Constantin Teleman - Examples & Counterexamples

Note Title 4/11/2007

in Geometric Langlands

Example! comes from joint work with E. Frenkel
- would follow easily from the conjectured equivalence of categories -
makes infinitesimal progress towards geometric Langlands

Counterexample : 1 April was less than two weeks ago!

Background: The Abelian case (GL_1)

A = an abelian variety, $A^\vee = Pic^0 A$ dual
$\Rightarrow$ universal line bundle $\mathcal{P} \longrightarrow A \times A^\vee$

Fourier-Mukai gives equivalence of categories
$$D\text{-}Coh(A) \longleftrightarrow D\text{-}Coh(A^\vee)$$

$$\begin{array}{ccccc} & & A \times A^\vee & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ A & & & & A^\vee \end{array}$$

$$\mathcal{F} \longmapsto Rp_{2*}(\mathcal{P} \otimes p_1^* \mathcal{F})$$

Let $V = Lie(A)$.

Fiber over $V^\vee$:

(0)

$$\begin{tikzcd} T^\vee A = A \times V^\vee \arrow[dr] & & A^\vee \times V^\vee \arrow[dl] \\ & V^\vee & \end{tikzcd}$$

$\Rightarrow D[\mathrm{Coh}(A \times V^\vee)] \longleftrightarrow D[\mathrm{Coh}(A^\vee \times V^\vee)]$
F-M over $V^\vee$.

(1) <u>Deformations</u>

$\mathcal{O}$ on $T^\vee A \rightsquigarrow \mathcal{D}$
$\leftrightarrow \mathrm{Sym}\,T$

RHS deforms to $A^\flat$ = moduli space of flat algebraic GL_1 bundles on A

$$\begin{tikzcd} V^\vee \arrow[r, hook] & A^\flat \arrow[d] \\ & A^\vee \end{tikzcd}$$

universal extension of $A^\vee$ by a vector space:

[$V = H^1(A^\vee, \mathcal{O})$
$\Rightarrow$ natural class $\mathrm{Id} \in H^1(A^\vee, V^\vee)$]

Have a Poincaré bundle $\mathcal{P}$ on $A \times A^\flat$ with flat connection along A.

__Theorem__ (Laumon, Polishchuk-Rothstein)
Fourier-Mukai gives an equivalence of categories $D(\mathcal{D}\text{-mod}(A)) \leftrightarrow D(\mathcal{O}\text{-mod}(A^\flat))$

($A \dashrightarrow A^\flat$: use de Rham cohomology,
$A^\flat \dashrightarrow A$: use usual pushforward)

(2). Have further deformations with more structure:
$\mathcal{L} \longrightarrow A$ ample line bundle

Replace $\mathcal{D}$ by $\mathcal{D}^{\mathcal{L}}$ diff ops on sections of $\mathcal{L}$ $= \mathcal{L} \underset{\mathcal{O}}{\otimes} \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}^{-1}$

- Same symbols as $\mathcal{D}$, $\operatorname{gr} \mathcal{D} = \operatorname{gr} \mathcal{D}^{\mathcal{L}} = \operatorname{Sym} T$.

$\mathcal{L}$ defines a (fractional) line bundle $\mathcal{L}^{-1}$ on $A^\vee$
(inverse curvature form).
$\mathcal{D}^{\mathcal{L}^{-1}}$: $\mathcal{L}^{-1}$ twist of $\mathcal{D}$ on $A^\vee$

__Theorem__ (Polishchuk-Rothstein) Equivalence of categories $D(\mathcal{D}^{\mathcal{L}}\text{-mod on } A)$
$$\simeq D(\mathcal{D}^{\mathcal{L}^{-1}}\text{-mod on } A^\vee)$$

- on level of $\mathcal{O}$-mod-les this is usual F-M.

Observe: $V^\vee \cong \bar{V}$ from curvature of $\mathcal{L}$

$\Rightarrow \quad \bar{V} = (\bar{V}^\vee)^\vee \qquad \bar{V}^\vee = T_1 A^\vee$

So $A^\vee \times V^\vee \cong T^\vee A^\vee$.

Examples of objects & Homs:

(0) • $\mathcal{O}$ on $T^\vee A \rightsquigarrow \mathcal{O}_P$ on $A^\vee \times V^\vee$

$P = \{1\} \times V^\vee \subset A^\vee \times V^\vee$

(P will survive deformation to A^b)

• $P_\alpha = \pi^{-1}(\alpha) \Rightarrow \mathcal{O}_{P_\alpha}$ on $A^\vee \times V^\vee$

goes to $\mathcal{O}(\alpha)$ $\alpha \in A^\vee$ line bundle on A.

