Some doubts problems

- Important distinction between algebraic Langlands: use theory of rational adeles, instead of Galois reps separate \rightarrow treat Eisenstein & cuspidal separately

Geometric story: they lie together, reducible reps can be deformed to others!

Today: 2 separate stories: G K V conjecture of irreducible GL, Beilinson-Drinfeld for geometric Eisenstein, satisfactorily for generic orbital Ls.

Problem 1: Take abelian toris to GL, deform it to be irreducible (using a universal deformation)

\rightarrow how to construct Langlands theory of this universal deformation? Can one find a deformation of geometric Eisenstein reps as Hecke eigenform?

- In general even for GLz: to be done very crudely C. V. technique but would be better to deduce in more uniform form

G reducible \Rightarrow \text{T} max torus, T good torus

Cas \text{\mathfrak{t}} \text{local system} p on X, generic

\Rightarrow E_{isp} \text{ Eisenstein series (Beilinson-Drinfeld): power series on } B_{reg}

E_{isp} = \bigoplus \text{E}_{isp} \text{ graded object labeled by } \lambda \text{ weights of } T

\text{version of infinite series}

Graded object \leftrightarrow \text{torus action: } \text{T} \text{ acts on } E_{isp}

\text{with character } \lambda

p \text{ a} \text{ torus } \Rightarrow p \text{ considered as } \mathfrak{g} \text{ local system}
Wahl: $V \rightarrow \text{local system } p \rightarrow \text{lagendeh factors, a complex } \mathcal{L}_p \text{ on } \mathbb{G}_m$.

As $p$ varies, fibers of $\mathcal{L}_p$ (complex of sheaves) should form complex of $\mathcal{O}$-modules on stalk of local systems. Even for fixed $p$ have rank-$1$ stalk since $p$ has automorphism $p$ grading is the central stalk on $\mathbb{G}_m$ (center of $p$ generic at $\xi_t$).

Fix $p_0$ & consider formal stalk $S$ of $p_0$'s infinitesimal close to $p_0$. $p_0$-stalk of $\mathcal{O}$-module by a game (crude) $V/\mathbb{G}_m$

$V = \text{Spec } \mathcal{O}$ complete local ring $\mathcal{O}$-def of complete red-ide local systems $\rightarrow V$ singular

$\mathbb{G}_m$ case $\Rightarrow$ quadratic case singular.

generally systems of quadratic cases

that of $\mathbb{G}_m$ shadowy $T \cap [A, B] = 1$

get $\mathfrak{m}$-local relations from $\mathfrak{m}$-def of center $\mathfrak{m}$

Problem: construct $\mathcal{L}_p$, on $(\mathbb{O}, \mathcal{O})$-module a $\mathcal{O}$-module. 

Lie algebra setting on also specify class of objects. 

Verdier stalk of $\mathcal{O}$-modules with $\mathfrak{m}$ action

Result: $B T < S$ should get Einstein series & $\mathcal{L}_p$

should be Herbe eg-nuch.

Technique to solve $\Rightarrow$ semi-ample flag $\mathcal{O}$

Finkenberg, Miura: geometric realization of $\mathfrak{g}$

get $\mathbb{G}_m$ variety $\mathcal{O} \rightarrow E_{\mathbb{G}_m}$.

Hint: $\mathfrak{g}$-determine many of local systems: controlled by cohomology of Lie algebra twisted by $\mathfrak{g}$ with $\mathfrak{g}$-action?
\((\gamma_2, \text{Flag II}) \Rightarrow \log: H^0(\cdot) \rightarrow \text{End} \, E_{\log}\)

Now we cut a few of the delta's \(\Rightarrow \) solve problem in char. 0 of coefficients...

To get away from char. 0 need not delta vanish but different entire vs. vs. not. Very needed but difficult to group should act

char. 0: have delta \(\mathbb{R} \mathbb{P}(X, \log_{\mathbb{P}_0})\) (i.e.,

want to convert rep. \(R\) to \(\text{End} \, E_{\log}; \quad E_{\log}\) should be isomorphism. Rankin-Selberg \(\text{L-element}\)

suggests this is an isomorphism.

