Doubled with triangulated categories! As basic but not as good as Hilbert spaces.

DG & Aoo category ... perfectly equivalent

nicer structures, also simpler, come naturally in derived category context.

k module ring

Complex: complex of k-modules $\cdots \to \mathcal{E}^n \to \mathcal{E}^{n+1} \to \cdots$

usual picture of complex. Alternatively $\mathcal{C} = \{ \mathcal{E}^i \}$,

$\delta^2 = 0$ i.e. $\mathcal{C}$ is a DG k-module: better AU psychologically!

complexes form a tensor category (symmetric monoidal category)

$\otimes : \{ kDGMod \} \times \{ kDGMod \} \to \{ kDGMod \}$

simpler to write in DG language:

$\mathcal{C} \otimes \mathcal{C}$ usual $\otimes$ on underlying k-modules with sum of $\delta$s, differential using (graded) $\delta^2 = 0$ form.

--- think uniformly in language of $\otimes$ categories

rather than in "complex" picture.

$(A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $A \otimes B \to B \otimes A$ functorial

isomorphisms satisfying identities.

Associativity for complexes obvious, commutativity is super-important.

$A \otimes \phi : \mathcal{C} \to \mathcal{C}$. \phi is p=deg a, q=deg b

$k$-module category.

\[ \text{DG - category = category enriched over } kDG - modules. \]

A category has objects $Ob = X$,

$\text{Hom} (X, Y) \in kDGmod$ morphism

composition $\otimes \text{Hom} (X, Y) \otimes \text{Hom} (Y, Z) \to \text{Hom} (X, Z)$ associative.

Identity:

$X \in Ob \Rightarrow \text{Hom} (X, X) = kDGmod \text{ morphism} k \to \text{Hom} (X, X)$

map $kDGmod A \to kDGmod A$ with $\delta z = 0$ & $dz = 0$

so $i dz \text{Hom} (X, X) dz = 0$ trivial.

Can consider $\otimes$ plain category: forget grading altogether

$\to k$-linear category.
Another categorical way to pass from an exact to usual category can replace $C$ by $\text{Set} \cap \text{Hom}(k, k) = \text{cycles}$ of degree zero...

**Examples**

1. $k$-$DG$-mod is a $k$-$DG$-category:
   \[ \text{Hom}(C, C) \] has natural $DG$-mod! grading comes from natural $k$-action on \[ \text{Hom} \]...
   replace whole $\text{Hom}$ by sum of homogeneous submodules \[ \text{Hom}(C, C) = \bigoplus \text{Hom}(C, C)_{n} \]

Morphism of complexes is a 0-cocycle in this dg hom \[ \text{Hom}(C, C') \] i.e., \[ \text{Hom}_{\text{complex}}(C, C') \in \text{Hom}(k, \text{Hom}(C, C')) \]

So $DG$-categories are categories with dg $k$-mod structure on hom sets so that compositions are $k$-linear, where does commutativity of $\circ$ come in? So can speak of dual $DG$ category \[ \text{D}G^* = \text{dual } DG \text{ category} \]

**Example 1**

- DG category with one object $\longleftarrow$ DG-algebra (unit $-$ implicit basefield $k$)
  2. A usual $k$-$DG$-category $= DG$ category with $k$-mod $k$-linear $\text{Hom}$ are dg $0$ $d=0$.
  3. Complexes of $R$-modules $R$ an algebra $\longrightarrow$ $k$-$DG$-category:
    - $R$-$mod = k$-$DG$-category $\text{mod}$ $R$ $k$-linear
      - Psychologically important that $\circ$ more accurately denoted by $\circ_R$ $\circ$ \[ \circ_R \]
      - Use conventions for $R$-$mod$ $\longrightarrow$
      - e.g., $X, \ldots X \in R$ for any object $X$ (usual $R$-$mod$ $\cong k$-$DG$-category $\text{End}_R X$ $\cong k$-$DG$-category $\text{End}_R X$)
      - If $R$ arbitrary can form formal limit $\mathbb{A}$ $\text{Hom}(X, X)$ $\text{get } k$-$DG$-algebra but without unit, only new idelephants
    4. DG-modules over a $DG$-algebra $R$ $\longrightarrow \mathbb{R}$ $\longrightarrow \mathbb{R}$ $\text{mod}$ $\text{R}$ $k$-$DG$-module $\text{mod}$ $R$ $\text{mod}$ $\text{R}$ $\text{mod}$
Another POV: a quasi-functor is a bimodule over $A$, $A$ will contain projms.
Another POV: the factors between DG categories
in replace at by a certain (canonical?) which
at least over a field!

**Triangulated Categories**

Example $K(C)$: homotopy category of complexes in
a k-linear category.

A triangulated category is a graded category with additional structure.
A "candidate triangle" is $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Additional structure: some candidate $\Delta$s are "distinguished" + axioms...

Don't require shift as structure; keep in this graded setting
ask for representability of suspension:

$$\text{hom}_C(0, X \otimes I) \xrightarrow{\sim} V \otimes \mathbb{Z} \Rightarrow \mathbb{Z} 
\text{ s.t. } f^{-1} \text{ exists.}$$

$\Rightarrow$ define map to unique generator, call $X \cong \mathbb{Z}$
$\Rightarrow$ weak action of $\mathbb{Z}$ on $C$.

$K(C)$: distinguished $\Delta$ is given to $X \xrightarrow{f} Y \xrightarrow{0} \text{Cor} f \xrightarrow{-} X(1)$

$\text{Cor} f = Y \otimes X^\otimes 1$

$\partial = \partial + d$.

