z}[u, u^{-1}]$. 
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...relate equivariant cohomology of loop space $LX$ with cohomology of $X$...
An Algebraic Take on Loop Spaces

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**Theorem (J. Jones)** The Hochschild homology of cochains on $X$ is $H^*(LX)$, and the cyclic homology of cochains on $X$ is $H_{S^1}^*(LX)$. 
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- Result: “derived loop spaces”
Why Derived Loop Spaces?

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- derived loop picture explains mysterious structures - e.g. Deligne conjecture
More Motivation

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- Same in geometric Langlands program (BZ-Nadler):
  Derived loops in flag manifolds on “Galois side” are Langlands dual to loop groups on “automorphic side”
Context

Our results hold for algebraic varieties or stacks.
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Basic idea: get a different notion of loop space by changing POV on $S^1$: 
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Basic idea: get a different notion of loop space by changing POV on $S^1$: replace $S^1$ by its combinatorial or algebraic avatars from homotopy theory.
Simplicial approach

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  ..so consider self-intersection $\Delta \cap \Delta$ in $M \times M$..
\( \Delta \cap \Delta \) very nontransverse intersection..expected dimension 0
Simplicial approach II

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- Result: $\mathcal{L}M$ is the odd tangent bundle, i.e., $TM$ considered as a supermanifold.
Functions on $\mathcal{LM}$

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- Calculation (Hochschild-Kostant-Rosenberg): For $R = \text{Fun}(M)$, get $\Omega^\bullet(M)$, exterior algebra of differential forms..
Cohomological approach

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Loop Spaces and Connections -- p. 14
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- Recall: tangent vectors are paths $\mathbb{R} \to M$ modulo $\epsilon^2$ terms.
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- Recall: tangent vectors are paths $\mathbb{R} \to M$ modulo $\epsilon^2$ terms.
- So maps $\mathbb{R}^{0|1} \to M$ are the odd tangent bundle $\mathcal{L}M$.
Derived loop rotation

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Cyclic homology

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Without localizing: keeping careful track of $\mathbb{Z}$-grading recovers Hodge filtration on de Rham cohomology (BZ-Nadler).
Consider $S^1$ equivariant cohomology of derived loops: imposing cohomological invariance under $d$. - goes by the name Cyclic Homology (A. Connes, B. Feigin-B. Tsygan)

**Theorem:** The cyclic homology $H^*_S(\mathcal{L}M)$ made periodic - inverting $u \in H^*(\mathbb{C}P^\infty)$, coincides with the de Rham cohomology of $M$ tensored by $\mathbb{Z}[u, u^{-1}]$.

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Get interpretation of calculus and de Rham theory in algebraic, singular, brave new and even noncommutative settings!
Equivariant vector bundles

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Bundle on $\mathcal{L}M$: look like pullback of bundle $E$ from $M$, 
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Loop Spaces and Connections – p. 18
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- $S^1$ equivariant bundles on $\mathcal{L}M$ give vector bundles with flat connection on $M$. 
Flat connections and loops

Can replace vector bundles by suitable sheaves: singular versions of bundles, can jump in rank
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**Theorem:** The category of $S^1$-equivariant sheaves on $LM$ is equivalent to the category of $\mathcal{D}$-modules on $M$

- more precisely, need to invert $u \in H^*(S^1)$, or keep track of gradings/Hodge filtrations everywhere (and work with derived categories..)
Dropping flatness

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- "Nonassociative" action of a group $G$: bunch of maps labeled by $G$ - i.e., action of free group $F(G)$ on underlying set.
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- **Solution**: $X$ pointed space, $F(X) = \Omega \Sigma X$, based loops on the suspension (James construction)
So $F(S^1) = \Omega S^2$...
Connections via loops

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**Theorem** \( \Omega S^2 \)-equivariant vector bundles on \( \mathcal{L}M \) are canonically identified with arbitrary bundles with connection on \( M \).

But where is curvature? and what’s the deal with \( \Omega S^2 \)??
Recall that $BS^1 = \mathbb{C}P^\infty$, and all $S^1$-equivariant gadgets “live” over the “equivariant point” $BS^1$. 
Freeing the circle

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The map $\Omega S^2 \to S^1$ is loops $\Omega$ applied to $S^2 \to BS^1$. 
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- Our “freeing” $S^1$ spaces means restricting to $S^2 \subset B S^1$.

- The map $\Omega S^2 \to S^1$ is loops $\Omega$ applied to $S^2 \to B S^1$.

- The “kernel” of $\Omega S^2 \to S^1$ is the looped Hopf fibration $\Omega S^3 \to \Omega S^2$. 


The birth of curvature

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**Theorem** Identifying $\Omega S^2 = F(S^1)$-equivariant bundles on $LM$ with connections $(E, \nabla)$ on $M$, the action of $\Omega S^3 = F(S^2)$ on $E$ is precisely the curvature of $\nabla$. 
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Flatness of $\nabla$ is equivalent to triviality of the $F(S^2)$ action.
The End

Thus the Hopf fibration may be considered the universal source of curvature.
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Thank you for listening!