If everything is ok, \(\log E\) is all proper \(\Rightarrow\) probably don't need more stupid algorithm of delta's with higher rank...

Q: What happens when \(\mathbb{P}_0 = \text{trivial}?\)

\(\text{GL}_2\) picture: (de Rham, etc., etc., etc., etc.)

\(\text{Noise hope:} \quad (\mathcal{O}, \mathcal{O}) - \text{module on} \, \text{LocSys}_{\mathbb{G}_m} \times \text{Bun}_{\mathbb{G}_m}\)

\(\text{flat} / \text{LocSys} \times \text{Hodge-\text{esquisse}}\)

\(-\quad \text{can realize this noise hope on open subspace} \quad \text{LocSys}_{\mathbb{G}_m} \times \text{Bun}_{\mathbb{G}_m} \Rightarrow (\mathcal{O}, \mathcal{O}) = \text{Hodge}\)

\(\text{P: difficult if reducible & irreducible case.}\)

\(\text{Isomorphic to each other -- eg trivial and difficult!}\)

\(\text{Theorem:} \quad \text{Maurin.
}

\(\text{M} \text{ on } \text{LocSys}_{\mathbb{G}_m} \times \text{Bun}_{\mathbb{G}_m} \text{ so that } \text{M is}\)

\(\text{codimension over Loc Sys}_{\mathbb{G}_m}, \text{Hodge-\text{esquisse}}, \text{etc.}\)

\(\text{Replace pictures by CM, vector properly, not flat} \)

\(\text{Loc Sys are l.c.i., in particular CM, so}\)

\(\text{no contradiction flat vs. CM!}\)

\(\text{on a smooth variety being flat } \Longleftrightarrow \text{CM,}\)

\(\text{good sing. flat } \Rightarrow \text{CM}\)
Does M define an equivalence of categories?
A: Which categories?  'til or not, etc?

Analogous situation: Fourier transform in $\mathbb{R}^d$ on
It obeys group & field $H^{*}$ which class of functions?  need to specify.
Generic representation $\to$ complexes
function spaces $\implies$ derived categories
need generic function analysis, consider various
classes of functions

test situations: as local systems with the inclusion
of some parts be fixed local monodromies
if at least one local monodromy nontrivial $\to$ easy
local systems. Need to think about group Cech
analysis: stack B with this case not
questions, loc sys singular so use
different notions of address/potential

Weka: check equivalence of derived categories in simplest
truly nonabelian situation, $\mathbb{P}^1$ with monodromies
to eliminate all reducible etc $\to$ remove all
singularities/infinite, some genericity.

To construct (triv): want to use Einstein
series construction... Grady tors symmetry
But for triv out of sys / symmetry
is G2 $\implies$ need exactly one B G2 not 17

F-dim vec of G2 decomposes onto T, labeled
by pairs of Higgs $& V_m = V_{mm}$ on L primes
(very! symmetric)
- so if your Einstein series carried G2 coh

Some trick to make vs get finite (kill above)
complex of D-modules
proof very technical & computational.
would love such base functional eqn. \[ f(x) \rightarrow \text{something would get made for an eqn.} \]
as if \( E \) is not a pole.

Suppose \( f(t) = \sum a_n t^n \quad g(t) = \sum b_n t^n \)
form series in two variables in opposite order.

What does \( f(t) = g(t^n) \)
mean? A finite series, Laurent polynomial.
More sophisticated; two Laurent expansions
of zero function,

\[ a_n \cdot b_m \rightarrow \text{root n of general progression} \]
\[ (\sum \text{finite series}) \leftrightarrow \text{pole of} \]
\( E \), \( s \) series when \( a_n \cdot b_m \) are equal,
so no pole equation, \( \text{V}_{\text{max}} = \text{V}_{\text{min}} \).

So don't have action of \( GL_2 \) since \( E \) is not a pole!