Another POV (Deligne):

$X \xrightarrow{f} X \xrightarrow{f} \cdots$

Forms a double complex (no chain), & care
is taken co-where ... really naive double complex!

$df = 0$

does not $(d + c_1) = 0$ so must introduce sits

Examples:

a. $K(C)$, $K(C\text{-mod})$, $R$ is acyclic.

b. $Ho(R \text{-DG-mod})$, $R$ a DG algebra.

c. $Ho(R \text{-DG-mod})$
T a triangulated category \( \Rightarrow T' \) a full subcategory.
A candidate image in \( T' \) is distinguished if it is so in \( T \).

Advantage of DG categories: Yoneda \( A \rightarrow A^0 \) -DG nod.

Triangulated core base \( T \rightarrow (\text{dg-categories}) \rightarrow \text{graded modules} \)

and a left \( T' \rightarrow \) cohomological functor, i.e. graded factors \( T'^0 \rightarrow \text{graded k-modules} \).

Hyper cohomological functors don't have natural triangulated structure.

**DG categories**

A DG category \( \rightarrow \) homotopy category \( \text{Ho}^*(A) \)

\( A \rightarrow A^0 \) - DG factors is a quasi-equivalence \( \Rightarrow \text{an equivalence} \) \( \text{Ho}^*(A) \rightarrow \text{Ho}^*(A^0) \).

Would like to localize "world" of DG categories in quasi-structure.

Triangulated category = graded category:

Yoneda: \( A \rightarrow A^0 \) - DG nod, \( \text{Ho}^*(A) \rightarrow \text{ho}(A^0 \text{-mod}) \).

**Def.** A candidate triangle \( \rightarrow \) has triangulated structure.

\( \text{Ho}^*(A) \) is said to be distinguished if it is distinguished in \( \text{Ho}^*(A^0 \text{-DG nod}) \).

\( A \) is said to be pre-triangulated if \( \text{Ho}^*(A) \) is triangulated when equipped with this structure, i.e.

\( \text{Ho}^*(A) \) is a triangulated subcategory of \( \text{Ho}^*(A^0 \text{-DG nod}) \).

- i.e. closed under cones & desuspensions.

Problem: the def is not self-adj. set two rules of distinguished triangle, opposite to each other, not clear if they agree. Do DG factors preserve these distinguished triangles? etc.

More convenient to reformulate correctly.
4. DG-cats are over a DG-category $\mathcal{A}$ as DG-models (DG-models of $\mathcal{A}$).

For any DG-category $\mathcal{A}$, the Yoneda embedding $\mathcal{A} \to \text{Ho}\mathcal{A}$ is continuous:

$\text{Hom}_\mathcal{A}(Y, Z) = \text{Ho}(\mathcal{A})(Y, Z)$.

"embedding": fully faithful (Ho(\mathcal{A})(Y, Z) \to \text{Ho}(\mathcal{A})(Y, Z))

$A = \text{DG category}, \quad \text{Ho}(A) = \text{homotopy category}\).

Ob $\text{Ob}(A) = \text{Ho}(A)$.

$\text{Hom}_{\text{set}}(X, Y) \to \text{Hom}_{\text{Ho}(A)}(X, Y)$,

$\text{Hom}(X, Y)$ maps $X \to Y$ as $\text{Ho}\text{Hom}(X, Y)$.

Other notation: $\text{Ho}(A) \equiv \text{Ho}(A)$.

$\text{Ho}(A)$ is the homotopy category, with morphisms elements of $\bigoplus \text{Ho}\text{Hom}(X, Y)$.

One can have property (not strictly) of being triangulated... if not can adjoin cone objects to make it triangulated.

**Def.** A DG-functor $F: \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if $\text{Ho}(F): \text{Ho}(\mathcal{A}) \to \text{Ho}(\mathcal{B})$ is an equivalence, i.e.,

1. $\text{Hom}(X, Y) \to \text{Hom}(F(X), F(Y))$ is a quasi-equivalence.

2. Existence of $\text{Ho}(F): \text{Ho}(\mathcal{A}) \to \text{Ho}(\mathcal{B})$ (enlarged version):

   $\forall Z \in \mathcal{B}, \exists X \in \mathcal{A}$,

   $\text{Ext}^{n}(Z, X) = 0$.

A homotopy inverse.

A, $\mathcal{A}_2$ DG-categories are quasi-equivalent if

$F: \mathcal{A}_1 \to \mathcal{A}_2$ is a quasi-equivalence.

$\mathcal{A}_1 \to \mathcal{A}_2$:

Not enough to define correct where on "work" of DG-categories... as equivalence of categories.

"Fuzzy notion" of quasi-functor $\mathcal{A}_1 \to \mathcal{A}_2$.

$A_1 \to \mathcal{A}_2$: These form (Albrecht Kock's) a DG-category in $\text{Ho}(\mathcal{K})$.
Need for \( A^0 \)-DG-mod: carries cones of morphisms
functor \( f \circ \bullet : \text{Cone}(f : X \to Y) \to \text{Cone}(f : A \to A) \)
on 
representable functor.

- Don't need full \( A^0 \)-DG-mod just cones (no iterated one).

