Rademacher Lectures

I. Physics on Number Fields

Die Grundlehren der
mathematischen Wissenschaften in Einzeldarstellungen
Band 169

Hans Rademacher

Topics in Analytic Number Theory

and from (41.2) by means of Abel's limit theorem

$$\lim_{z\to 0} Z(s, z) = \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} .$$

The last two equations together yield

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\,\Gamma(s)\cos\frac{\pi s}{2}\,\zeta(s) ,$$

which is equivalent to (40.5). The functional equation is thus proved again first for $\operatorname{Re}(s) < 0$, and then by analytic continuation for all s.

42. Connection between the ζ-function and a ϑ-function

There are some more proofs of Riemann's functional equation known (see e.g. [74]). Riemann himself provided two in his memoir, and a third one was edited by Siegel from Riemann's manuscripts [69]. Among Riemann's proofs the second one has so far been the most fruitful one, and we shall give it here, in view of its importance. We change the proof slightly, using Mellin's formula, which will save us some discussions of convergence. We have from (23.5)

$$e^{-\pi n^2 u} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(z)\,(\pi n^2 u)^{-z}\,dz, \qquad a > 0, \quad u > 0$$

and thus

$$\sum_{n=1}^{\infty} e^{-\pi n^2 u} = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{a-i\infty}^{a+i\infty} \Gamma(z)\,(\pi n^2 u)^{-z}\,dz .$$

For $a > 1/2$ we can interchange summation and integration, since after (21.51) the integral and sum converge absolutely. With

$$\psi(u) = \sum_{-\infty}^{\infty} e^{-\pi n^2 u}$$

we obtain thus

$$\psi(u) - 1 = \frac{2}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(z)\pi^{-z}\,\zeta(2z)\,u^{-z}\,dz , \qquad a > 1/2 . \tag{42.1}$$

Then Mellin's theory (§ 27) permits us to write

$$2\pi^{-z}\,\Gamma(z)\,\zeta(2z) = \int_0^{\infty} (\psi(u) - 1)\,u^{z-1}\,du , \tag{42.2}$$

i.e.

$$\boxed{\text{Zeta} = \text{Mellin of Theta}}$$

Our goal: give a QFT perspective on this formula & its generalizations.

The Unreasonable Effectiveness of QFT in Number Theory:

model theory of L-functions on partition fns of QFTs

Physics on what space??

Arithmetic Geometry

Rings of integers $\mathcal{O}_F \supset \mathbb{Z}$:

$\longleftrightarrow$ number field $F/\mathbb{Q}$ finite

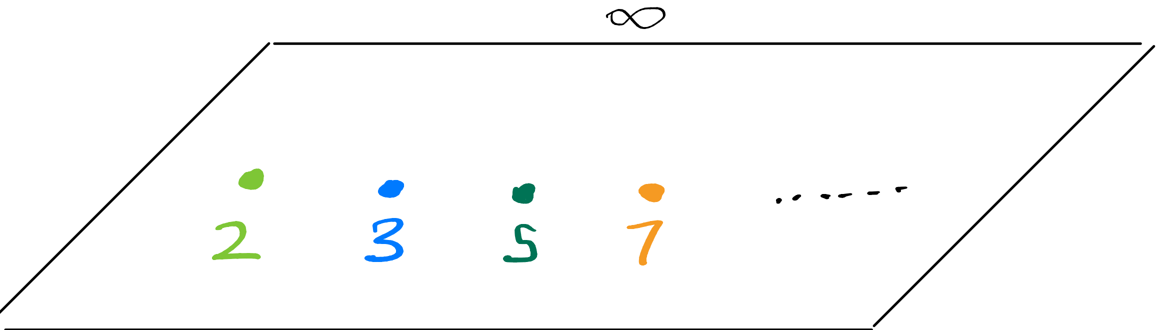
eg $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{5}]$ ...

Try to understand $\mathcal{O}_F$ as ring of functions on a geometric object, $\operatorname{Spec} \mathcal{O}_F$

[Grothendieck]

Spec $\mathbb{Z}$:

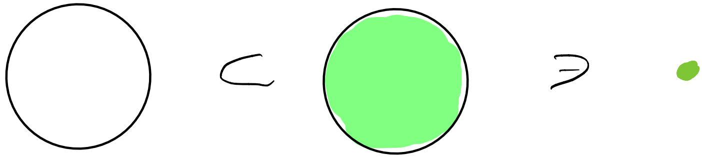
Points :



Functions !

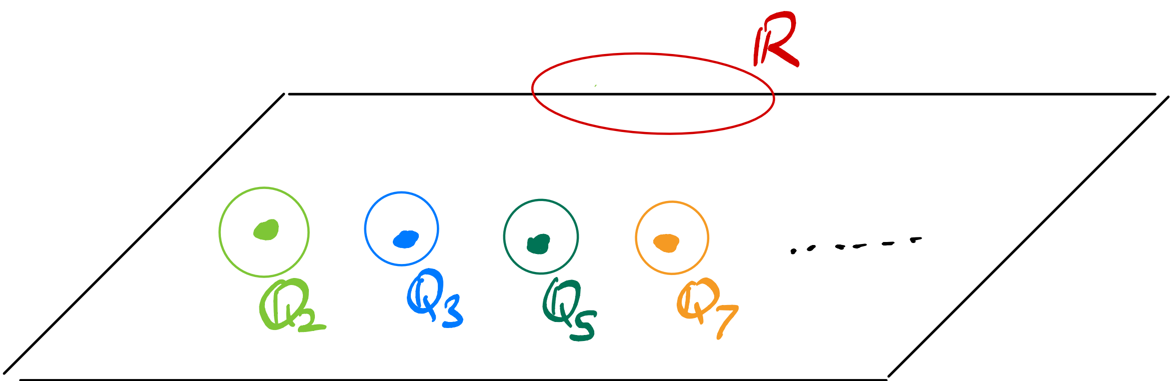
Taylor & Laurent series

$$\operatorname{Spec} \mathbb{Q}_p \qquad \operatorname{Spec} \mathbb{Z}_p \qquad \operatorname{Spec} \mathbb{F}_p$$



$$a_{-N}p^{-N} + \dots + a_0 + a_1 p + \dots \xleftarrow{\text{allow poles}} a_0 + a_1 p + \dots \xrightarrow{\text{set } p=0} a_0$$

⤳ local fields :