Milnor-Eisenbud etc: Every \( GL_2 \) action of the Lie algebra
Login -- so don't get stack or stack \( B GL_2 \).
Maybe this is wrong object.

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Familiar notation: \( \tilde{f}(g) = \int_R \tilde{K}(\tilde{s}, \tilde{x}) \)
inverse Fourier: at a minus sign
Special version: replace kernel sheaf by
Verdier dual.

**Orthogonality:** \( \int e^{i(s-y)x} dx = \Gamma(s-y) \)
--- need such both ways, one
as \( \text{B}_{\text{GL}_2} \) in \( D\)-module sense \& one
as \( \text{L}_{\text{GL}_2} \) "\( D\)-module."

**Orthogonality** / \( D\)-module: Lysenko (or \( B_{\text{GL}_2} \)):
--- need of capital \( D\) or \( \mathbb{Q}\) and dualized

**Vectors over functions**: \( GL_2 \) answer by
Routine. Systems defined \( \Rightarrow \) geometric vector for \( GL_2 \).
Orthogonal to our local tools: no standard tools to replace.

Lysoko: only irreducible data at work. Direct counter to no orthogonal (or see) or orthogonal scalar square.

Factor on represented to our subject.

An important technical tool — another corollary of $\text{Bun}_B$ or $\text{Specs of rational maps}$ different for $\text{Direct germ}$'s.

\[ G = B \quad \text{Bun}_B \to \text{Bun}_B \]

\[ \varphi = \text{const of relative compactification of } \]

this representable morphism:

\[ \varphi^{-1}(F) = \text{B-structures on } F \quad \text{sector of } (G/B) = \Gamma(X, (G/B) \gamma, ) \]

\[ \varphi^{-1}(F) = \Gamma_{\text{quasi}}(X, (G/B) f) \quad \text{quasi-secures} \]

Fibers of $\text{Bun}_B \to \text{Bun}_G$ will be compact but not algebraic.

$\text{Bun}_G$ not algebraic, since just schemes, as fibers.

\[ \text{SprC} \quad \text{Bun}_B(S) = \{ F \in \text{Bun}_B \quad \text{rational sector of } (G/B) \}

\text{rational sector = sector on } X \times S \to X \quad \text{D finite } S

\text{eg } F = \text{triv} \Rightarrow \text{all rational sectors to } G/B

\text{any } \text{Bun}_B(F) \text{ becomes trivial on an empty open subset, so just get rational maps to } G/B

\text{study topological object } \text{Rat}(X, Y) \text{ of variety } X \to Y$
\[ \text{Rat}_{x \to y}(S) = \text{set of rational maps } X \times S \to Y \]
\[ = \lim_{D \to X \times S} \text{Map}(X \times S - D \to Y) \]
\[ \text{in } \mathcal{S} \quad \text{(require } D \text{ have zero dimension)} \]

Suppose \( \mathbb{G} = C \)
\[ S/C = S - \text{point} \quad \text{of } \text{Rat}_{x \to y} \text{ defining map } \]
\[ q_x : S(C) \to \text{Rat}_{x \to y}(C) \]

We will call such \( q_x \) to be continuous.
At strongest topology s.t. all \( q_x \) are continuous.

\text{Example: } \text{Rat}(\mathbb{P}^1, \mathbb{P}^1) : \text{next mainly} \]
\[ \text{rational maps from } \mathbb{P}^1 \text{ to } \mathbb{P}^1 \text{ are regular} \]
\[ = \text{locus of regular maps, split as disjoint union of maps of fixed degree} \]
\[ \text{Here we'll only get attention by degree} \]

\[ \text{Rat}_{\leq 1}(\mathbb{P}^1, \mathbb{P}^1) : \mathbb{Z} \to \frac{a z - b}{c z + d} \]
\[ \leftrightarrow \text{not fix } (a, b) \text{ up to rescale, norm} \]
\[ \Rightarrow (a : b : c : d) \in \mathbb{P}^3 \quad \text{If not fix is degenerate} \]
\[ \text{ad } = a c + b d \text{ get degree } O \text{ map} \]
\[ \text{quadratic in } \mathbb{P}^3 \quad \text{O}(\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3) \]
\[ \text{ad } = (a : b : c : d) \mapsto \left\{ \frac{a}{c} = \frac{b}{d} \right\} \mathbb{P}^1 \]
\[ \text{So projection on this } \mathbb{P}^1 \text{ is just the map we want} \]

