Rademacher Lectures

II. Characters & Geometric Quantization

Today: explore aspects of geometric quantization as a way to organize linear invariants

- in particular, characters

Geometric Quantization

Symplectic manifolds

$$\Big\downarrow$$

quantum theories

Relative Geometric Quantization

Symplectic manifolds

with Hamiltonian G-actions

Kirillov Orbit method

quantum theories

with unitary G-actions

Higher Geometric Quantization

(n-shifted)

Pantev
T-V-V

Symplectic manifolds

$\Big\downarrow$

(n+1)-dimensional

quantum theories

Polarized Geometric Quantization

$$
\begin{tikzcd}
 & \text{manifolds} \arrow[dl, "T^*"'] \arrow[ddl, "\text{linearization (e.g. } L^2)"] \\
\text{Symplectic manifolds} \arrow[d, dashed] & \\
\text{quantum theories} &
\end{tikzcd}
$$

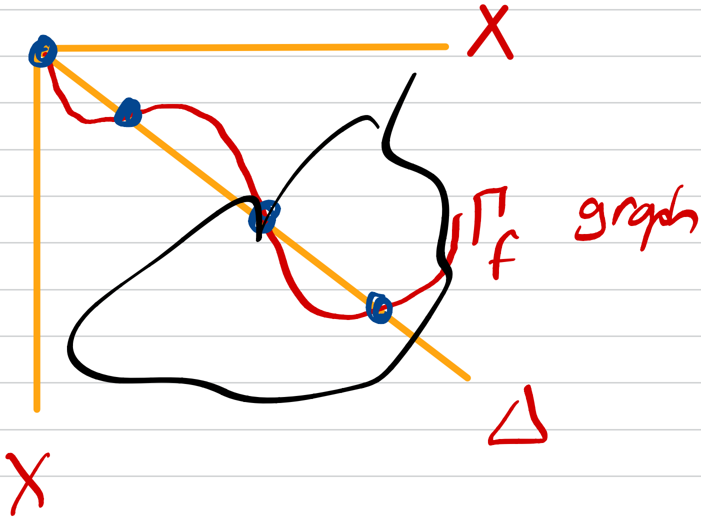
Fixed points & traces

I. Geometry:

$f : X \longrightarrow X$ self-map

$\rightsquigarrow$ fixed points

$$X^f = \{ x \in X : f(x) = x \}$$



$$\boxed{X^f = \Gamma_f \cap \Delta}$$

This makes sense for <u>relations</u>

$Z \subset X \times X$,

$$\text{f.p.}(Z) := Z \cap \Delta$$

. . . or even

<u>correspondences</u>

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

$$f.p.(Z) := Z \times_{X \times X} \Delta$$

$$= \left\{ x \in X, z \in Z : \begin{tikzcd} & z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x & & x \end{tikzcd} \right\}$$

- II. Algebra:

V f.d. vector space $/k$

matrices

$$\begin{tikzcd}
 & V \otimes V^* \arrow[r, "\text{eval}", "v\otimes v^* \mapsto v^*(v)"'] \arrow[d, "\wr"] & k \\
k \arrow[ur, "1 \mapsto \sum e_i \otimes e^i"] \arrow[r, "1 \mapsto \mathrm{Id}_V"] \arrow[urr, bend left=60] & \mathrm{End}\, V \arrow[ur, "\mathrm{Tr}"'] &
\end{tikzcd}$$

linear transformations

- Tr = sum of diagonal entries, in any basis ...

- Recover $\dim V = \mathrm{Tr}(\mathrm{Id}_V)$

III. Physics

Notion of <u>size</u> of quantum (or stat.) mechanical system — the <u>partition function</u>

$\mathcal{H}$ Hilbert space

H Hamiltonian

$$\rightsquigarrow \boxed{Z(T) = \mathrm{Tr}_{\mathcal{H}}\left(e^{iTH}\right)}$$

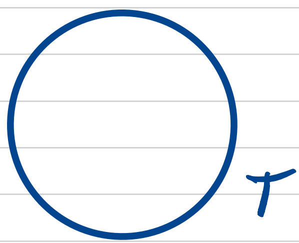
$\sim$ "number of states weighted by energy"

QFT (path integral) perspective:

$$Z_{[0,T]} : \mathcal{H} \xrightarrow{e^{iHT}} \mathcal{H}$$

time evolution,

Attach ends (make time periodic)!