Mazur ('60s):

improved picture –

arithmetic topology

based on Galois theory /

Grothendieck's étale topology,

not just ring theory

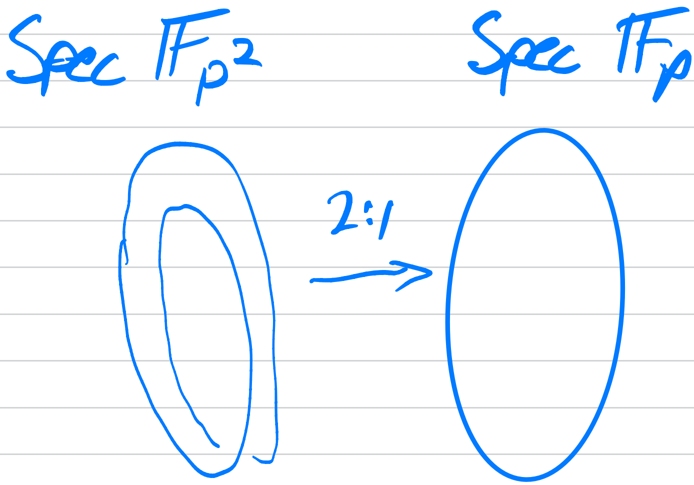
$\mathbb{F}_p$ has cyclic Galois group

$\longleftrightarrow$ finite extensions $\mathbb{F}_{p^n}$

S^1 has cyclic fundamental group

$\longleftrightarrow$ finite covers

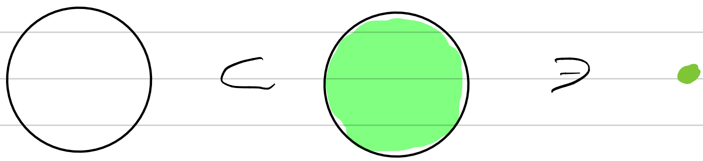
$\Rightarrow$ Picture $\operatorname{Spec} \mathbb{F}_p$ as a circle



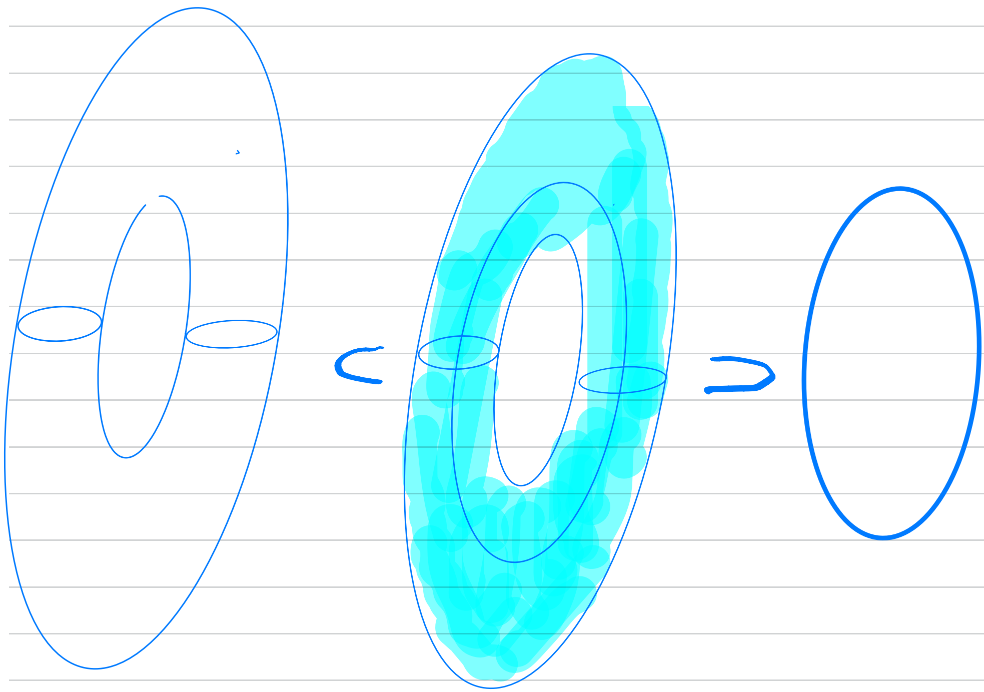
Adds a dimension to all our pictures!

Spec $\mathbb{Q}_p$ Spec $\mathbb{Z}_p$ Spec $\mathbb{F}_p$

Old:



New:



Spec $\mathbb{Q}_p \rightsquigarrow$ torus [Klein bottle]

Gal $\mathbb{Q}_p \twoheadrightarrow \langle$ longitude, meridian $\rangle$

Frobenius, monodromy

"Knots & Primes" dictionary

[Mazur, Morishita, Kapranov, Reznikov]

Number Fields $\longleftrightarrow$ 3-manifolds

primes $\longleftrightarrow$ knots

extensions $\longleftrightarrow$ covering spaces

Galois group $\longleftrightarrow$ Fundamental group

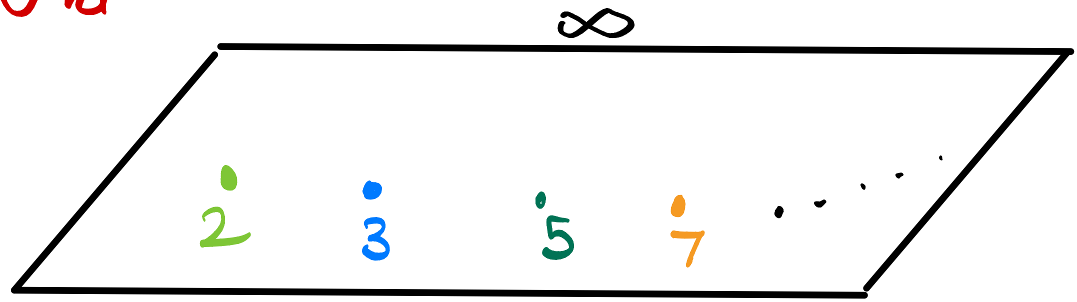
Quadratic reciprocity $\longleftrightarrow$ Linking

Class Field Theory $\longleftrightarrow$ Poincaré duality

Iwasawa Theory $\longleftrightarrow$ Alexander polynomial

Spec $\mathbb{Z}$

Old:



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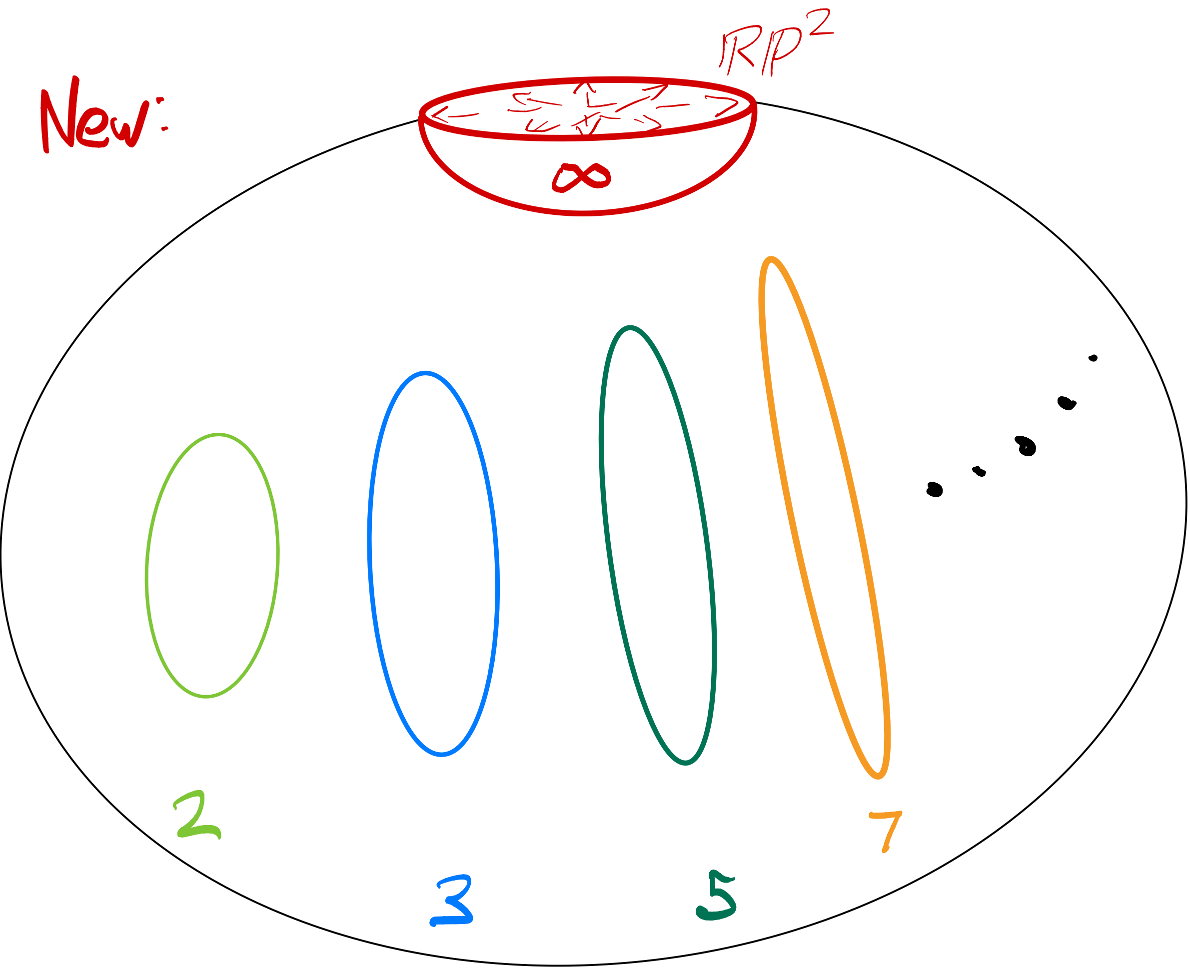
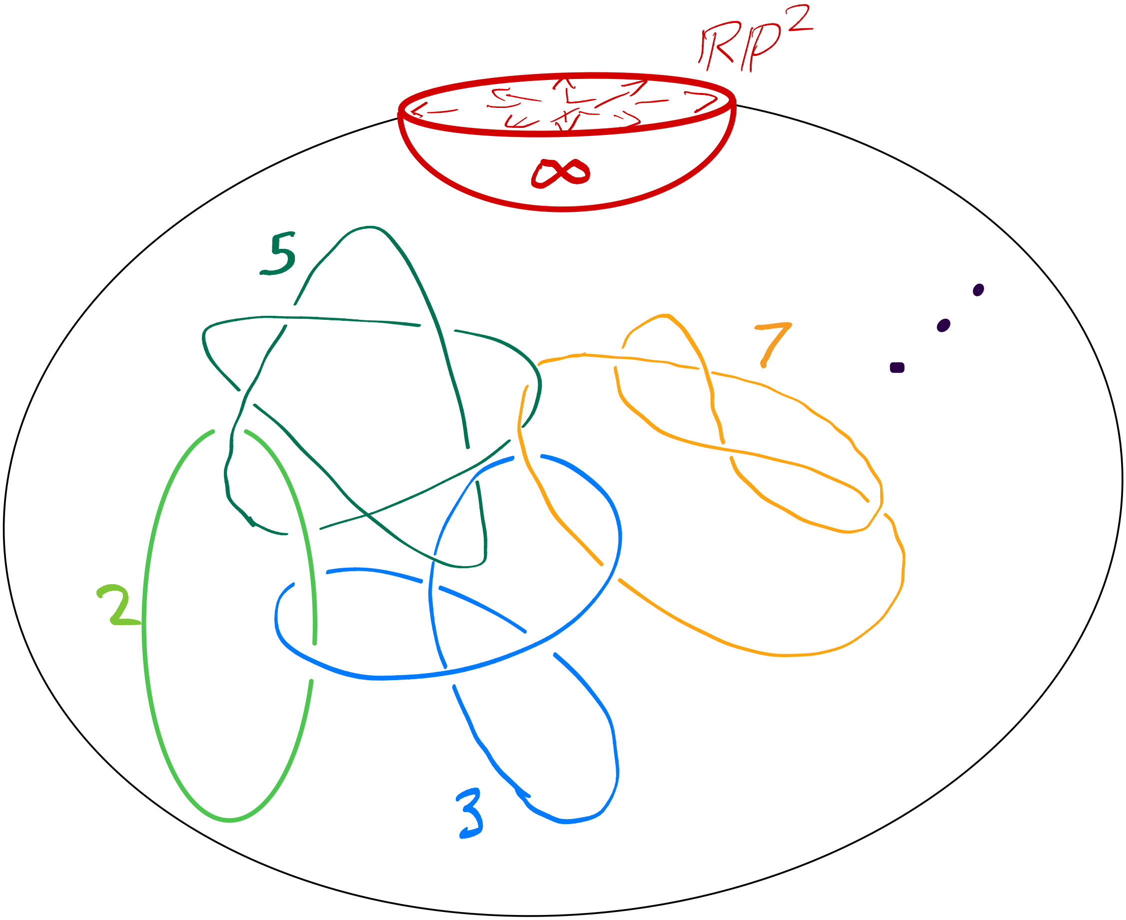


– or maybe better:



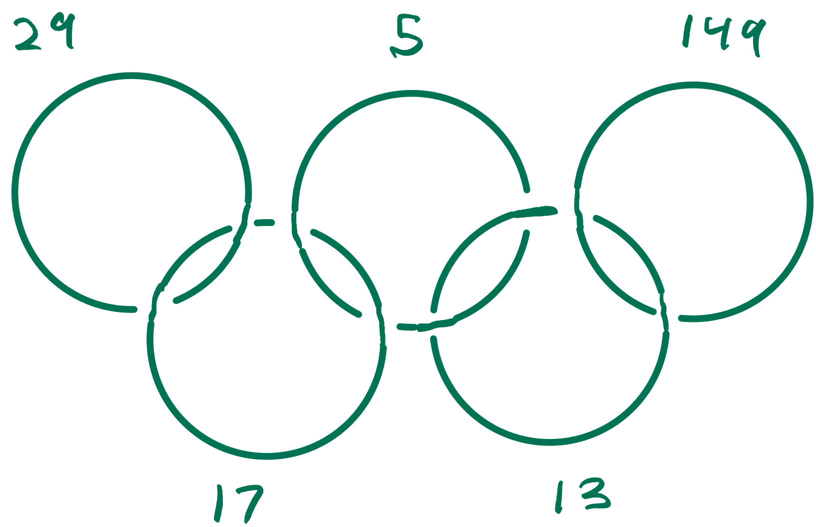
Linking $\longleftrightarrow$ Quadratic reciprocity

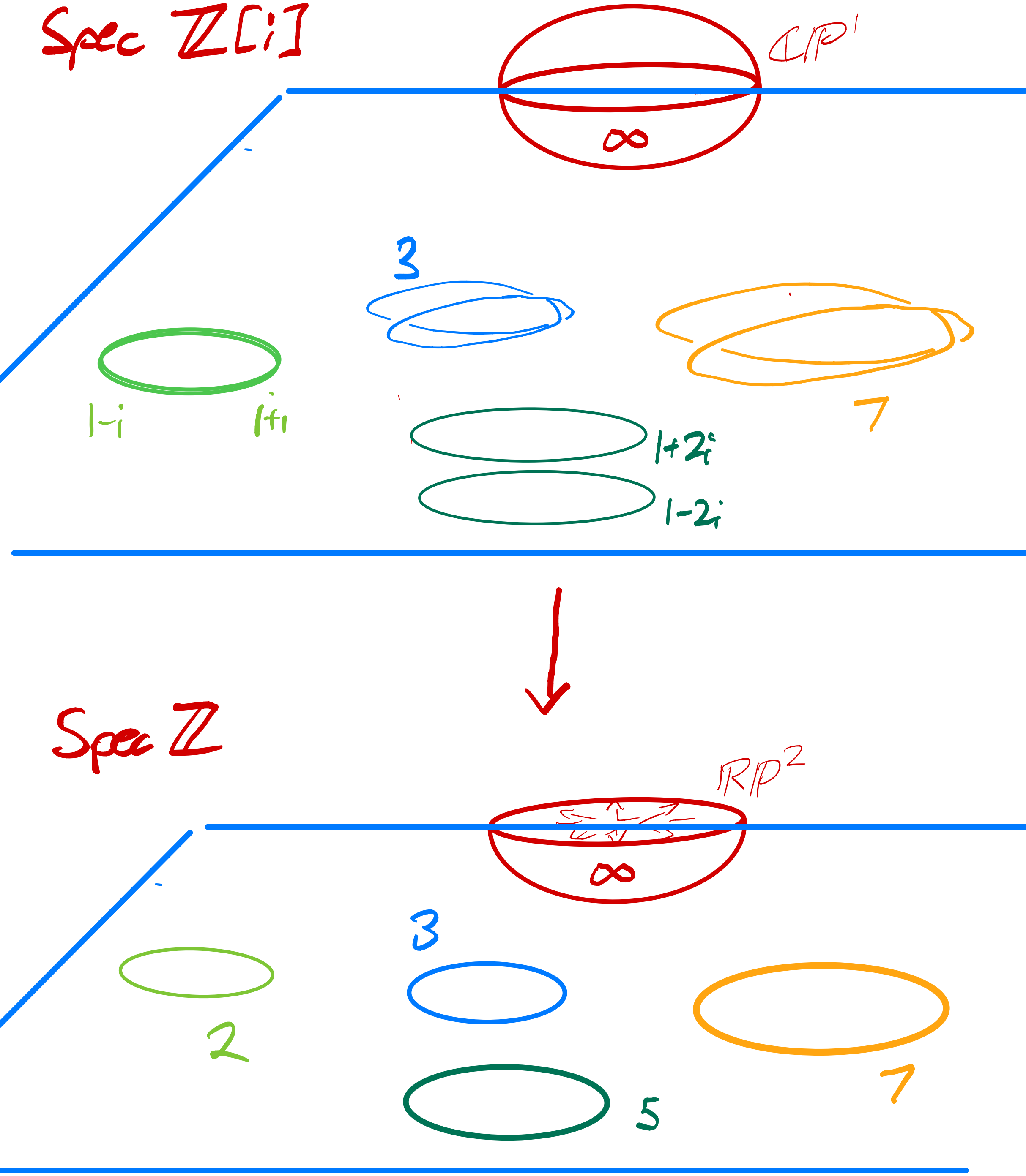
Gauss

Roughly: (eg for primes $\equiv 1 \bmod 4$)

p, q linked $\longleftrightarrow$ p is a square mod q

e.g. "Olympic primes"




Spec $\mathbb{Z}[i]$
$\mathbb{C}P^1$
∞
3
$1-i$
$1+i$
7
$1+2i$
$1-2i$
Spec $\mathbb{Z}$
$\mathbb{R}P^2$
∞
3
2
7
5

Arithmetic Physics

If number fields $\Longleftrightarrow$ 3-dimensional spaces, can we do physics here??

[Kapranov, M. Kim]

First take: doubtful!

No clear analog of classical mechanics

(PDE on number fields??)

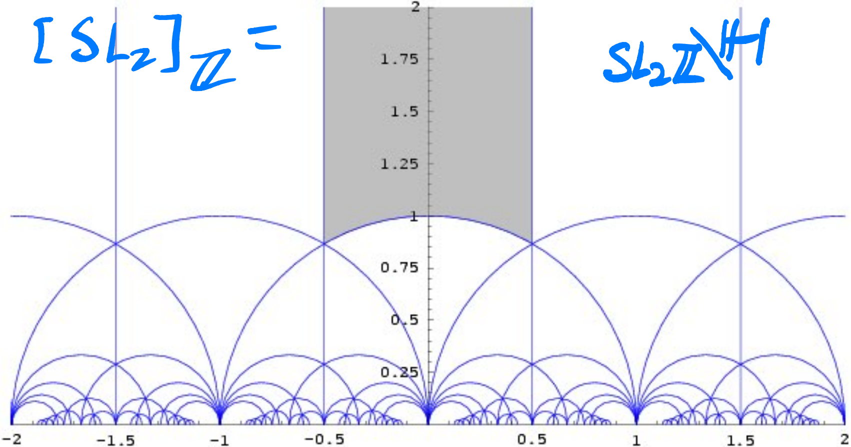
But: long history of arithmetic $\underline{\text{quantum}}$ mechanics