\[ \text{Rat}_{\leq 1} = \text{take } \mathbb{P}^3, \text{ contract quadratic } \]
\[ \text{to } \mathbb{P}^1 \text{ under map } \frac{a}{c} = \frac{b}{d} \]
\[ \text{forbidden in algebraic geometry, but} \]
\[ \text{fine as topological space.} \]
equivalence relation on \( B_{\text{inB}} \) which is (1) proper (two projects are proper) (2) rest is discrete (in P)

\[ \text{e.g. G(3)}: \text{rl 2 realize} \quad L, A_1, A_2 \subseteq L \quad \text{s.t.} \quad A_1 \cap A_2 = \emptyset \]

\( B \) cop P \( \rightarrow P \) is idm of two

- a row polynomial of degree (e.g.) P
- \( \text{to a} \) subgraph in projective space for equivalent points \( \text{pf} = 9g \).

Please states etc. to define on algebraic variety with proper equivalence relation — 5th degree of states, with Verider qualities etc.
- states of states, upstairs with descent data
- two notions of equivalence ! & x

\[ R \subseteq X \times X \text{ proper equivalence relation, } R \xrightarrow{P} X \xrightarrow{\Pi} X/R \]

Define \( T(X,F) \) on \( X \) with \( \exists \)

\[ \left( P^* F \xrightarrow{\ast} P^* F \right) \]

Problem: define what is \( \Pi^* F \)?

\( \ast \) descent data on \( \Pi^* F \).

Suppose \( X/R \) exists in usual sense — use proper base change

\[ R \xrightarrow{P} X \xrightarrow{\Pi} X/R \]

Cotpole \( P^* \Pi^* F \) by \( \ast \)

\[ \text{Adjustment: } T(X, \Pi^* F) = \Pi^* F \]

\[ E = \text{ker} (\Pi^* F) \]

Suppose \( X/R \) exists, how \( \Pi^* F = F \). Now

\[ \Pi^* \text{ write empirical scheme } \Rightarrow \text{ use proper base change} \]
for $F \rightarrow H_\mathbb{Q}^F \rightarrow \cdots$  

$\Rightarrow$ make $F$ in terms of direct images of $\mathbb{Q}$

given in terms of $F$

want to describe $H_\mathbb{Q}^F$ from $F$ & descent data

in case everything exists.

resolve $F$ using $F$'s, but by base change.

however pullback of $\mathbb{Q}$-sheaves
do not exist still have $\mathbb{Q}$-cycles on the tower

want a simplicial set

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**Generic Example:**

- take "iteration pullback" of perverse sheaf on $\text{Bun}_G$ to $\text{Bun}_P$, & pushforward to $\text{Bun}_G$

- consider pushforward to $\text{Bun}_G$ before $\text{Bun}_B$

- "Normal theory": fibers of $\text{Bun}_B \rightarrow \text{Bun}_G$

are contractible

Another potential application:

$C = \mathbb{C}P^1$ :  

$\text{Bun}_B \rightarrow \text{Bun}_G = \mathbb{C} \rightarrow \mathbb{C}^1$

rank 2 + fixed rank 1 singularity

can construct complex on $\text{Bun}_G$ assoc to inv loc syst now need to descend it to $\text{Bun}_G$.

Use fact that fibers are projective space hypersurfaces.

can push both stack on $\text{Bun}_G$ will to $\mathbb{C}P^1$.

or (I need positivity to descend

descent for complexes more complicated

- need contractibility
If don't impose irreducible local systems would probably get nonprocesse complexes; would be nice to have something with contractible fibers over B that related to B... 