$Z(T) = Z($  $)$

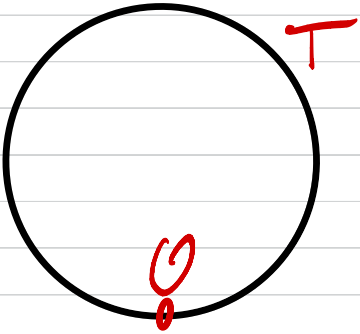
attached to S^1_T

– volume of space of maps from S^1_T (weighted by action)

(loop space)

Insert observable $\mathcal{O} \circlearrowright \mathcal{H}$:

$\langle \mathcal{O} \rangle_{S^1_T} :=$



$\longmapsto \mathrm{Tr}\left(e^{iHT}\mathcal{O}\right)$

1-point function

(~ expectation value of $\mathcal{O}$)

Topological QM:

$H = 0$,

$Z = Z(S^1) = \dim \mathcal{H}$,

$\langle \mathcal{O} \rangle_{S^1} = \mathrm{Tr}_{\mathcal{H}}\, \mathcal{O}$

How do these worlds interact?

$$
\begin{tikzcd}
\text{Spacetimes} \arrow[rr, bend left, "\text{QFT}"] \arrow[dr, "\text{Fields}"'] & & \text{Algebra} \\
& \text{Geometry} \arrow[ur, "\text{Linearization}"'] &
\end{tikzcd}
$$

How do these worlds interact?

$$\begin{tikzcd}
\underset{\amalg}{\text{Spacetimes}} \arrow[rr, bend left, "\text{QFT}"] \arrow[dr, "\text{Fields}"] & & \underset{\otimes}{\text{Algebra}} \\
& \underset{\times}{\text{Geometry}} \arrow[ur, "\text{Linearization}"] &
\end{tikzcd}$$

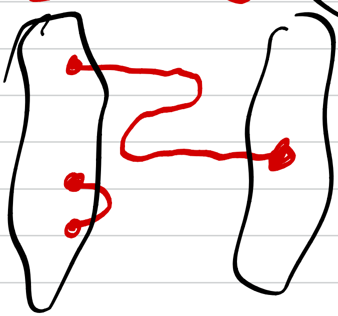
How do these worlds interact?

$$\begin{tikzcd} \text{Spacetimes} \arrow[rr, bend left, "\text{QFT}"] \arrow[dr, "\text{Fields}"'] & & \text{Algebra} \\ & \text{Geometry} \times \text{correspondences} \arrow[ur, "\text{Linearization}"'] & \end{tikzcd}$$

Spacetimes

$\parallel$

bordisms



Algebra

$\otimes$

linear maps

$V \longrightarrow W$

Geometry

$\times$

correspondences

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

Toy model

$$\begin{tikzcd} \vec{} \arrow[rr, bend left] \arrow[dr] & & V = \mathbb{C}[X] = \mathcal{H} \\ & X \arrow[ur] & \end{tikzcd}$$