- theory of $\underline{\text{modular}}$ & $\underline{\text{automorphic}}$ $\underline{\text{forms}}$

$$\underbrace{L^2(\Gamma \backslash \mathbb{H})}_{\text{Hilbert space}} \circlearrowleft \Delta, \text{ Hecke ops.}$$

Hamiltonian, observables

$[SL_2]_{\mathbb{Z}} = SL_2\mathbb{Z}\backslash\mathbb{H}$



Theory of automorphic forms

Study L^2 - or, algebraically cohomology - of arithmetic locally symmetric spaces, like

$$[SL_2]_{\mathbb{Z}} = SL_2\mathbb{Z} \backslash \underbrace{SL_2\mathbb{R} / SO_2}_{\mathbb{H}^2}$$

- ie a quantum system associated to our "space" $M = \operatorname{Spec} \mathbb{Z}$

- Can replace $SL_2\mathbb{Z}$ by $\Gamma < SL_2\mathbb{Z}$ defined by congruences, ie modify at some primes

  - ie variant associated to link $L \subset M$

- Can vary number ring - e.g.

$$\mathbb{Z} \rightsquigarrow \mathbb{Z}[i]$$

$$[SL_2]_{\mathbb{Z}[i]} = $$

$$SL_2\mathbb{Z}[i] \backslash \underbrace{SL_2\mathbb{C} / SU_2}_{\mathbb{H}^3}$$

- ie vary "space" $M = \operatorname{Spec} \mathcal{O}_F$

- Can vary group

$SL_2 \rightsquigarrow GL_1, \quad GL_n,$
$SO_n, \quad Sp_n, \quad E_8, \ \ldots$

– i.e., vary "quantum gauge theory" we are studying

What kind of "physics"
is this?

and why bother?

Electricity & Magnetism

Quantum Maxwell theory

attaches to M 3-manifold
(space)

$\longmapsto$

Hilbert space of states

on spacetime $M \times \mathbb{R}$:

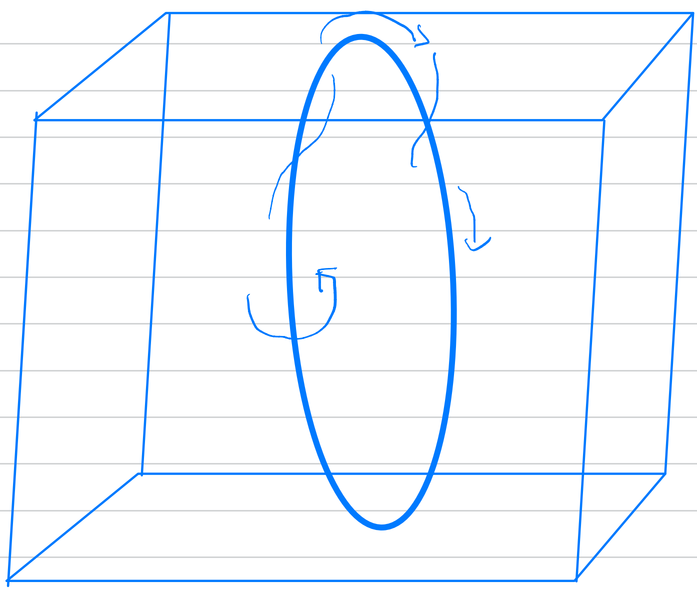
$$L^2\left(\left\{ \begin{array}{c} U(1)\text{-connections} \\ d+A \\ \text{on } M \end{array} \right\} \Big/ \text{gauge eq.} \right)$$

$\circlearrowright$

algebra of observables

Lots of other structures,
such as <u>defects</u>

e.g. allow solenoids — singularities
of gauge fields on link $L \subset M$



Very rich... let's simplify:

This has two (pale) shadows depending only on the topology of space, & carrying commutative algebras of observables

A. (Magnetic)

L^2 (gauge fields)

$\rightsquigarrow$

locally constant functions

H^0 (gauge fields)

(or H^* ...)

- remembers only data of magnetic fluxes $H^2(M; L, \mathbb{Z})$

[topology of space of gauge fields / magnetic Gauss law]

& observables: magnetic monopoles

B. (Electric)

All Connections

$\rightsquigarrow$

flat connections

- remembers only phase shifts

$$\pi_1(M \setminus L) \longrightarrow U(1)$$

[monodromy / Aharonov-Bohm effect]

& observables: measuring phase

Electric-Magnetic Duality

Symmetry $E \leftrightarrow B$ $\quad (F \leftrightarrow *F)$

of Maxwell theory

$$\rightsquigarrow$$

$$H^0\left[\begin{array}{c}\text{space of}\\ \text{connections}\end{array}\right] \simeq \mathbb{C}\left[\begin{array}{c}\text{space of}\\ \text{flat connections}\end{array}\right]$$

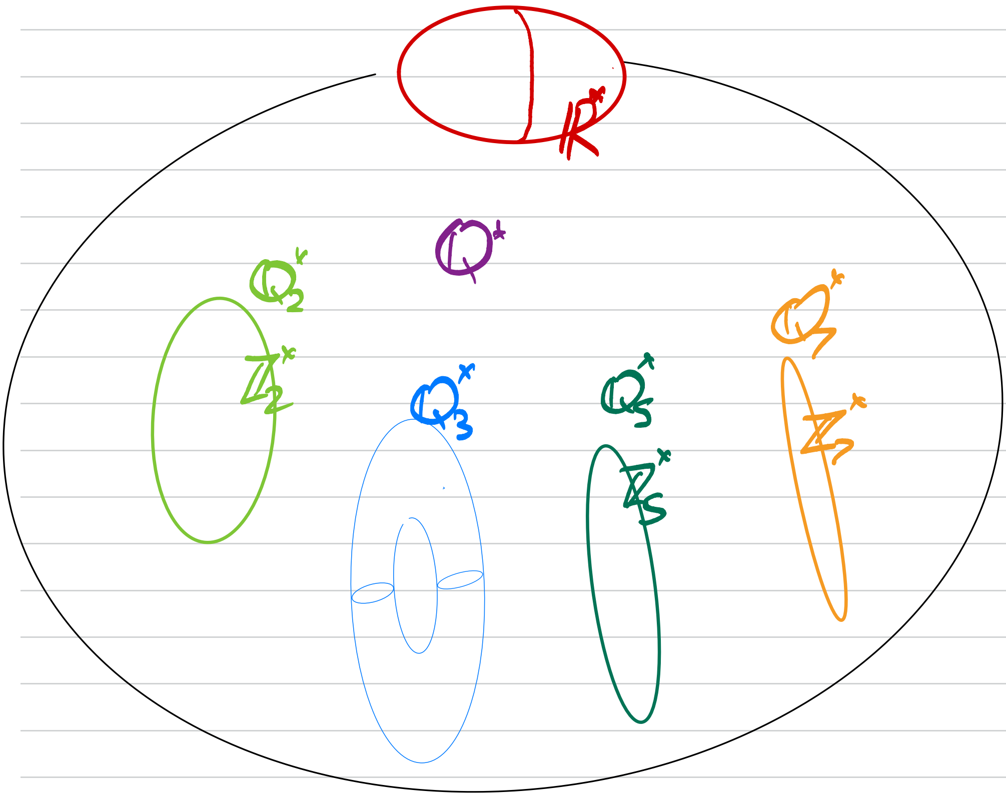
magnetic states — create monopoles

electric states — multiply by phase

($\Longleftrightarrow$ Poincaré duality on M, L)