\( \mathcal{C} \) pre-waddell category (have susp of maps but not are direct sus)
- So formally add direct sus if so several ranks:

\[ \mathcal{C} \to \mathcal{C}^{\omega}-\text{mod}. \]

Or conceptually just take all categories with objects \( \mathcal{O} : \mathcal{C} \to \mathcal{C} \) to see what the
for direct sus morphisms just matrices (rather than "linear transformations" in other defs).

\[ \text{Step 1: Replace } A^0 \text{-DG-mod by } A \subset A^0 \text{-DG-mod} \]

- Full subcategory: smallest full DG-subcategory s.t. \( A \to A \) and

\[ \text{M} \to \text{M}, \quad \text{M} \to \text{M} \text{ iterated } \Rightarrow \text{M}^1 \to \text{M}^2 \to \text{M}^2 \to \text{M}^3 \to \text{M}^3 \to \]

\( \text{M} \to \text{M} \) stable wrt semi-split extension:

\[ 0 \to M_i \to M_2 \to M_3 \to 0 \]

- Semi-split extension (of DG-modules) is split in sense of

graded modules.

- So cone \( 0 \to N \to \text{Cone}(M \to N) \to M/N \to N \to 0 \)

- Determines failure of splitting to be closed.

- Its differential is \( F \).

More concretely

\[ \text{M} \to \text{M} \quad \leftrightarrow \quad \exists \quad \text{M}_{i=0} \subset \ldots \subset \text{M}_{i} = \text{M} \]

finite filtration s.t. \( \text{M}/\text{M}_{i-1} = \text{ker}(\text{g}: \text{M} \\ \text{M}/\text{M}_{i-1} = \ldots \text{representable up to shift}, \quad \text{dg} \}

- the extensions \( 0 \to M_i \to M_j \to M_j \to 0 \)

- automatically semi-split because representable functors

are projective objects.

By suppose \( A \to A \) is an algebra (i.e., only \( A \)-object)

- As \( A \) \-object \( \to \text{mod-}A \) \( \text{h}_A = A \) as

\( A \)-module, which is free \( \to \text{projective} \).

\[ \text{Step 2: Fix the splittings } 0 \to M_i \to M_j \to M_j/N \to 0 \]

So \( M_i = \oplus \text{h}_A [i] \) as graded modules.
As a $D_5$-module $M = (\bigoplus_{i} a_i \otimes [r_i] \ V_i = d + w )$

d: standard differential on direct sum of dg modules

$w = (w_{ij}) \quad w_{ij} \in \text{Hom} (a_j, a_i) \ [r_i, r_j]$

of degree 1 (wrt grading)

1. $w$ is strictly upper triangular $w_{ij} = 0$ if $i > j$
2. $\nabla_2 = 0$ is flat connection

(Maurer-Cartan eqn.)

$D_6$ A pre

A pre-triangulated hull of $A$ : "$D_6$ category of twisted complexes"

$D_6$ A pre

are formal expressions $\sum_{i=1}^n (a_i, [r_i], w)$

$w = (w_{ij})$ as above.

$\text{Hom} (\bigoplus a_i \otimes [r_i], \bigoplus a_i \otimes [r_i])$ as graded module

ie ignore $d$, hence $w$ is just space of natural

$f = (f_{ij}) \quad f_{ij} \in \text{Hom} (a_j, a_i) \ [r_i, r_j]$

composite automorphism $f' f = \nabla \ f = f - f \nabla w(-1) \text{ super commutator}$

symbol $\nabla$ has degree 1 $\text{ch}r = d r + w'(r - (-1)^{\text{deg}} r w)$

"twisted complexes" --- Tohoku-Tsy

This definition is clearly self-dual, & clear that

$D_6$ functors extend automatically to pretriangulated hull

$f: X \to Y \text{ in } A \Rightarrow \text{Core}(f) \in \text{pretr. cauchy object}$

$\text{Core}(f) = (Y \otimes X[7], (f'))$

$X \in A$, $X \in \text{pretr}$

Def

$X \to Y$ in $\text{Ho}(A)$ is distinguished if it is isomorphic to $X \to Y \to \text{Core}(f)$

$X$ is pretriangulated if $A: X \to Y$ in $A$, Core$(f)$

is homotopy equivalent to an object of $A$ & so for $X \otimes J$ not?

Exercise 1. Notion of distinguished $\Delta$ is self dual

2. $F: A \to B$ $D_6$-functor $\Rightarrow$ $\text{Ho} \ F: \text{Ho} A \to \text{Ho} B$
3. \( A \) is a pretriangulated DG-category
4. \( Ho (A^{tr}) \) is equivalent to \( \Delta \)-category by \( Ho(A) \)
5. \( A \) is pretriangulated if \( A \rightarrow A^{tr} \) is a quasi-equivalence
6. \( A \rightarrow A^{tr} \) (pretr) is an equivalence of DG-categories.
7. If \( F : A \rightarrow B \) is a quasi-equivalence then it induces a quasi-equivalence \( F^{tr} : A^{tr} \rightarrow B^{tr} \)

Bondal-Kapranov notation \( A^{tr} := Ho^* (A^{pretr}) \).

A DG-category \( \rightarrow \) triangulated category.

Do derived categories, e.g. \( D(A) \)-modules or \( D_k \)-modules, come from a DG-category? 2-categorical constructions are complicated objects take care now.

A DG category \( \Rightarrow D (A) := \text{derived category of } A \)-modules \( A = \text{Ho}^* (A^{pretr}) \) (cycle DG-triangulated)

Importance of this example (B. Kellers) have yields \( A \rightarrow A^{tr} \)-derived exacts \( \Rightarrow \) fully faithful \( A^{tr} = Ho^* (A^{pretr}) \rightarrow D(A) \)

fully faithful --- so any of the \( A^{tr} \)'s embed into such a derived category of modules.