functions on a finite set X

$k \quad k \quad k \quad k \quad k$

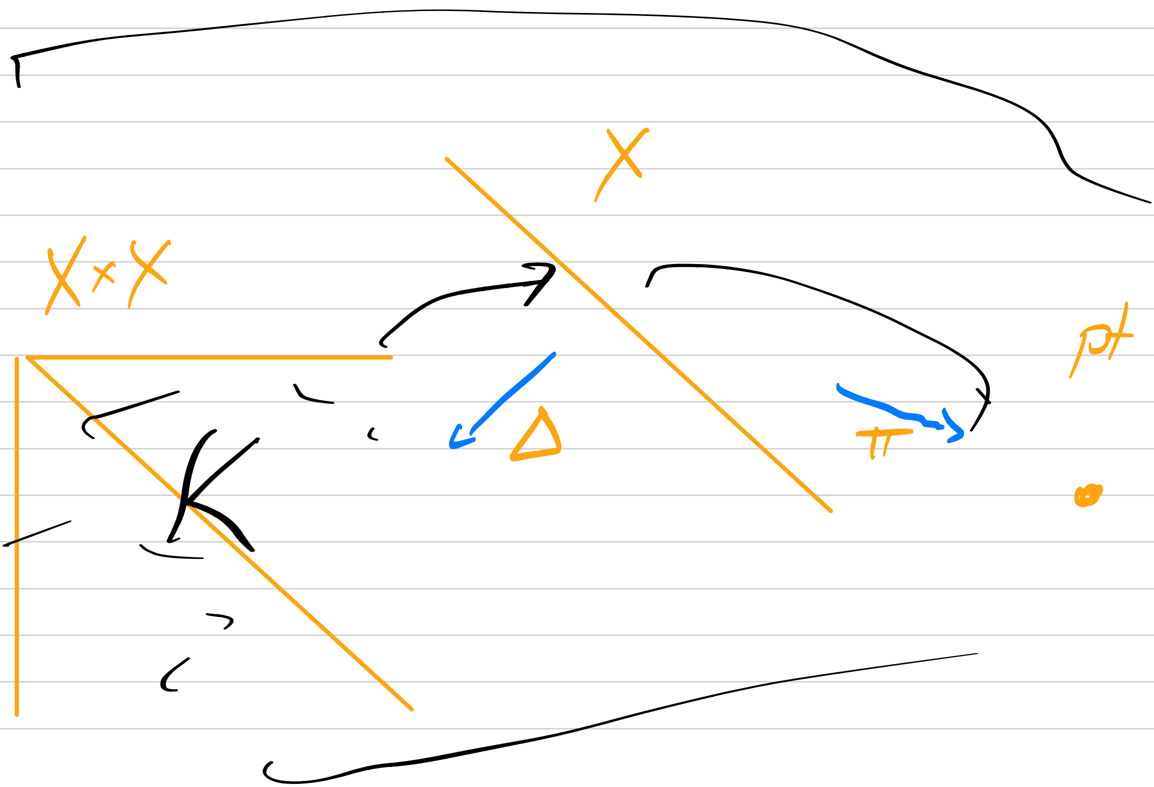
$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \qquad X$

~ i.e. pick basis $\{\delta_x \in V\}_{x \in X}$

$$\Rightarrow \operatorname{End} V \simeq k[X \times X]$$

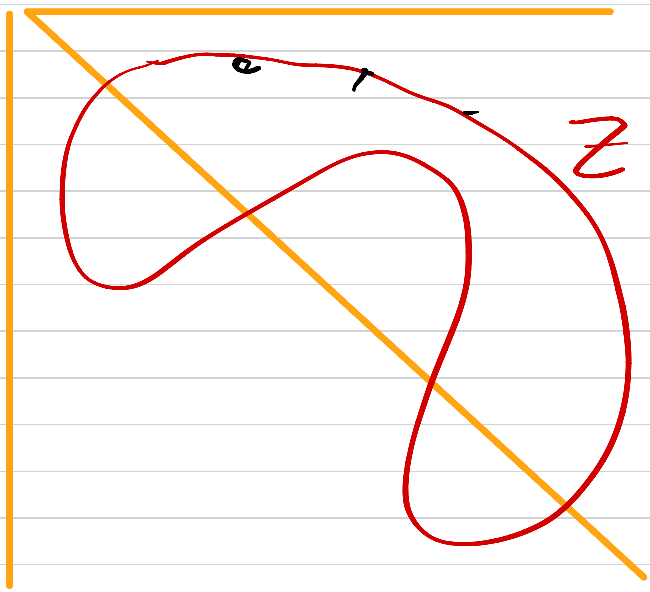
$$\operatorname{Tr}(K) = \sum_x K_{xx}$$

$$= \int_\pi \Delta^* K$$



The ur-fixed point theorem:

$Z : X \dashrightarrow X$,



$$\boxed{\mathrm{Tr}(\Gamma_Z) = \# \, \mathrm{f.p.}(Z)}$$

or in language of integral transforms

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

$$Z_* = \int_{\pi_2} \circ \, \pi_1^* : k[X] \circlearrowleft$$

$$\boxed{\mathrm{Tr}\left(Z_* \right) = \# \, \text{f.p.}(Z)}$$

Very general proof:

- Notion of dual of object $V \in \mathcal{C}$

a symmetric monoidal category $\mathcal{C}$:

$\exists\ V^\vee$ & maps

$$1 \xrightarrow{\text{unit}} V \otimes V^\vee \xrightarrow{\text{ev}} 1$$

satisfying

$$V \longrightarrow \overbrace{V \otimes V^\vee}^{\text{ev}} \otimes V \longrightarrow V \quad = \quad V \xrightarrow{\text{id}} V$$

$$V \otimes \underbrace{V^\vee \otimes V}_{\text{unit}} \longrightarrow V$$

- unique if it exists

- Lets us define traces

$$\begin{tikzcd} & V\otimes V^\vee \arrow[r] \arrow[d, "\wr"] & 1 \\ 1 \arrow[r] \arrow[urr, bend right] & \mathrm{End}\,V \arrow[ur, "Tr"'] & \end{tikzcd}$$

$Tr(A) \in End(1)$

eg $\quad$ dim $V = Tr(Id_V)$

- preserved by all symmetric monoidal functors

- Any space is canonically self-dual if we allow correspondences:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ * & & X \times X & & * \end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad}_{\text{unit}} \qquad\qquad \underbrace{\qquad\qquad\qquad}_{\text{ev}}$$

- $X \longmapsto k[X]$

$$\text{spaces} \longrightarrow \text{Vect}$$

takes $\_ \times \_$ to $\_ \otimes \_$

$\Longrightarrow$ taking functions <u>preserves</u> <u>traces</u>

$$\mathrm{Tr}\left(\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}\right) = Z \times_{X \times X} X$$

$$= \mathrm{f.p.}(Z)$$

$$\Longrightarrow \quad \mathrm{Tr}(Z_*) = \#(\mathrm{Tr}(Z))$$

$$= \# \, \mathrm{f.p.}(Z)$$

$\blacksquare$

This proof applies to any

$$\begin{array}{ccccc} \text{Spacetime} & \longrightarrow & \text{Geometry} & \longrightarrow & \text{Algebra} \\ \sqcup & & \times & & \otimes \end{array}$$

$$\text{Spacetime} \xrightarrow{Z} \text{Algebra}$$

eg $\dim Z(pt) = Z(S^1)$

eg. taking cohomology

$$\left[X \circlearrowleft f \right] \longmapsto H^*(X) \circlearrowleft H^*(f)$$

$\Longrightarrow$ Lefschetz fixed point formula

$$\boxed{\mathrm{Tr}(H^*(f)) = H^*\Big(\underset{\substack{\| \\ * \leftarrow X^f \rightarrow *}}{\mathrm{Tr}(f)}\Big) = \chi(X^f)}$$

Grothendieck:

$X / \mathbb{F}_q \Rightarrow$ Frobenius map

$$F \circlearrowright \bar{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$$

$$(\bar{X})^F = X(\mathbb{F}_q)$$

$$\rightsquigarrow$$

$$\boxed{\# X(\mathbb{F}_q) = \mathrm{Tr}\left(F \circlearrowright H^*_{ét}(\bar{X})\right)}$$

<u>Characters</u> : add symmetry group G

<u>Geometry</u> $G \circlearrowright X \Longrightarrow$

collection $\{ g \longmapsto X^g \}_{g \in G}$

of fixed points

<u>Algebra</u> $G \circlearrowright V \Longrightarrow$

function $\{ g \longmapsto \chi_V(g) := Tr_V(g) \}_{g \in G}$

G-analog of dimension

(e.g. $\chi_V(1) = \dim V$)

Physics

n-dim QFT with G symmetry $\iff$ finite, for now

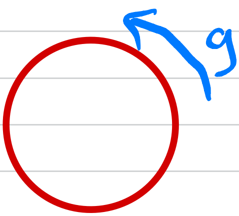
can evaluate on spacetimes

with background gauge fields

In our topological QM setting:

this means more circles

as inputs to partition function:

$S^1_g =$ 

$g \in G \longmapsto$

g-twisted circle

(circle w/ monodromy g)

S^1_g labels a G local system on S^1, $S^1_g \in \mathrm{Loc}_G(S^1)$:

a locally constant map to $\bullet/G = BG$

point with G-symmetry

- On general manifolds:

$$\mathrm{Loc}_G M \cong \{\pi_1 M \longrightarrow G\}/G$$

G-twisting for every loop.

$$\mathrm{Loc}_G S^1 = \frac{G}{G} :$$

- depends on g up to conjugacy
- carries residual symmetry by centralizer of g

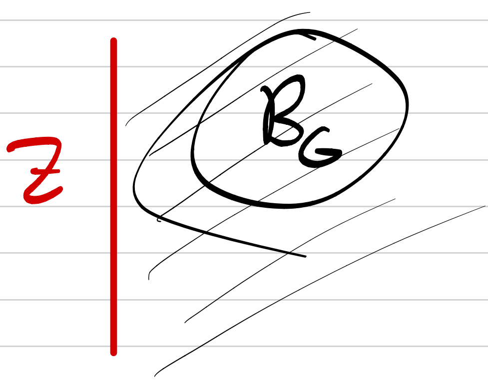
So partition function for a G-TQM is a <u>class function</u> $g \longmapsto Z(S^1_g)$

2d TQFT POV

Modern perspective on (G-)symmetry in n-dim QFT Z:

(symTQFT, topological phases of matter . . .)