General contractibility statement: \[ \text{[Contractibility]} \Rightarrow \text{Y vanish} \]

- Cohomology with trivial coefficients, vanishes
- Both make sense in this general context.

Let \( X \) = irreducible variety (e.g., curve), \( E = k(V) \)
\( Y \) = scheme of finite type / \( E \)

\[ \Rightarrow \text{associate fiber space (factors)} \]
\[ Y_E : k \text{-} \text{sections} \rightarrow \text{sets} \]
- "kernel" to choose a model for \( Y \)

As generic fiber of \( Y \rightarrow X \) scheme of finite type

\[ Y_E(S) = \text{family of reduced sections of } Y \rightarrow X \]
\[ = \lim_{D \text{ proper of } X} \text{Max } ((X \times S) \setminus D, Y) \]

- Independent of model \( Y \).

To be checked: Suppose \( Y \) is irreducible \& \( V \in \mathbb{V} \)
\[ \exists \ U \subset Y \ \text{Zariski open isomorphic to an open subset of } \mathbb{A}^n \]
\[ \text{Then } E \text{ is contractible.} \]
\[ \lim_{U \text{ closed of } Y} \lim_{U \text{ closed of } Y} \text{ is zero.} \]

(\dagger): If \( Y = U \cap V; \Rightarrow Y_E = (U \cap V)_E \) hopefully still goes!

- Case with spaces of reduced ryzk to affine schemes

\( U \subset \mathbb{A}^n \) properly \( \Rightarrow \) claim \( U_E \) is contractible.
3. Microlocalization

DeRham setting: $E$-modules

Parallel picture for perverse sheaves (Kashiwara ... not suitable to complex algebraic context ... requires reconstructable sheaves)

& modified by MacPherson-Vilonen-S.Gelfand]

$T^*M \to 0$

$\Pi_{\geq t}$

$P(TM)$

$M$ smooth algebraic variety: $P(TM) := (T^*M \to 0) / G_m$

Contact variety, with sheaf of associative filtrable algebras $E$, complet $\mathbb{R}$-wrt filtration $\text{gr } E = T^*C$

$T^*C$ is $\mathbb{Z}$-graded, with Poisson bracket of degree -1

Define $E$: sections over open affine subset (issue $M$ affine — construct local $M \times \mathbb{R} \times P(-)$ pair)

$P$ genus $F$ on $T^*M$, look of complement $\{ F = 0 \}$

$\Gamma_0(C, E) = \text{completion wrt filtration}$

$(\mathcal{D}_M, \text{invert all derivatives wrt } C)$

Suppose were model differential $F \Rightarrow$

invert (in complete) operator of form $F + G$ where $G$ is of lower degree

Equivalent loc. locally: write differential via symbols

$\Sigma a_{\alpha}(x) (\partial^\alpha x) \mapsto \Sigma a_{\alpha}(x) \xi^\alpha$

Symbol of a product expressed via symbol of factors via formula which notes sense for symbols more general for than polynomials! $F(x, \xi)$ infinite sums of long functions $x, \xi \Rightarrow$ explicit control...

Let $P: P(TM) \to M$ natural map $P^*D \to C$
\( N \text{ Dm-mod-le} \Rightarrow \Sigma \text{Dm} \quad \overset{\rho}{\rightarrow} \quad \rho^*N \text{ microlocal of } N. \)

Would like to apply to \( M = \text{Bun}_X \text{ --- k.C.} \)

What at smooth variety be a group...

Assume we can take notion of \( E \)-module on \( \text{Bun}_X \).

\[ T^*\text{Bun}_G = \{ (F, \eta) : F G \text{-bundles } \alpha X \eta \in H^0(C, \text{Log}_{\alpha} \otimes \omega_G) \} \]

Def \( \eta \) is weakly generic if \( \eta(x) \) is regular for \( x \in X \) generic (regular Hitchin invariant)

Problem: Can a local \( E \)-system \( p \) on \( X \) compare/identify/construct

the restriction of the micro/localize of \( p \) to \( T^*_p(\text{Bun}_G) \) weakly generic

Hecke correspondence on \( T^*_\text{Bun}_G \) preserve Hitchin fibration & more; if \( (F, \eta) \)

corresponds to \( (F', \eta') \) \( \mapsto F \mapsto \) identical generically & \( \eta, \eta' \) agree there.