Exercise: A quasi-equivalence \( A \rightarrow B \) yields an equivalence \( D(A) \rightarrow D(B) \), hence restriction factor is a quasi-equivalence functor of ind-sheaves.

\( D(A) \underset{Res}{\rightarrow} D(B) \) always adjoint --- in this case only inverses are isomorphisms.

\( D(A) = Ho^* (\_?) \) ? DG-category containing \( A \)

Arrow (learn to bro guilds through quartz Cass?)
- Spaltenstein, Arras, Hillel, Hirsch
Will define $A \rightarrow A^0$-$D_\ast$-module : category of
semi-free $D_\ast$-modules \\
so that $Ho^0(A) \rightarrow D(A)$ is an equivalence.

Well known (in bounded setting) : $R$ a usual ring,

\[ \text{derived category of bounded above complex of } R\text{-modules} \]
\[ \cong \text{category of bounded above complex of } R\text{-modules} \]

by $6$-clique boundedness

Example : $R = \mathbb{Z}/4\mathbb{Z}$

\[ 0 \rightarrow R \rightarrow R \rightarrow R \rightarrow \ldots \]
acyclic complex but not homotopic to zero ;

apply $\otimes \mathbb{Z}/2\mathbb{Z}$

\[ \ldots \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \rightarrow \ldots \]

with nonzero cohomology !

So to eliminate boundedness need projectivity assumption on whole complex not its terms !

Exercise : if $A$ has finite homological dimension, (i.e. finitely projective) then can remove boundedness in above equivalence.

( in particular acyclic complexes of projectives will be null homotopic )

Def : a $D_\ast$-$A^0$-module $P$ is free if it is $D_\ast$-isomorphic to $0$ (i.e. $\otimes$-free $\Rightarrow$ projective). (in case of not a $D_\ast$-algebra! module generated by $e_i$'s of degree $-r_i$ with $d(e_i) = 0$).

$P$ is semi-free if $0 \rightarrow P \otimes R < \ldots$ exhaustive filtration by $D_\ast$-modules such that each quotient is free.

Free $\Rightarrow$ projective so exaherbas are semi-split. (not split for differential !)

A $A^0$-$D_\ast$-module introduced before is just the $D_\ast$-category of finitely generated semi-free $D_\ast$-modules !

so (semi-free modules) $A^\infty$ is infinite version of $A$.
A category of ind-objects (systems $\text{L} \rightarrow \text{X}$) or certain $\infty$-modules (representable functors)

I abstr. categ of ind-objects. Funct $I \rightarrow \infty$ consider homotopy colimit of this funct $\Delta^I$. 
Consider simplicial object. Colimit of such $I$ of weak (Aoo-)functors live in $\infty$.

Inspired by (generalized) cell complexes -- ie don't necessarily after in correct order.

A DG-algebra. A DG-module $P$ over $A^\otimes$ is free if $\exists \ O \subset P \subset P_i \subset \ldots \ (\forall i \geq 0 \ P_i = P)$.

$P_i$ freely generated by $P_i \sim$ homogeneous generators, $g_j \in J$, so $\text{deg } g_j \in P_i - 1$.

Of course, $\Rightarrow P_j/P_i$ is free as dg module.

Adv: of above formula: semi-groups makes sense in non-linear situation. Eg for DG algebras...

one def: exists nilpotent, of course queinites won't be algebras. $P_i = P_i \sim X_j$ purely graded by $P_i \sim$ generators $X_j$, differentials are defined in subalgebra of old generators.

G for algebras over any operad...

Eg topological semi-groups: attach by cell and add cell in topological sense (map from sphere), generate semi-group by old semi-group & this cell with attaching map. Construction of algebraic free semi-group & topological algebra alg.

Eg Steenrod operad: to what is it an answer? Contact semi-free resolution of the assemble operad (in topological sense).

Theo: $\exists$ model structure on $A^\otimes$ DG-mod (co-nbd eg or semi-free operad).
The results also hold for objects C (e.g., morphism 0 → P are cokernels) are retracts of semi-free objects.

**Example 3** A banded above complex of free $R$-modules

is semi-free.

\[ \text{so \ a \ banded \ above \ complex \ is \ semi-free.} \]

**Example 4** $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is not semi-free (can't find basis vector annihilated by the differential!)

**Theorem** $\text{Ho}(\mathcal{A}) \to \text{DG}(\mathcal{A})$ is an equivalence.

**Proof** In banded above setting, we can use Lemma 1. Anything has banded above projective object.

**Lemma 1** $M \in \mathcal{A}^\mathbb{Z}$ is a direct sum of $\text{Hom}(P, M)$ for $P$ in $\mathcal{A}$.

A similar result can be obtained using the axioms not involving $\text{Hom}$.

**Lemma 2** $P \xrightarrow{f} M$ is acyclic if every morphism $P \to M$ is homotopic to $0$.

Every morphism $P \to M$ is homotopic to $0$.

**Exercise** $M$ is homotopically projective if it is homotopy equivalent to a semi-free module.

**Proof of Lemma 2** $P = \bigoplus P_i$, $P_i/M_i$ free, $f : M \to P$, i.e., $f \in \text{Hom}(P, M)$, $\deg f = 0, \partial f = 0$.

Construct boundary: $H_0 = 0, H_i \in \text{Hom}(P, M)$, $\deg h = -1$.

Choose $H_i$ such that $H_i : f_i = f_i' - i$.

Correct $H_i = \tilde{H}_i - H_i : \text{Hom}(P, M)$, $\deg \tilde{H}_i = 1$. 

Corresponding $H_i$. 