Arithmetic version

Space of gauge fields
ramified along link L
$\rotatebox{90}{$\subseteq$}$
(3)



describe by gluing data
up to gauge transformations

$$\Rightarrow \quad [GL_1]_{\mathbb{Z};L} = $$

$$\left(\mathbb{Q}_2^* / \mathbb{Z}_2^* \times \underline{\mathbb{Q}_3^*} \times \mathbb{Q}_5^* / \mathbb{Z}_5^* \times \mathbb{Q}_7^* / \mathbb{Z}_7^* \times \cdots \times \underline{\mathbb{R}^*} \right) \Big/ \mathbb{Q}^*$$

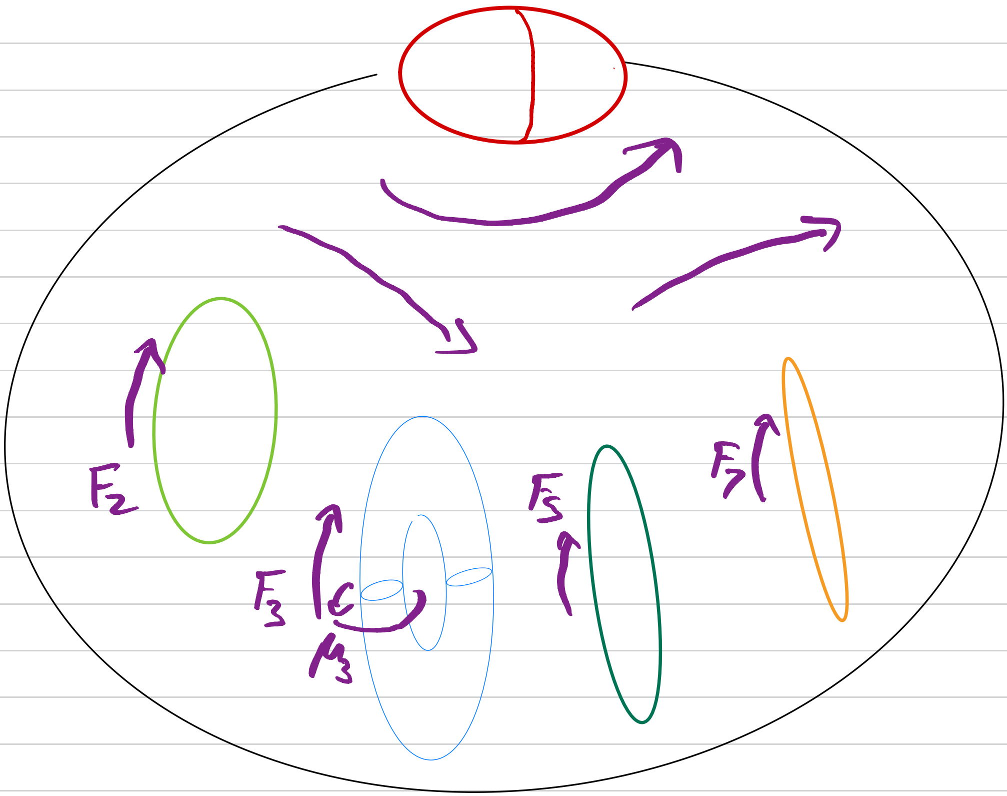
idèle class group

(ramified at L):

"line bundles on $\operatorname{Spec} \mathbb{Z}; L$"

Source of Dirichlet & Hecke characters

Space of phases (monodromies):



Characters $\mathrm{Gal}\,\bar{\mathbb{Q}}/\mathbb{Q} \longrightarrow \mathbb{C}^*$ ramified along L

Electric-magnetic duality $\rightsquigarrow$

Class Field Theory:

$H^0[\text{class group}] \simeq \mathbb{C}[\text{Galois characters}]$

magnetic states $\circlearrowright$ translation

electric states $\circlearrowleft$ multiplication

$\Longleftrightarrow$

Poincaré duality for Spec $\mathcal{O}_F$ in étale cohomology

[Artin – Verdier]

Nonabelian version

Maxwell $\rightsquigarrow$ Yang-Mills

$U(1) \rightsquigarrow G$ compact Lie group

E-M duality $\rightsquigarrow$ Montonen-Olive S-duality conjecture

[for SUSY variants]

Switches G for its dual $\check{G}$

$GL_n \longleftrightarrow GL_n$

$SL_n \longleftrightarrow PGL_n$

$SO_{2n} \longleftrightarrow SO_{2n}$

$SO_{2n+1} \longleftrightarrow Sp_{2n}$

...

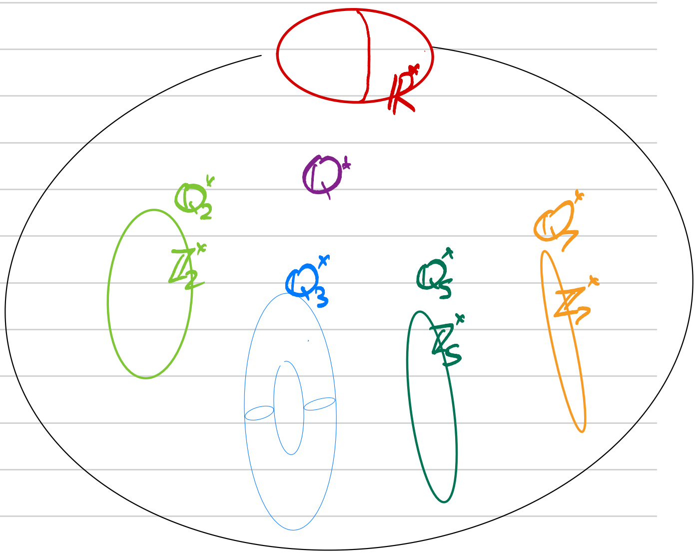
Topological Shadow:

A	B
Magnetic	Electric
Topology of gauge fields	Alg. Geom. of monodromies
[moduli of bundles]	[character varieties]
Monopole creation	Measure monodromy
['t Hooft loop]	[Wilson loop]