\( X \mapsto Y \text{ map of smooth varieties } \Rightarrow \)

Lagrange in \( T^*X \times T^*Y \).

Correspondence \( Z \leq X \times Y \Rightarrow \text{Corr of } Z \leq T^*(X \times Y) \)

\( \Rightarrow \text{Lagrange correspondence between } \)

So for Hecke correspondence, if \( (F, \eta) \), \( (F', \eta') \)

\( \mapsto \text{generically } \eta \leftarrow \eta' \)

So if one is generic so is

So on consider Hecke correspondences restricted to generic

Hitchin operators provide for.
Theorem. \( \text{Sing Supp} \left( \mathcal{F} \right) \subseteq \text{zero fiber of } \mathcal{H} \). Fiber of \( \mathcal{H} \) is nilpotent with unique fixed point.

Let \( \mathcal{H} \) be a fiber bundle with fiber \( \mathcal{F} \). The fiber of \( \mathcal{H} \) is nilpotent with a unique fixed point.

If \( \mathcal{H} \) is well-behaved, then for \( \mathcal{G} \) (or \( \mathcal{G}_n \))

For \( \mathcal{H} \) on \( \mathcal{O} \), this property is clear by definition.

Zero fiber: \( \mathcal{H} \) is not nilpotent everywhere.

(at least set theoretically -- carefully with multivalues)

Global nilpotent core

We're looking only at \( \mathcal{H} \) which is regular nilpotent generically.

"Kashiwara" \( L \in \mathfrak{P}(\mathcal{M}) \) Legendrian & smooth

\{ Describe \( \mathcal{E}\)-moduli set -moderately separated \}

\( \text{on } \mathcal{L} \)

| local system on \( \mathcal{L} \) |

Careful: need to replace \( \mathcal{E}\)-moduli by twisted \( \mathcal{E}\)-moduli & correspondingly change \( \mathcal{E} \).

Similarly should consider \( \mathcal{E}\)-moduli-twisted local systems.

So we're in generic part of global nilpotent core, describe Hecke \( \mathcal{E}\)-moduli as local systems on smooth Legendrian -- restrict to smooth local systems.

Conjectural (partial) answer for \( G = GL_2 \):

Higgs bundle: \( (L \text{ rank } 2, \eta : L \to L \otimes \mathcal{O}_X) \)

nilpotent: \( \eta^2 = 0 \), \( \eta \) is not identically zero

What is \( (L, \eta) \) a smooth point of the nilpotent core? (as reduced scheme)
Assume: if all the zeros of \( \eta \) are simple.
Number of zeros can be arbitrary \& labels
the irreducible component for which \((L, \eta)\)
is generic. --- oo many components
(can guess correct \( g \) & \( b \)).

Suppose \( \eta \) has \( k \) simple zeros.

Conjecture: Fiber of \( \eta \) over \( \text{local system at } (L, \eta) \)
(generating \( \eta \)) of microlocalization \& \( \eta \) is

\[ \beta \otimes \ldots \beta \]

Conjecture: \( \beta \) irreducible component of Langlands functor
achieves as Radon transform (Freeness for homogeneous
functors).

Contact: complex on \( \beta \) should be effective relative
or \( \alpha \) dual to \( \text{Ext} (L, \eta) \) on \( \alpha \) dual of \( \beta \)-module.
Apply Radon to \( \text{Sp} \beta \) get more
sheaves on \( \beta \).