Theorem: $\text{Ho}(\mathcal{A}) \to \text{DG}(\mathcal{A})$ is an equivalence.
$\delta(\delta) = 0$.

But this DG-module of hom's is acyclic:

$\text{A}/\text{P}$ is a direct sum of free (shifted) modules, so then

is a product of shifted copies of $M \Rightarrow$ acyclic.

**Proof of Lemma:**

Sublemma: $N \to M$ morphism of DG modules

\[ \Rightarrow \exists \text{ factorization } N \to N' \to M \]

1. $N' \to M$ acyclic

2. $N'/N$ semisimple

Sublemma $\Rightarrow$ kernel constant surjective $\phi: P_1 \to M$

$P_i$ semi-free $\Rightarrow H^i \phi \to H^i M$

Then apply sublemma for $N=P_i \Rightarrow P \to M$

Surjective $\Leftrightarrow$ isomorphism on cohomology:

$H^i M$ a generator, make module with no generator

degree of correct degree, $d_e = d_e'$ $d_e = 0$.

$\Rightarrow$ surjectivity on module, similarly for cohomology.

Left to convince.

**Proof of sublemma:** Enough to consider $N \to N_1 \to M$, $N_i$ semi-free

with $\ker (H^* N \to H^* M) = \ker (H^* N \to H^* N_1)$

Then build $N' = \bigcup N_i$ and $N' \to N_1 \to N_2 \to M$

$\ker (H^* N \to H^* M)$.

kick homs generators $(H^* N_1)$

... odd refs to fill empty.
2.-category of DG-categories --- partial answer

Today: All DG-categories are pre-triangulated unless mention.

Typical example of 2-category: \( \text{Cat} \) -- have objects: category, morphisms: functor, 2-cell: natural transformation

We'll define 2-category \( \text{DG-cat} \). Objects: (small) DG-categories \( A, B \in \text{DG-cat} \Rightarrow \text{T}(A, B) \) category

of quasi-functors ... Ob \( \text{T}(A, B) \) are 1-morphisms of our 2-category, morphisms in \( \text{T}(A, B) \) are 2-morphisms of \( \text{DG-cat} \).

Composition is functor \( \text{T}(A, B) \times \text{T}(B, C) \rightarrow \text{T}(A, C) \)

Associativity here is structure, not property:

- Weak associativity:
  \[
  \begin{array}{ccc}
  A & \xrightarrow{F} & B \\
  \downarrow & \searrow & \downarrow \\
  C & \xrightarrow{G} & D
  \end{array}
  \]

- is an isomorphism

In all three -- *associativity constraint

1. Pasty axiom for composable functors

\[
(F_1 F_2) F_3 F_4 = F_1 (F_2 F_3) F_4
\]

\[
F_1 (F_2 (F_3 F_4)) = (F_1 F_2) (F_3 F_4)
\]

\[\forall A \in \text{DG-cat} \exists ! \text{Ho}(A) \in \text{T}(A, A) \text{ s.t.} \]

3. Isomorphism \( \text{id}_A \Rightarrow F \) functorial with \( F \)

\[ F \text{ similar for } \text{Fut} \Rightarrow \text{Fut} \]

Here \( \text{T}(A, B) \) will not be triangulated.

- Problem: should really replace \( \text{T}(A, B) \) by \( \text{DG-categories} \)

Possible over a field, not over ring...

\[ A \in \text{DG-cat} \Rightarrow \text{Ho}(A) \text{ triangulated category} \]

will give a 2-functor \( \text{Ho} : \text{DG-cat} \rightarrow \text{Cat} \)
Stupid 2-category: $\text{DG-cat}$
- Objects are $\ast$-categories.
- Morphisms are $\text{DG-functors}$, 2-morphisms are closed morphisms of $\text{DG-functors}$.

A closed morphism is $\eta: F \to G$ s.t. $\forall A : \eta(A, F(A)) \to G(A)$ is closed.

Problem: want to invert quasi-equivalences, but this notion of closed morphism not invariant under quasi-equivalence.

Ex. 1: arbitrary $\text{DG-category}$ $\mathcal{C}$ with $\text{DG-algebra}$ $k$.

As $\text{DG-algebra}$ on $\mathcal{C}$ with one object $T(k, A) = T(k, \text{pretr}, A)$

Answer: $H_0(X)$

Ex. 2: $T(A, k) = k$-version of $\text{pretr}$

$k_3 = \text{semifree envelopes}$

Answer: $T(A, k_3) = \text{derived category of } A$-modules

(well $A$-mod-$k_3$ is just $\text{DG-functor } A \to \text{complexes}$)

Here get derived cat. homotopy category

... does't change under quasi-equivalence unlike homotopy cat.

$= H_0(A^0)$

$k^\perp\text{ semifree } A$-modules

$= D(A)$

Ex. 3: $T(A, k^{\text{pretr}}) = D(A)$ full subcategory

$= \{ F: A \to \text{complexes s.t. } F(a) \text{ is quasi-isomorphic to a finite complex of } k \text{-modules of finite rank } \}$

Keller's def $k$ = field

$T(A, B) := D(A \otimes B^0)$ derived category of $\mathcal{C}_k$

..."ind quasi-functors"

$N_\ast T(A, B)$ give wedge $\text{Fun} : H_0(A) \to D(B^0)$

$= H_0(B)$
\[ T(A, B) = \{ M \in \text{Mod}(A, B) : \forall a \in \text{Hom}(A) \text{ isomorphic to an object of } \text{Hom}(B) \} \]

- can define \( \text{DGcat} = \overrightarrow{\text{DGcat}} \) where homomorphisms are ind quasi-factors in \( I(A, B) \) ...