• $\mathrm{Hom}_{T^\vee A}(\mathcal{O}, \mathcal{O}) = \Gamma(T^\vee A, \mathcal{O}) = \Gamma(A, \mathrm{Sym}\, T)$
$= \mathrm{Sym}\, V$

$\mathrm{Hom}_{A^\vee \times V^\vee}(\mathcal{O}_P, \mathcal{O}_P) = \Gamma(P, \mathcal{O}) = \mathrm{Sym}\, V.$

— naturally isomorphic as algebras.

- $\mathrm{Hom}_{T^\vee A}(\mathcal{O}(\alpha), \mathcal{O}(\alpha)) = \mathrm{Sym}\, V$

  $\Gamma(P_\alpha, \mathcal{O}) = \mathrm{Sym}\, V$ no changes.

- $\mathrm{Ext}_{T^\vee A}(\mathcal{O}, \mathcal{O}) = H^*(T^\vee A, \mathcal{O})$

  $= H^*(A, \mathrm{Sym}\, T) = \mathrm{Sym}\, V \otimes \underset{\Lambda^\bullet \bar{V}^\vee}{\underset{\parallel}{H^{0,*}(A)}}$

- $\mathrm{Ext}^{\bullet}_{A^\vee \times V^\vee}(\mathcal{O}_p, \mathcal{O}_p)$

  $= \Gamma(A^\vee \times V^\vee, \mathcal{O}_p) \otimes \Lambda^\bullet(\text{normal bundle})$

  $= \mathrm{Sym}\, V \otimes \Lambda^\bullet \bar{V}^\vee.$ isomorphic algebras.

Note Ext^1 classifies first order deformations

$\bar{V}^\vee \leftrightarrow$ parallel motions of the fiber P.

Other side $\bar{V}^\vee \cong H^1(A, \mathcal{O})$ = infinitesimal defs of line bundle $\mathcal{O}$ on A giving deformations on $T^\vee A$.

If we didn't know FM, what of the equivalence could we construct?

If $\mathcal{M}$ is an $\mathcal{O}$-module on $A^\vee \times V^\vee$ supported on P (ie an $\mathcal{O}_P$-module)

$\mathcal{M} \longmapsto \Gamma(P; \mathcal{M}) = \underline{M}$ Sym V-module

Can localize $\underline{M}$ on $T^\vee A$, $\mathcal{O} \otimes_{\Gamma(T^\vee A, \mathcal{O})} M = \widetilde{M}$

almost tautological that $\widetilde{M}$ is F-M transform of $\mathcal{M}$

$\Rightarrow$ Baby version of classical B-D construction:

Prop Get an equivalence of abelian categories

$\mathcal{O}$-modules on $T^\vee A$ "globally presented by their sections" $\longleftrightarrow$ $\mathcal{O}$-modules supported on P.

Cute but not very useful:

Exts in these abelian categories are not the Hom groups in $D(\mathrm{Coh}\, T^\vee A)$

BUT...

Remainder of the Ext calculation + ε $\Longrightarrow$

<u>Theorem</u> There is an equivalence of triangulated categories

$$\left\{\begin{array}{c}\text{full subcat of}\\ \mathcal{O}\text{-modules on } T^\vee A\\ \text{whose cohomologies}\\ \text{are extensions of}\\ \text{globally presented}\\ \mathcal{O}\text{-modules}\end{array}\right\}^{\circ} \longleftrightarrow \left\{\begin{array}{c}\text{full subcategory of}\\ D\text{-}Coh\,(A^\vee \times V^\vee) \text{ of}\\ \text{objects whose}\\ \text{cohomologies are}\\ \text{extensions of } \mathcal{O}_p\text{-modules}\end{array}\right\}^{\circ}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad D$ of formal nbhd of P

ε: need to know deformations are unobstructed (ie vanishing of higher Whitehead brackets) + need more info to pin down the isomorphism. (Ext^1 pins down first order)

(1) $\mathcal{D}$ on A, $\mathcal{O}$ on A^b.