Arithmetic A-side:

Arithmetic locally symmetric spaces:

$$[GL_1]_{\mathbb{Z};L} = \left(\mathbb{Q}_2^\times/\mathbb{Z}_2^\times \times \underline{\mathbb{Q}_3^\times} \times \mathbb{Q}_5^\times/\mathbb{Z}_5^\times \times \mathbb{Q}_7^\times/\mathbb{Z}_7^\times \times \cdots \times \underline{\mathbb{R}^\times} \right) \Big/ \mathbb{Q}^\times$$



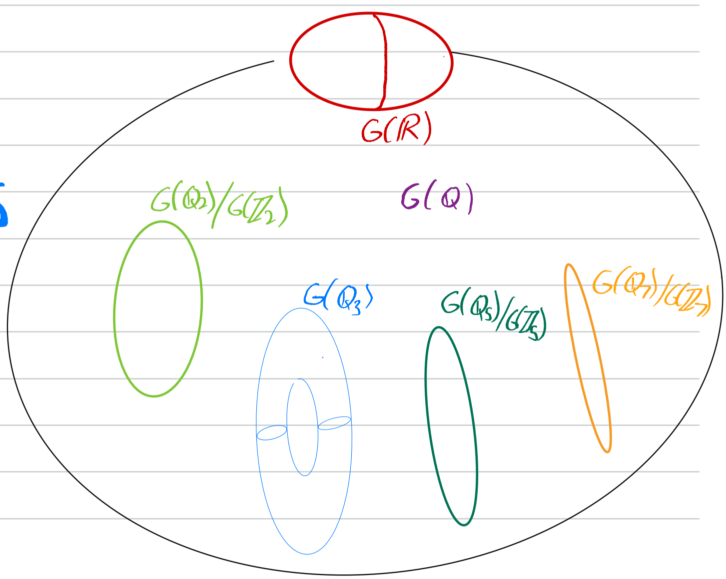
Arithmetic A-side:

Arithmetic locally symmetric spaces:

$$[G]_{\mathbb{Z};L} \cong \left(G(\mathbb{Q}_2)/G(\mathbb{Z}_2) \times \underline{G(\mathbb{Q}_3)} \times G(\mathbb{Q}_5)/G(\mathbb{Z}_5) \times G(\mathbb{Q}_7)/G(\mathbb{Z}_7) \times \cdots \times \underline{G(\mathbb{R})} \right) \Big/ \mathbb{Q}^*$$

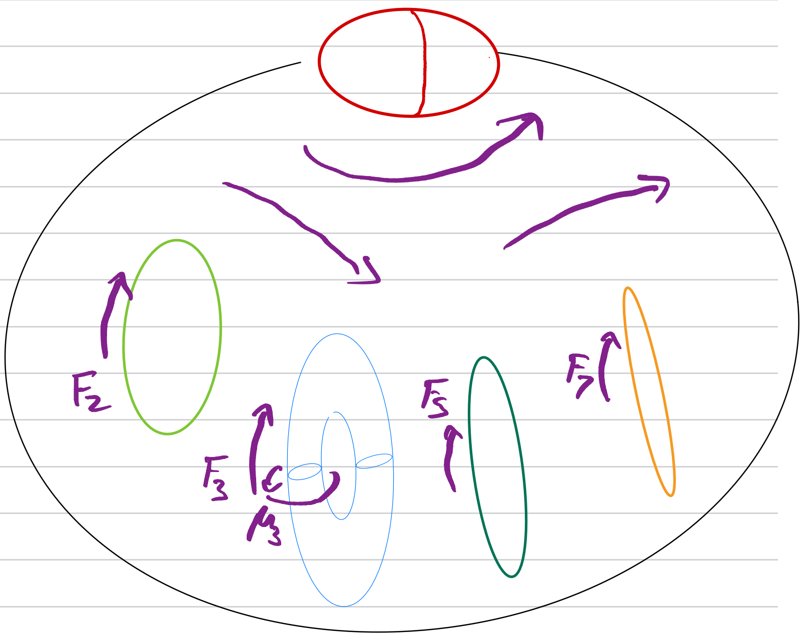
$\simeq$ "G bundles on Spec $\mathbb{Z}$"

(rel. L)



Physical B-side:
Character varieties of fundamental groups

$$\mathrm{Loc}_G(M \setminus L) = \left\{ \pi_1(M \setminus L) \longrightarrow \check{G}(\mathbb{C}) \right\} \big/ \check{G}(\mathbb{C})$$

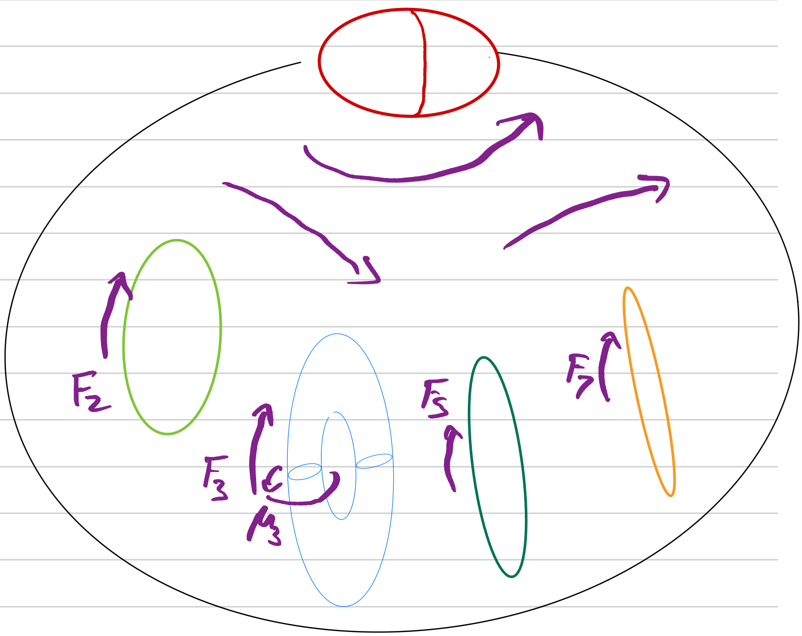


Arithmetic B-side:

Character varieties of Galois groups

$$\mathrm{Loc}_G(\mathcal{O}_F, L) :=$$

$$\left\{ \begin{array}{c} \mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q} \longrightarrow \check{G}(\mathbb{C}) \\ \text{ramified only at } L \end{array} \right\} \Big/ \check{G}(\mathbb{C})$$



E-M duality $\Longleftrightarrow$
Langlands Correspondence

very poetically:

A	B

$$H^*([G]_{\mathcal{O}_F}) \simeq \mathbb{C}[\mathrm{Loc}_{\check{G}}(\mathcal{O}_F)]$$

automorphic	Galois
magnetic	electric
Hecke operators	multiplication by traces
(convolution / monopole / 't Hooft operators)	(multiplication / phase / Wilson operators)

What can number theory learn from physics?