Composition: \( M \circ I(A, B) = D(A \circ B^0), N \circ D(B^0) \Rightarrow M \circ B^0 \circ N \in D(A \circ B^0) \)

Note: over a field can lift anything from \( D^0 \) to \( \text{DGcat} \) ... can choose all cohomology trivial over a field! \( M, N \) semi-free don't need to derive \( M \circ B^0 \circ N = M \circ B^0 \circ N \) semi-free.

Note over field: only \( \circ \) will be well defined under quasi-equivalence (without use of above assumption ...)

Any \( k \) - everything works under additional assumption of "DFeness":

Def: \( k \) is homotopically flat if all hom complexes are homotopically flat.

Def: A complex \( C^\cdot \) is said to be homotopically flat if \( C^\cdot \circ (acyclic) \) is acyclic.

Homotopically Flatness & Projectivity: Recall \( M \) is projective if

\( \text{Hom}(M, -) \) is exact,

complex \( C \) is homotopically projective if \( (D) \text{ functor} \)

\( \text{Hom}(C, -) \) preserves acyclics \( \Leftrightarrow \) preserves quasi-isomorphisms

A semi-free complex is homotopically flat (\( \Leftrightarrow \) of \( k \)

semi-free modules \( \otimes \) commutes with \( \text{Hom}(A, B) \))

Stable homotopical projectivity \( \Rightarrow \) homotopically flatness

Suppose \( C \) is homotopically flat \& acyclic \( \Rightarrow \)

\( (\otimes \text{acyclic}) \) is acyclic ...
A DG category can have a semi-free resolution, hence in particular a homotopically flat resolution.

\[ A \rightarrow A_i \rightarrow A_{i-1} \rightarrow \cdots \rightarrow A_0 \]

We need to compare categories \( T(A_i, B_i) \) (don't need to resolve with \( A_i, B_i \)).

\[ \text{Ob } T(A, B) = \prod_{i} T(A_i, B_i) \]

Don't have to be strict in defining objects in a category since only care about categories up to equivalence.

or even \( \prod_{i} T(A_i, B_j) \).

or etc.

\[ A_i \rightarrow A, \quad B_i \rightarrow B \]

\( \forall i, j \), there exist canonical objects \( F_{ij} \in T(A_i, A_j) \) up to equivalence.

\[ i, j \in I \text{ with coproduct } i \oplus j \in I \]

\( i, j \oplus k \in I \), with \( F_{ij} F_{jk} 

\[ T(A_i, A_j) = \bigoplus_{i,j} D(A_i \otimes A_j) \]

(\( \forall i, j \))

\( A_i \rightarrow A_j \rightarrow A_k \rightarrow \cdots \rightarrow A_0 \rightarrow \text{Bounds} \)

\[ (a, b) \in A_i \otimes A_j \]

\[ \text{Hom}(T_i(a) \otimes T_j(b)) \]

\( \Rightarrow \) canonical \( F_{ij} \).

So this allows us to define DG category without imposing strictness, which is often enough.

\[ \text{eg. dg complexes of abelian groups} \]

Problem doing all this on DG level:

- Get not concrete complexes but complexes up to canonical homotopy equivalence, should instead

DG category, to 2 objects assign many models of hom complexes & compatible family of homotopies.
equivalences ... even weaker than Ab category ... so
kind of $\mathcal{D}$-category, ... so for now
$T(A,B)$ is only triangulated not $D$.

Note: category $A$ is quasi-equivalent to
category of pairs $X,Y \in A$ and map $X \to Y$
with quasi-isomorphism, i.e.,

horizontall diagoal of $A$.
Each of the two preorders is a homotopy
equivalence with $\mathcal{A}$.

Kontsevich model

There is a canonical 2-functor $\mathbf{D}G\mathbf{a}$ 
(assumed only flat versus homology).
$F : \mathbf{A o B}^0 \rightarrow \mathbf{D}G\mathbf{a}$

analogy of bimodule case to algebra $A \rightarrow B$

- namely $\mathcal{B}$ is a $(\mathcal{A} o \mathcal{B}^0)$-module.

- in categorical version $\mathcal{M}_{\mathcal{E}}: \mathcal{A} \times \mathcal{B}^0 \rightarrow \text{complexes}$

is $\mathcal{M}_{\mathcal{E}}(a,b) = \text{Hom}(b,F(a))$.

which actually maps in $T(a,b) \in I_{\mathcal{B}}(a,b) \Rightarrow D(\mathcal{A} o \mathcal{B}^0)$

and the functor $\mathcal{B}(a) \rightarrow \mathcal{B}(a,b)$ carry from $\mathcal{M}_{\mathcal{E}}$ is

just $\mathcal{B}(F)$.

So we have the 2-functor on morphisms.

Stere: any quasi-functor can be visual as functor
$A \rightarrow \mathcal{D} o \mathcal{B}$-modules with quasi-operations $\mathcal{D} o \mathcal{B}$. 

j quasi to representable one

--- localization of factors $A \rightarrow \mathcal{B}$. ---

$\text{Ext}^1_{\mathcal{A} o \mathcal{B}^0}(\mathcal{M}_{\mathcal{E}}, \mathcal{M}_{\mathcal{E}}) = \text{Ext}^1_{\mathcal{A} o \mathcal{B}^0}(\mathcal{M}_{\mathcal{E}}, \text{Hom}(\mathcal{F}, \mathcal{G}))$

$\text{Hom}(\text{Hom}(\mathcal{F}, \mathcal{G}))$ is the $\mathcal{D} o \mathcal{A} o \mathcal{B}^0$-module

$(a, a') \rightarrow \text{Hom}(\mathcal{F}(a'), \mathcal{G}(a))$
What is $\text{Hom}_A$? For algebra $A$ as a $A$-bimodule.