$$\mathrm{Ext}^*_{D\text{-mod}\,A}(\mathcal{D}, \mathcal{D}) = H^*(A, \mathcal{D}) = \mathrm{Sym}\, V \otimes \Lambda \bar{V}^\vee$$

$$\mathrm{Ext}^{\bullet}_{A^\flat}(\mathcal{O}_P, \mathcal{O}_P) = \mathrm{Sym}\, V \otimes \Lambda V^\vee$$

Note: something changes a bit with deformations:

$\pi^{-1}(\alpha) =: P_\alpha$ is no longer canonically isomorphic to P, so self-ext is noncanonically $\mathrm{Sym}\, V$

$P_\alpha \longleftrightarrow \mathcal{D} \otimes_{\mathcal{O}} \mathcal{O}(\alpha)$ as $\mathcal{D}$-modules.

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}\otimes\mathcal{O}(\alpha), \mathcal{D}\otimes\mathcal{O}(\alpha)) \simeq \Gamma(A, \mathcal{O}(\alpha)^{-1}\otimes\mathcal{D}\otimes\mathcal{O}(\alpha))$$
$$= \mathcal{D}^\alpha \text{ diff ops on } \mathcal{O}(\alpha)$$

(isomorphic to $\mathcal{D}$ after choice of connection on $\mathcal{O}(\alpha)$).

(2) Further deformation:

<u>Prop</u> $H^*(A, \mathcal{D}^{\mathcal{L}}) \simeq \mathbb{C}$, in degree zero
$= \mathrm{Ext}_{\mathcal{D}^{\mathcal{L}}}(\mathcal{D}^{\mathcal{L}}, \mathcal{D}^{\mathcal{L}})$

$A^\flat$ side: neighborhood of P in $A^\flat$ looks like T^*P, & deformation to $\mathcal{D}^{\mathcal{L}}$ $\longleftrightarrow$ quantizing T^*P.

Prop (case (1)) This recovers FM in a formal neighborhood of $P \subset A^0$

[Note $\mathrm{Ext} = \mathrm{Sym}\, V \otimes \Lambda V \simeq \Omega^\bullet(P)$]

Blanket theorem (w/ E. Frenkel)

All this holds for any reductive G

$P \longleftrightarrow$ subvariety of Beilinson-Drinfeld opers

On Bun_G look at $\mathcal{D}^{\sqrt{K}}$

$$\mathrm{Ext}_{\mathcal{D}\text{-}\mathrm{Bun}_G}(\mathcal{D}, \mathcal{D}) \xrightarrow{\sim} \Omega^\bullet_{alg}(P)$$

algebraic differential forms on opers

& deformations are unobstructed to all orders

$\Rightarrow$ extend the B-D construction of $\mathcal{D}$-modules from P to a formal neighborhood

(construct FM kernel in formal nbhd of P)

— Not yet a good understanding of how "Parallel motions of P" are parametrized.

Part 2

Study of the FM transform between coherent sheaves on $T^*\mathrm{Bun}_G$ & on $T^*\mathrm{Bun}_{G^\vee}$

(Cohomology computations work for all G. "global" issues – gerbes, action of center, etc for groups other than GL_n – see Donagi–Pantev).

<u>Recall</u>

$$\begin{tikzcd}
T^*\mathrm{Bun}_G \arrow[dr, "\chi"'] & & T^*\mathrm{Bun}_{G^\vee} \arrow[dl, "\chi^\vee"] \\
& \mathcal{H} = \bigoplus \Gamma(\Sigma, K^{\otimes d}) &
\end{tikzcd}$$

Σ = curve, K = canonical bundle

d = exponents + 1 = $1, \ldots n$ for GL_n.

Generic fibers are (almost) dual abelian varieties. GL_n: Jacobians of spectral curves.

$p \in \mathcal{H}$ $\quad p = (p_1, \dots p_n) \in \bigoplus_{d=1}^{n} \Gamma(K^{\otimes d})$

Spectral curve given by $\{\tau^n + p_1 \tau^{n-1} + \dots + p_n = 0\}$
$\tau \in \mathrm{Tot}(K)$.

Structure sheaf $\mathcal{O}_S$ of spectral curve, as algebra over Σ, is $\mathrm{Sym}\, T$ / principal ideal generated by above equation.
$\Rightarrow$ resolve

$$\mathrm{Sym}\, T \otimes T^n \xrightarrow[(p)]{} \mathrm{Sym}\, T \longrightarrow \mathcal{O}_S \longrightarrow 0$$

map given by characteristic polynomial p above.