I. Symmetry E-M duality is fundamentally <u>symmetric</u>

II. Structure QFT encodes very rich structures, e.g.

defects
operator products
dualities

Key example: Riemann zeta

$$\frac{\pi^{s/2}}{\Gamma(s/2)}\int_0^\infty y^{s/2}\,\Theta(y)\,dy \qquad\qquad \prod_p \frac{1}{1-p^{-s}}$$

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}}$$

Integral representation (automorphic)

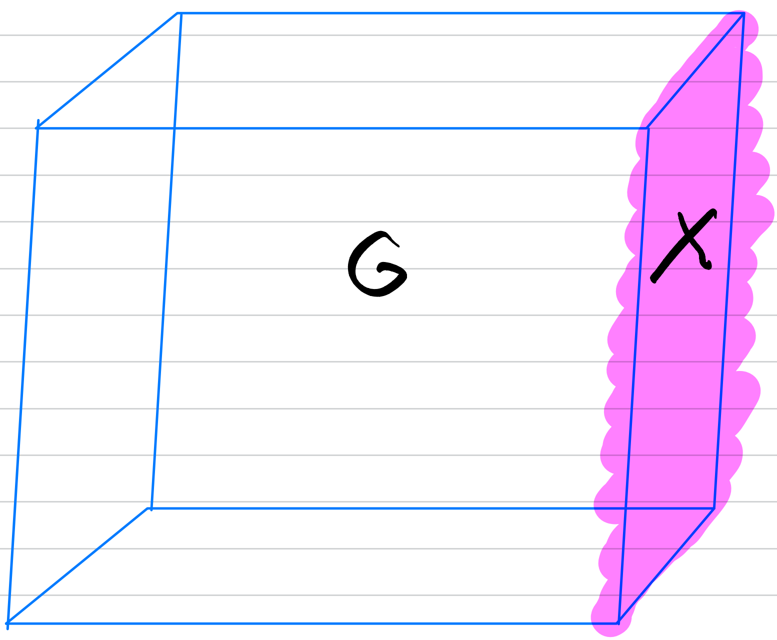
Euler product (Galois)

measurements on the two sides of Class Field Theory

Physics : add matter!

Study gauge theory with boundary:

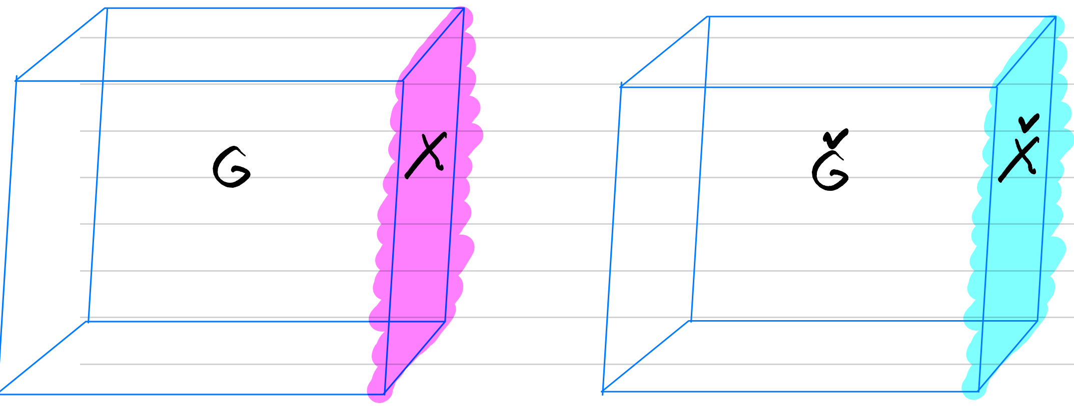
charged conducting material



E-M duality matches

dual boundary conditions /

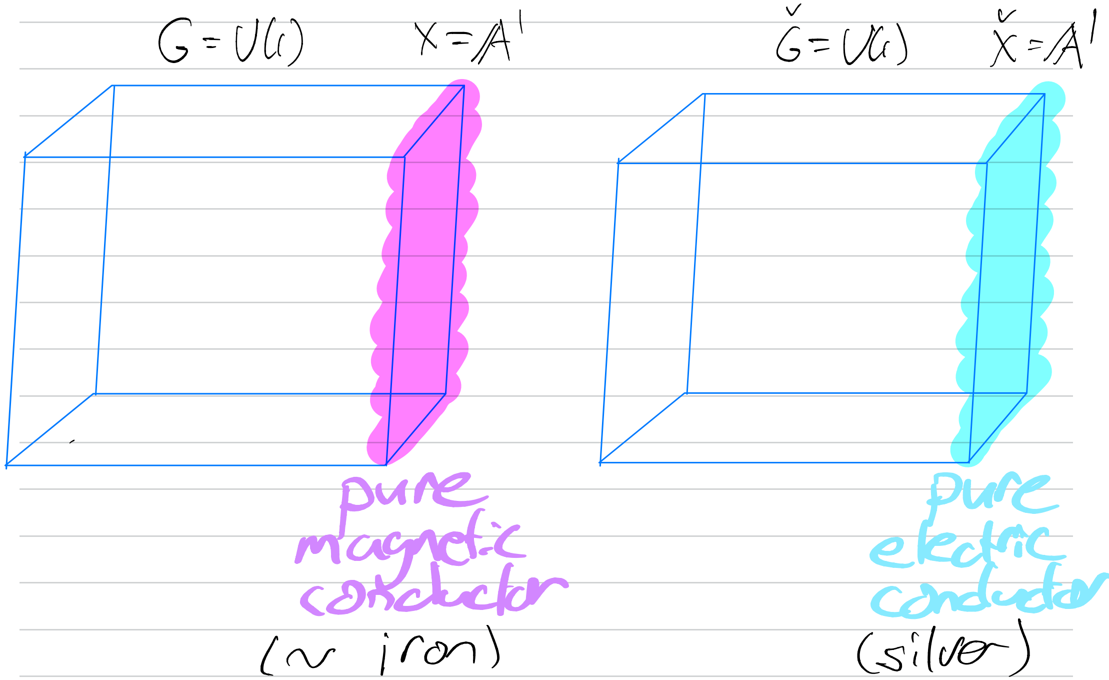
charged materials



$\Rightarrow$ produce matching

measurements on dual sides

Riemann's integral formula
à la BZSV



magnetic observable = period integral

electric observable = L-function

$\int(\theta\text{-function}) = \zeta =$ Euler product


E
M
Maxwell
K+W
Spectral
Automorphic
Langlands