$\text{Hom}_A$ is the category whose DG $A \otimes_A$-module 
$(a, a') \mapsto \text{Hom}(a, a')$ (i.e. $\text{Hom}(\text{Id}, \text{Id})$)

This gives 2-morphisms for any DG bimodule $M_A^C$, with

This gives the full subcategory of

with pullback and pushout.

Ex: If $A$ is semi-free, then $\text{Ob}_{\text{DG}(T(A, B))}$ is

Note $\text{Hom}_A = \text{diagonal bimodule}$

$\text{Ext}^i(\text{Hom}_A, -) = \text{H}^i(A)$ (cohomology)

- gives full description of DG $A$ for $A$-

semi-free by a base exercise... in general must

use semi-free resolutions.

A semi-free, i.e. $\text{Ann} \text{cochains of } A$ is semi-free $A$-

standard $\text{Ann}$ has a semi-free resolution.

- has a semi-free resolution

- use semi-free resolution.

- can compute $T(A, B)$ in terms of this resolution.

$\text{std}(A)$:

Oh $\text{Ob}_{\text{DG}(T(A, B))}$ are DG-factors $\text{std}(A) \rightarrow R$

aka $\text{Ab}$-factors $A \rightarrow B$ ...

$\text{std}(A)$ is named, counter to two adjoint factors,

maps to the identity ("twisting cocycles" of Quillen)

gives way to understand $\text{Ab}$-factors.

$\Rightarrow$ $\text{DG}$ model this way for $T(A, B)$ ...

\[ \text{std}(A) \]

\[ \text{std}(A) \text{ is semi-free, i.e. } (A^*) \text{ is semi-free.} \]
Example of comparison of Keller & Kontsevich models:

\[ T(A, k) \quad (k_{pers} = \text{parameters with } A = \text{DG algebra for simplicity}) \]

Keller: semi-free DG - A-module M s.t. \( \text{dim}_k H^*(M) < \infty \)

Kontsevich: finite complexes of fin dim k-vector spaces with a weak action of A.

Weak action: For any \( a \in A \) have an endomorphism \( fa : C \to C \) (depending linearly on \( a \)), \( f_0 a \neq a f_0 \) necessarily, but \( f_0 = a f_0 = d(a) \), some combo of \( d \) which is compatible with boundary, etc...

- Come naturally to this by trying to construct f.d. model of Keller module \( M \). \( M \) is homotopy equivalent to a f.d. complex \( C \), get for \( a \in A \) an endomorphism up to homotopy, will satisfy \( A \)-gerbe relations.

Again find \( \text{dim} \) by losing strictness of action.

Quotients of DG categories

Verdict: \( T \) triangulated, \( Q = T \) full triangulated subcategory get \( I \to T/Q \) quotient triangulated category with universal property w.r.t. such quotient.

What is DG version: \( B \in A \) DG categories [Keller]

2-categorical version: [advantages: works over rings, disadvantage: less precise, weaker form of uniqueness]
Theorem-definition A 2-categorical quotient of $A$ and $B$ is a pair $(C, \xi)$ of a DG category $C$, a quasi-functor $\xi \in \text{Hom}(A, C)$, s.t. the above equivalent properties hold:

1. $\text{Ho}(\xi) : \text{Ho}(A) \to \text{Ho}(C)$ identifies $\text{Ho}(C)$ with $\text{Ho}(A)/\text{Ho}(B)$, i.e., $\text{Ho}(A) \to \text{Ho}(C)$ (B killed in $\text{Ho}(C)$).

   (Note that $\text{Ho}(C)$ is an equivalence).

2. For $C \in \text{DG Cat}$ the functor $\text{Tor}(\xi, C) \to \text{Ker}(\text{Tor}(A, C) \to \text{Tor}(B, C))$ is an equivalence. [When $B \to A \to C$ is zero.] (Verdier setting: universal property $\text{Ker}(\text{Tor}(A, C) \to \text{Tor}(B, C)) = \text{Tor}(\xi, C)$ every further killing $B$ factors through $C$.)

Such $(C, \xi)$ exist in 2-categorical sense.

Not obvious that either implies the other.
Recall $\text{Ho}(A^+) = T(C^+, A^+)$ while second property formulated in terms of $T(A^+, -)$.

In brief each $T(\cdot)$ is actually a DG-category.

Worse precise version, what is tight in shifting case.

Problem is not to prove but to formulate question.

It is anything.

Suppose $T(A, C)$ exists for a DG category $D(A, C)$
- above formulation means have collection of objects $\xi, \eta \in D(A, C)$ together with a homotopy class of homotopy equivalences $\xi \to \eta$.
- So more precise version need either specific $\xi$ or functor contractible space of morphisms from the object $\xi, \eta$ (i.e., $\text{Prod}(\xi, \eta) \to \text{Hom}(\xi, \eta)$).

$k$ not a field don't have $D(A, C)$ yet...
Def. A DG quotient $A$ and $B$ is a diagram

\[ \xymatrix{ T(A, C) = T(C, C) } \]

s.t. $(C, T)$ is a 2-categorical quotient.

Replacing $T$ by $D(C) : DGC(C, C') \to \text{ker}(\text{DG}(C) \to D(C'))$

is a quasi-equivalence (automatic from definition 1, just need to define $DGC(C, C')$)

2 constructions. $T :\text{new (Donald)}$, works if $A$

is homotopy flat.