<u>For GL_n</u> can fix a base point in each fiber by chosing the trivial line bundle on each curve.
(fiber over p = something containing Pic of spectral curve).

$\Rightarrow$ associated vector bundle on Σ is $\pi_* \mathcal{O}_S =: E$

Can see $E = \pi_* \mathcal{O}_S \simeq \bigoplus_{i=0}^{n} T^{\otimes i}$ (det $\neq 1$)

Higgs field on E is $\theta = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \\ 1 & & & \\ & 1 & & 0 \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} : E \to E \otimes K$

This defines the Hitchin section of $T^* Bun_{GL_n} \to \mathcal{H}$.

Sometimes more canonical to use
$E = K^{\frac{n-1}{2}} \oplus \cdots\cdots \oplus K^{\frac{1-n}{2}}$, so det $= 1$
(ie SL_n bundle)
.. may need to choose $K^{\frac{1}{2}}$ to define this.

We know part of the Poincaré kernel for FM transform --- a line bundle over the locus where S is smooth.
Can extend this to a slightly bigger set:

For GL_n - let $R \subset T^* Bun_{GL_n}$ be the open subset of pointwise regular Higgs field. & S its complement

$\Rightarrow \mathcal{P}$ extends to a line bundle on

$T^*Bun_{GL_n} \times_H T^*Bun_{GL_n} \longrightarrow S \times S$

Can write a formula in terms of det of spectral curve.

[GL_n : regular Higgs field $\Longleftrightarrow$ Higgs sheaf is a line bundle]

For general G, know how to extend to a slightly smaller subset.

"Geometric" construction of P :

- P is $\mathcal{O}$ on $T^*Bun_G \times_{\mathfrak{h}} \sigma(\mathfrak{h})$ Hitchin section.

Recall functions on $\mathfrak{h}$ act by Hamiltonian flows on T^*Bun_G (linear flows along fibers).

Restrict to linear functions, a copy of $\mathfrak{h}^\vee$.

(Hitchin) Thm $R^1\chi_*\mathcal{O}$ = vector bundle canonically isomorphic to $\mathfrak{h}^\vee$ ... as if χ was a fiber bundle with fibers abelian varieties

($\mathcal{H}^\vee$ = fiberwise tangent bundle).

$R^1 \chi_* \mathcal{O}$ gives infinitesimal deformations of line bundles.

Exponentiating gives a class in $H^1(T^*Bun_G, \mathcal{O}^*)$, ie a line bundle.

<u>Prop</u> If you move the Hitchin section σ by $\exp(h)$ $h \in \mathcal{H}^\vee$
$\Rightarrow \wp$ becomes
$\mathcal{O}(\exp h) \in Pic(T^*Bun_G \times_{\mathcal{H}} \exp h \cdot \sigma(\mathcal{H}))$

What is meaning of transformed Hitchin section
$\exp h \cdot \sigma(\mathcal{H}) \subset T^*Bun_G$?

$\sigma(\mathcal{H})$ itself represented all spectral curves + trivial line bundles on them.

$h \in \mathcal{H}^\vee = \bigoplus H^1(\Sigma, T^{\otimes m})$
defines an infinitesimal line bundle on $T^*\Sigma$.

Deformed Hitchin section gives all spectral curves & restriction of this line bundle.

(Pretty good indication that this story deforms to NC case... $\sigma(\mathcal{H}) \rightsquigarrow Op_G \subset Flat_{G^\vee}$

transformed sections ... local foliation of $Flat_{G^\vee}$).
(conjectural)

Can define $\mathcal{P}$ on $T^*Bun_G \times_{\mathcal{H}} T^*Bun_{G^\vee} \setminus S \times_{\mathcal{H}} S$

e.g. for a coherent sheaf whose support does not meet S can define FM transform on <u>all</u> of $T^*Bun_{G^\vee}$

E.g. $\mathcal{O}_\sigma$ structure sheaf of $\sigma(\mathcal{H}) \longmapsto \mathcal{O}$

$\mathcal{O}_{\exp h \cdot \sigma} \longmapsto \mathcal{O}(\exp(-h))$ line bundle

Check we get same self-exts:

$$\mathrm{Ext}_{T^*\mathrm{Bun}_G}(\mathcal{O}_\sigma, \mathcal{O}_\sigma) = \mathbb{C}[\mathcal{U}] \otimes \Lambda^\bullet \mathcal{U}^\vee$$

= diff. forms on $\mathcal{U}$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\mathrm{Ext}_{T^*\mathrm{Bun}_{G^\vee}}(\mathcal{O}, \mathcal{O}) = H^*(\mathrm{Bun}_{G^\vee}, \mathrm{Sym}\, T)$$

canonically!

theorem of Frenkel-Teleman

Good check! this detects information about all of $T^*\mathrm{Bun}_{G^\vee}$.