Radon transform on singular supports:
\[ \text{Radon}(T \beta) = \{ x \in \beta \mid \beta \text{ is hyperplane}\} \]
Self-dual variety --- see for \( \beta \) (dual \( \beta \)-module);
So \( \beta = \text{Radon}(T \beta) \)

2. Radon \( (SSupp(N)) \rightarrow SSupp(\text{Radon}(N)) \)

Under this identification
So Lasser calculates \( SSupp \) of \( \text{Radon}(N) \) using this construction:

In fact, all microlocal/\( \text{Ext} \) below cell under
Radon \( \Rightarrow \) can compute microlocalization of
the Langlands kernels.
η \rightarrow \text{nilpotent} \Rightarrow \text{flag of } F \text{ generically} \\
\Rightarrow \text{flag } \eta \text{ exists}, presented by \eta \in \Gamma (\tau, \eta) \approx \mathbb{C}^{\tau} \\
\eta \text{ has canonical filtration (central series)} \\
\text{gr } \eta \in \mathfrak{g} = \mathfrak{h} / [\mathfrak{h}, \mathfrak{h}] = \mathbb{C}^{\tau} \\
\oplus \mathbb{C} v \text{ vertex of Dynkin diagram} \\
\text{So get invariant of } \eta \text{ when one vertex of Dynkin diagram is deleted labeled by such } v \\
\text{So can formulate generalization of G2 to, say, } \text{to all } G \\
\text{despite lack of direct (natural) } \\
\text{lie in context.} \\
\text{Nice feature of this picture: answer is local, involves only points where } \eta \text{ not regular} \\
\text{might expect such factorization can outside of smooth case, factorization for maps from curve to nilpotent cone, labeled by fundamental weights?} \\
\text{but acts come up naturally for other groups in context of this scheme on } \mathcal{S} \\
\text{supp from local systems?} \\
\text{Answer seemingly independent of choice, such as passage to } \mathbb{P} \\
G_2: \text{fiber of flag locus through generic points has rank } 2. \ \text{Can attempt to write as torsor product of } 35 - 3 = 2 \text{dim vector spaces.} \\
\text{but this is only } 0 \text{ section! No genuine part of nilpotent cone!} \\
\text{Claim is that this picture literally holds} \\
\text{on generic part of nilpotent cone!} \\
\text{Should be related to Whitham flows.} \\
\text{classical catastrophe forms for } G_2 \text{ have multiplicities one.} \\
\text{No such for general } G \text{ in classical catastrophe theory.}
So this suggests a weakly generic locus has some multiplicity are results...

What's the relation to Whittaker series?

Conjecture: generic series on $Bun_G$ with

Whittaker coeff. $= 0$ — does it correlate

with $D$-twists close singular support misses

weakly generic locus?

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4.

Motivic Beilinson

\[
\{\text{Triangulated category of motives}\} \to \{\text{Derived category of Gal E/k - motives}\}
\]

$p$ on normal local system on $X$, rank $n$

\[ \Rightarrow p \text{ on } Bun_G(n) \]

$F \in Bun_G(n) \Rightarrow (\mathcal{L} p)_F \in \{\text{Derived category of Gal E/k - motives}\}$

(C) If $p$ is “motivic” can we define

\[ (\mathcal{L} p)_F \in \{\text{Triangulated category of motives}\} \]

— motivic automorphic reps?

Example of motivic local system: Take mod-$k$

ell-adic variety over base, almost regular

of good reduction.

More correctly: look at $p$ of Artin type

(finite image representations)

Simplest case: $n = 2$, $F$ rank 2 with half-dual,

Define complex of Galois modules: first

choose sufficiently regular rank 1 subsheaf $A$

of $F$ \[ \Rightarrow \text{Artin algebra of motives, clearly of motives above} \]

\[ (\mathcal{L} p)_F \to A \text{ of motives} \]

— need to show independent of choice, for different $A$. Need issue of complement, maybe some relies.
or complexes defined using standard factors from $p$ so are motive but agonize to show independent of $x$ is not motive in acting uses fact not local systems on $P^n/k$ is constant need motive version of this (basic case $P'$) is any motive local system on $P'$ constant?

Below: Any $\mathbb{P}^1$ thus a constant local system is constant

For central varieties

case of $P'$: central bundle $\Rightarrow$ Morse Atkin is a moving flag if $2$ complex conjugate give full space $\Rightarrow$ rep $P' \rightarrow$ upper half plane ... which must be constant.