$\text{Ob } C^{\text{op}} = \text{Ob } A^{\text{op}}$, add new morphisms:

\[ \forall X \in B, \text{ add } Ex : X \to X, \text{ of degree } -1 \]

\[ dEx = \text{id}_X \] $C$ is freely generated by $A$ and the $Ex$.

If $A$ is homotopy flat this gives desired answer.

Rather than inverting quasi-isomorphisms, part of killing objects is by killing all morphisms to & from this object (homotopically) -

enough to kill identity morphisms.

II. Keller's definition: consider inclusion $A \to B$.

$B \to B^{\perp}$ full subcategory: we have

$\text{Ext}^{1}(B^{\perp}, M) = 0 \forall B \in B^{\perp}$

Fact: $H^0(B^{\perp}) \cong H^0(A)$, $H^0(B^{\perp})$ is an equivalence.

$H^0(A) / H^0(B^{\perp})$ fully faithful.
So define \( \tau \in C = \mathcal{L}(\mathcal{B}) \colon \exists \alpha \in \Delta \) s.t. \( \triangle \) with \( \mathcal{N} \subset \mathcal{B} \)

--- essential image of \( H(\mathcal{C}) \rightarrow H(\mathcal{B}) \)

Here consider \( \mathcal{D} \) valued \( \mathcal{A} \rightarrow C \) but need \( \mathcal{N} \subset \mathcal{B} \)
--- try to assign \( \mathcal{M} \) to \( \alpha \) using a choice
but with compatible space of cores
(possibly equivalent s.s.s.)

Equivalently, consider \( \mathcal{N} \subset \mathcal{B} \)
\( f \) of deg 0 \( (P : \mathcal{N} \rightarrow \mathcal{B}) \)
so that \( \text{Core}(f) \subset \mathcal{B} \)
--- Core \( \mathcal{A} / \mathcal{B} \) is exact
\( \text{Core}(f) / \mathcal{B} \) is exact
\( \text{fl} / \mathcal{B} \) is a quasi-isomorphism

Restrict to \( \mathcal{B} \) - think of \( \mathcal{A} \) as a representable functor by \( \mathcal{D} \)
\( \mathcal{D} \rightarrow \mathcal{B} \) and take a semi-free resolution \( P \rightarrow \mathcal{B} / \mathcal{B} \)
--- this is our choice
but unique
in strongest homotopical sense
(constantly space of cores)

Not self-dual definition - so would like universal property to identify all.

How might you come to this data? Why semi-free \( \mathcal{D} \)-modules?
--- believe \( \mathcal{D} \)-module exists:
\( \mathcal{A} \rightarrow \mathcal{C} \) with property

Yoneda:
\( C \rightarrow \mathcal{C}^0 \rightarrow \mathcal{D} \cdot \mathcal{C} = \mathcal{D} \cdot \mathcal{A} \)
\( A \rightarrow \mathcal{D} \cdot \mathcal{A} \) nat.

\( f \cdot (C) \rightarrow H(\mathcal{A} : \mathcal{D} \cdot \mathcal{A} \rightarrow \mathcal{D} \cdot \mathcal{A}) \)
--- fully faithful, so natural to look
for \( C \) in \( \mathcal{A} \).

In fact \( \mathcal{A} \) (infinite torsor) is implicit in Verbera:

\( \text{Ob } T / \mathcal{Q} = \text{Ob } T \), morphisms: invert \( \mathcal{Q} \) quasi-isomorphy
--- i.e. morphisms with core in \( \mathcal{Q} \).
So morphisms are zigzags in \( T / \mathcal{Q} \), \( \mathcal{E} \in \mathcal{Q} \),
But also to take \( \mathcal{E} \) and \( \mathcal{M} \).
Verdict: \( X \to \mathcal{C} \to \mathcal{E} \to \mathcal{D} \to (X, Y) = \)

\[ \text{Imm } \mathcal{C} \xrightarrow{c_y} \mathcal{E} \]

over all "injective registers" \( c_y \) given by

\[ c_y = \text{category with objects } 0 \text{-maps } X \to Y \]

if maps \( X \to Y \), give map \( X \to Y \) and category \( c_y \).

\( c_y \) is a filling category.

-- need for group structure on the limit of \( c \).

\( c_y \) commute with finite projective limits (i.e. pullbacks).

Claim this is same infinity as \( \mathcal{A} \to \mathcal{A} \)

-- can write in homotopy category but Keller's construction takes care of this via semifun.

\[
\text{Exer. a, a' } \in \mathcal{A} , \quad \mathcal{P}_{a,b} \xrightarrow{p_{a,b}} \mathcal{A} \text{ semifun. nodis. }
\]

\[
\text{Show directly that } \mathcal{E} \text{ as follows: } \\
\text{Hom}_{\mathcal{A}}(a', a) = \text{Imm } \text{Hom}_{\mathcal{E}}(a', a) \\
\text{and } \\
\text{Keller } \text{Hom}_{\mathcal{A}}(a', a) \text{ of } \mathcal{P}_{a,b} \text{ (i.e. projective limit)} \\
\text{up to homoto. }
\]

\[
\text{write } \mathcal{P} = \mathcal{P}_{a,b}, \text{ with } \mathcal{A} \text{ nodis. filling (filling family)}
\]

-- see codensity of \( \mathcal{E} \). The limit over filling in

\text{direct limit of codensity -- slightly different filling categories}

\text{get hom to from one to another by direct construction of one sequence -- make it is an equivalence.}