(this follows from Fishel-Grojnowski-Teleman)

Would like to claim fully faithful functor from D-Coh (with support in R) to D-Coh $(T^*\mathrm{Bun}_{G^\vee})$

[only follows for a neighborhood of σ, not checked on all of R)

Eg. GL_n : $\mathcal{O}(1)$ on Bun_{GL_n} = $\det^{-1}$ of cohomology

$\longrightarrow$ gives $\mathcal{O}(1)$ on $T^*\mathrm{Bun}_{GL_n}$

[On abelian varieties $\mathcal{O}(1) \longrightarrow \mathcal{O}(-1)$]

Claim: on R the transform of $\mathcal{O}(1)$ is $\mathcal{O}(-1)$.

Check: $\mathrm{Ext}(\mathcal{O}, \mathcal{O}(1)) = H^*(T^*\mathrm{Bun}_G, \mathcal{O}(1))$ = sections of a line bundle on Hitchin space

$\qquad\qquad ||$

$\mathrm{Ext}(\mathcal{O}_\delta[-\dim \mathrm{Bun}_G], \mathcal{O}(-1))$ = free rank 1 module on $\mathbb{C}[\mathcal{H}]$

Try $\mathrm{Ext}(\mathcal{O}(1), \mathcal{O}) = H^*(T^*\mathrm{Bun}_G, \mathcal{O}(-1))$

$$= H^*(\mathrm{Bun}_G, \mathcal{O}(-1) \otimes \mathrm{Sym}\, T)$$

$$||$$

$$0$$

BUT $\mathrm{Ext}(\mathcal{O}(-1), \mathcal{O}_\delta[-d])$ = rank 1 free module over $\mathbb{C}[\mathcal{H}]$ in degree d ...

Counterexample to geometric Langlands!

appears to deform to D-module setting...

... Serre duality fails on Bun_G since it's of infinite type....

$\Rightarrow$ can't give local definition of category of coherent sheaves, need constraints on behavior at infinity ... on for Atiyah-Bott strata.

[Evidence for fully faithful functor
$D(\mathrm{Coh}(Flat_{G^\vee})) \longrightarrow D(Bun_G, \mathcal{D})$
is strong, but not equivalence...]

- problem: Bun_G not proper, so base change doesn't work.

<u>Opers</u> An oper on Σ is

- a bundle E with hol. connection ∇
- a full flag $0 \subset E^1 \subset E^2 \subset \dots \subset E^n = E$

s.t. $\nabla E^k \subset E^{k+1} \otimes \Omega$

$gr\,\nabla : E^k / E^{k-1} \xrightarrow{\sim} E^{k+1}/E^k \otimes \Omega^1$

.... associated graded of an oper is a Higgs field θ in the Hitchin section.

Want: a local foliation of $Flat_{GL_n}$ of which Op are a leaf.

<u>Rephrase the definition</u>

an oper is actually a $\mathcal{D}$-module quotient of $\mathcal{D} \otimes_{\mathcal{O}} K^{\frac{n-1}{2}}$ of rank n, by a principal ideal.

Recall: a spectral curve was a quotient of $\mathrm{Sym}\, T$ of rank n.

<u>Ideal</u> defining an oper is an NC spectral curve.

Want: line bundle on this coming from a line bundle on $\mathrm{Spec}\,\mathcal{D}$

... ie a locally free $\mathcal{D}$-module of rank 1.

- classified by $H^1(\Sigma, GL_1(\mathcal{D}))$

infinitesimal variations: $H^1(\Sigma, gl, \mathcal{D})$

$= H^1(\Sigma, \mathcal{D})$

$\simeq \bigoplus H^1(\Sigma, T^{\otimes n})$ (noncanonically)

Unfortunately these don't integrate to honest deformations.

But classes from $H^1(\Sigma, T)$ do integrate to deformations of the curve.

Result of a def of $\mathcal{D}$ on Σ will be a $\mathcal{D}_{T^*K}$-mod deformation of $\mathcal{D}_{T^*K}$ on Bun_G Fourier dual to movement of subvariety of opers