$(\mathbb{P})_{/P}$ stacks \( X \xrightarrow{\pi} X \rightarrow P \in C \rightarrow \tilde{C} \)

\( Y_{/P} \) a variety with action of $S_n \times G^n$ (symmetric product of $X$)

$(\mathbb{P})_{/P} = R^\infty(\mathbb{P}_{/P} \rightarrow \tilde{C})$ - isotype quotient

where $p$ gives rise to $\rho^{(i)} = p \otimes \ldots \otimes p$ rep of $S_n \times G^n$

Motive over $\mathbb{A}$ replace $R^\infty(\cdot \rightarrow \tilde{C})$

by motive of the variety - object of additive category $G$ acts on $Y \Rightarrow$ acts on it; motives get direct sums/molds corresponding to isotypic components. So we have motives associated to $J/A$

... to show independence of $A$ use description of local systems on $P^n$, doesn't help us noticeably

Classical version of Atkin's heur: Atkin's formula

for Atkin rep, long as that (polynomial not Laurent polynomial)

(Atkin's conjecture)
Interpretation of Motivic Artin conjecture:
show certain cycle actually reduces to zero for
motivic reasons

\[ \text{Motivic Deligne Theorem?} \]

Artin lift to a function field is holographic

Deligne's Theorem: For \( n \geq d(2g-2) \), \( R\mathcal{T}_X(\overline{\mathbb{Q}}) \) vanishes.

Artin lift to a function field is holographic (polyhedral)
almost all coefficients zero, 
all coefficients greater than \( c(2g-2) \) are zero.

Deligne shows most coefficients are zero.

\[ \text{Deligne: } \mathbb{A} \in \mathbb{P} \overset{\text{Pic}^n X}{\rightarrow} \mathbb{P}^n \]

\[ \text{Fiber } (R\mathcal{T}_X(\overline{\mathbb{Q}}))_x = R\mathcal{H}^n(\text{fiber}_{x(\overline{\mathbb{Q}})}, \mathbb{P}^n_{\overline{\mathbb{Q}}}) \]

\[ \text{\Rightarrow reformulate on level of fibers: } \mathbb{P} \overset{\text{Pic}^n X}{\rightarrow} X \]

From finite etale cover \( \pi : \tilde{X} \rightarrow X \)
\[ \text{Pic}^n(\tilde{X}) \overset{\pi^*}{\rightarrow} \text{Pic}^n(X) \]

\[ \text{\Rightarrow smooth for sufficiently generic } \alpha \text{ (in case } 0 \text{ for general reasons, here for all characteristic)} \]

\[ (R\mathcal{T}_X(\overline{\mathbb{Q}}))_\alpha = H^n(\tilde{X}(\overline{\mathbb{Q}}), \mathcal{O}_\tilde{X}) \]

\( \sigma \)-isotropic condition

\[ \text{Deligne ... this } \sigma \text{-isotypic case is zero.} \]
What is motivic version? to bip o of $\text{Sp} \times C^\times$ there corresponds an element in the

\begin{align*}
\text{group algebra} \quad & G \in \mathcal{E} \quad [H=\text{Sp} \times C^\times] \\
\text{Assume} \quad & (\mathcal{X}^\times)^\text{smooth}
\end{align*}

Deligne $\iota_o$ acts trivially on $H^\ast(\mathcal{X}^\times, \mathbb{Q}_\ell)$

- each element in group algebra gives cycle in $Y \times Y$
- where group $G$ acts on variety $Y$
- so $\iota_o$ gives cycle $\iota_o \in \mathcal{X}^\times \times \mathcal{X}^\times$.

Deligne's theorem says $\iota_o$ is homologically equivalent to zero.

Q: Is $\iota_o$ rationally equivalent to zero?

- would imply acts trivially on motives...
- should follow from standard conjectures, but not clear generally!

Q: If Voevodsky motive has motivic realization $H$ then motive should be zero?

- - follows from some general conjectures
- - implies above.