$\Longleftrightarrow$ promote to boundary theory for $(n+1)$d gauge theory B_G



Gauge theory linearizes spaces of gauge fields

G finite $\longrightarrow$ local systems Loc_G

2d spacetimes M $\xrightarrow{\mathcal{B}_G}$ Linearize $\mathrm{Loc}_G M$

$$\boxed{X = \mathcal{F}(M) = \mathrm{Loc}_G M}$$

2d theory $\mathcal{B}_G$ attaches a Hilbert space to 1-manifolds

$Z_G(S^1)$ time

$- \; L^2(\frac{G}{G} = \mathrm{Loc}_G S^1)$ in 2d YM

[N.B. for G infinite,

many different <u>kinds</u> of

G-symmetry $\longleftrightarrow$ many

kinds of G-gauge theory ...

Our discussion of $\mathcal{B}_G$

still makes sense for G

<u>algebraic group</u> like $GL_n\mathbb{C}$

— <u>algebraic</u> G-symmetry /

BF-theory]

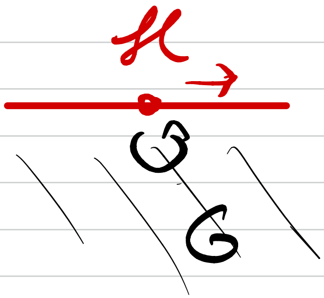
G- quantum mechanics Z

$\Longleftrightarrow$

Rep. $\mathcal{H} = Z(pt)$ of G

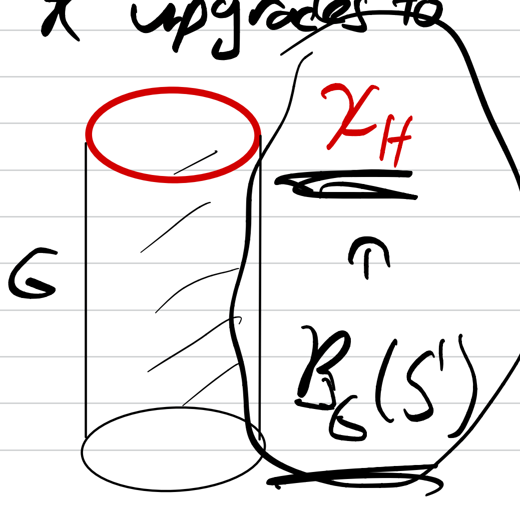
$\Longleftrightarrow$

Boundary theory for $\mathcal{B}_G$



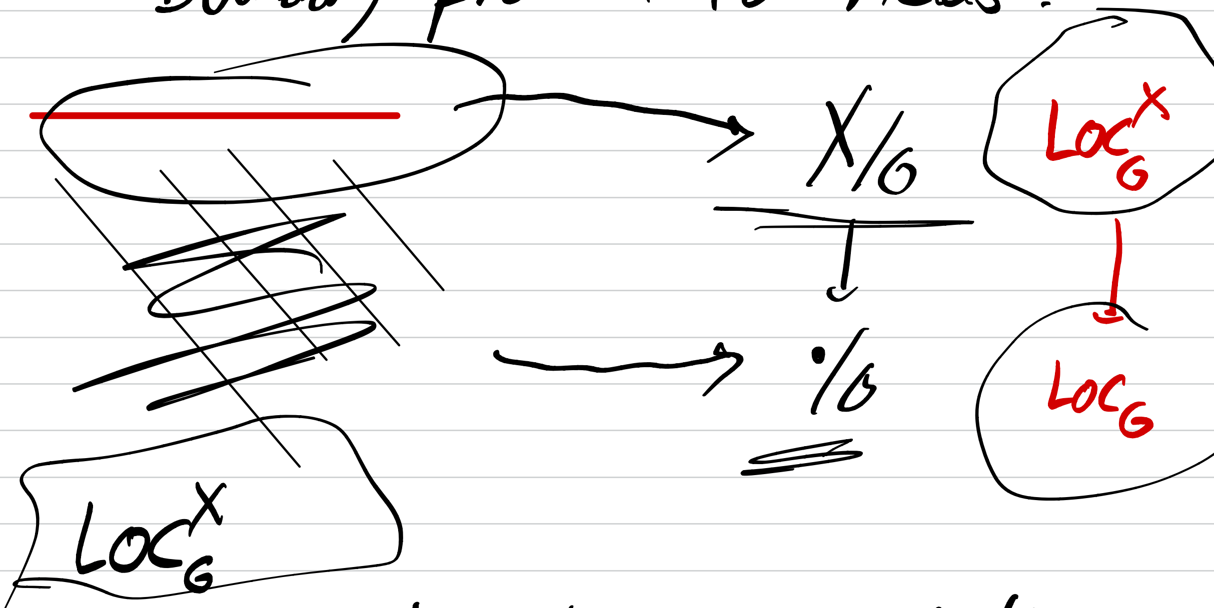
Functor on spacetimes w/ red boundary marked by Z

Partition function of $\mathcal{H}$ upgrades to boundary state: the character



$X \circlearrowleft G \Rightarrow$

boundary problem for fields:



= G-local system ρ on bulk

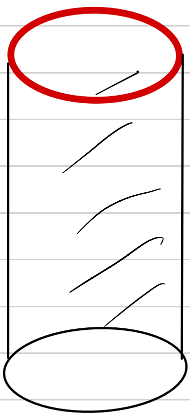
+ ρ-twisted map to X on boundary

(section of X-bundle associated to ρ)

$\Rightarrow$ character formulas for linearization of $G \circlearrowright X$:

$$G \circlearrowright \mathbb{C}[X]$$

G



$$\begin{array}{ccc} \mathrm{Loc}^X_G(S^1) & = & \left\{ \begin{matrix} g\in G \\ x\in X^g \end{matrix} \right\} / G \\ \Big\downarrow \pi_{S^1} & & \Big\downarrow \\ \mathrm{Loc}_G(S^1) & = & \dfrac{G}{G} \end{array}$$

- organize all fixed points X^g

(with their $C_G(g)$ actions)

into family over G/G

Atiyah-Bott:

Character of linearization

$$V = \mathbb{C}[X] = Z(pt) \in T_G(pt) = \mathrm{Rep}\,G$$

$$\boxed{\chi_V = \pi_{S^1 *} 1 = \int_{\pi_X} d\mathrm{vol}}$$

$$= Z(S^1) \in B_G(S^1) = \mathbb{C}\left[\frac{G}{G}\right]$$

Relative version of

dimension $\dim Z(\cdot) = Z(S^1)$

Example $X = G/K$

$$
\begin{tikzcd}
\mathrm{Loc}_G^{G/K} S^1 = \mathrm{Map}(S^1, {}_G\backslash X = {\bullet}/K) = \arrow[d] & \frac{K}{K} \arrow[d] \\
\mathrm{Loc}_G S^1 \arrow[r, phantom, "="] & \frac{G}{G}
\end{tikzcd}
$$

$\Rightarrow$ Frobenius character formula for induced representations

Example $X = G/B$ flag variety

$B = \begin{pmatrix} \diagdown & * \\ & \diagdown \end{pmatrix}$ Borel

$$\begin{tikzcd}
\mathrm{Loc}_G^{O/B}(S^1) = \frac{B}{B} \simeq \frac{\tilde{G}}{G} \arrow[d] = \left\{ \begin{array}{l} g \in G + \\ \text{flag preserved} \\ \quad \text{by } g \end{array} \right\} \\
\frac{G}{G}
\end{tikzcd}$$

recover Grothendieck–Springer resolution

$\rightsquigarrow$ Atiyah - Bott proof of Weyl character formula

Higher Quantization

Real power of this perspective:

Algebra

Vect

Cat

2Cat

& G-actions on them

Higher Quantization

Real power of this perspective :

Algebra		Geometry
Vect	$\rightsquigarrow$	Fun
Cat	$\rightsquigarrow$	Shv
2Cat	$\rightsquigarrow$	ShvCat

These arise as "higher linearizations of (G-) spaces

Higher Quantization

Real power of this perspective:

Algebra	Geometry	Physics
Vect	Fun	TQM
Cat	Shv	2d TQFT
2Cat	ShvCat	3d TQFT

- values (on a point) of higher dimensional TQFTs

(w/ G symmetry / relate to gauge theories)

Higher Quantization

Real power of this perspective:

Algebra	Geometry	Physics
Vect	Fun	TQM
Cat	Shv	2d TQFT
2Cat	ShvCat	3d TQFT

$\rightsquigarrow$ attach more concrete traces / characters / partition functions to higher dimensional manifolds

2d TQFT Z w/ G-symmetry

defines [G finite or algebraic]

$\boxed{0.}$ Category w G-symmetry:

$$Z(\cdot) \in B_G(\cdot) =$$

G-cat = categories / $Loc_G^{\bullet} = {}^{\bullet}/G$

2d TQFT Z w/ G-symmetry

defines [G finite or algebraic]

$\boxed{1.}$ a <u>character</u> sheaf

$$Z(S^1) \in \mathcal{B}_G(S^1) =$$

$$\mathrm{Vect}\left(\mathrm{Loc}_G S^1 = \frac{G}{G}\right)$$

QC if G algebraic

2d TQFT Z w/ G-symmetry

defines [G finite or algebraic]

$\boxed{2.}$ a 2-character on surfaces:

$$Z(T^2) \in \mathcal{B}_G(T^2)$$

$$= \mathbb{C}\left[\mathrm{Loc}_G T^2 = \{g, h \in G \text{ commuting}\}/G\right]$$

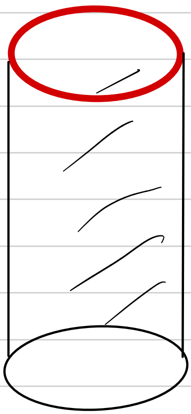
- a function on commuting pairs

$G \circlearrowright X$

$\Rightarrow$ category $Shv(X)$

2d TQFT measuring maps into X

G



$\rightsquigarrow$ character sheaf

$\chi \in Shv(Loc_G S^1)$

given by

$$\boxed{\chi = \pi_* \mathbb{C}}$$

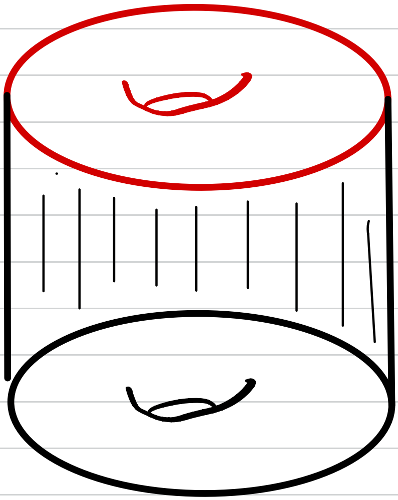
for

$$\{g \in G,\ x \in X^g\}/G$$
$$\downarrow \pi$$
$$\frac{G}{G}$$

eg Springer sheaf ($X = G/B$)

2-character

$2\chi \in \mathbb{C}[\mathrm{Loc}_G T^2]$



Given by

$$\begin{array}{c} \mathrm{Loc}_G^X(T^2) \\ \downarrow \\ \mathrm{Loc}_G T^2 \end{array}$$

i.e.

$$2\chi(g,h) = \mathrm{vol}\,(X^{g,h})$$

joint fixed points

$SL_2\mathbb{Z}$-invariant

This will be our prototype for L-functions, as "3-characters" attached to a 3d TQFT with G-symmetry

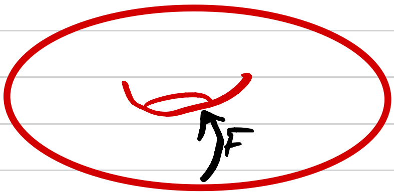
$\in \mathbb{C}[\mathrm{Loc}_G(\mathrm{Spec}\, \mathcal{O}_F)]$

Lusztig (reinterpreted)

G reductive group defined $/\mathbb{F}_q$

Evaluate 2characters on $S^1 \times \underline{S^1_F}$

twist by Frobenius



$\boxed{\begin{array}{c}\text{Tr}(F) \text{ on character sheaves} \\ \| \\ \text{characters of finite group of} \\ \text{Lie type } G(\mathbb{F}_q) = G^F\end{array}}$