Rademacher Lectures

III. L-functions as Partition Functions

Q: How many positive integers are there?

i.e., what is $\sum_{n=1}^{\infty} 1$?

A. $-\frac{1}{2}$

$\left(\begin{smallmatrix}\text{class \#}\\ \text{of } \mathbb{Q}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{rank of}\\ \mathbb{Q}^\times\end{smallmatrix}\right)$

roots of unity in $\mathbb{Q}$

$= \zeta(0)$, ζ = Riemann zeta

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

... defined by analytic continuation from $\mathrm{Im}\ s \gg 0$.

Two lives of ζ:

$$\zeta(s)$$

$$\frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi n^2 y} \right) y^{s/2} \frac{dy}{y} \qquad\qquad \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

Two lives of ζ:

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty \left(\sum_1^\infty e^{-\pi n^2 y} \right) y^{s/2} \frac{dy}{y}$$

$$\zeta(s) = \prod_{\substack{p \\ \text{prime}}} \frac{1}{1-p^{-s}}$$

A	B
analytic	arithmetic
automorphic	spectral
	Galois

Two lives of ξ!

$$\xi(s)$$

$$= \int_0^\infty \left(\sum_{1}^{\infty} e^{-\pi n^2 y} \right) y^{s/2} \frac{dy}{y}$$

$$= \frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

Two lives of ξ:

$$\xi(s)$$

$$= \int_0^\infty \left(\sum_{n=-\infty}^{\infty} \phi(nx) - 1 \right) x^s \frac{dx}{x} \qquad = \mathcal{M}(\phi) \prod_{\substack{p \\ \text{prime}}} \frac{1}{1-p^{-s}}$$

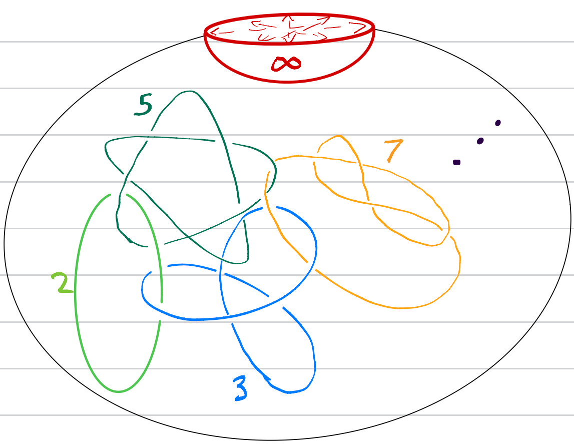
$$\phi = e^{-\pi x^2} \qquad \mathrm{Mellin}(\phi) = \frac{\Gamma(\frac{s}{2})}{\pi^{s/2}}$$

$\rightsquigarrow$ equality of distributions on $\mathbb{R}$

Our goal:

Argue for interpretation of both sides as partition functions of arithmetic QFTs

— ie as some kind of volume



B. Arithmetic

$$\zeta(s) = \prod \frac{1}{1-p^{-s}}$$

Dedekind — number field F

Dirichlet — character χ

$$\zeta_F(s) = \prod \frac{1}{1-N(\mathfrak{p})^{-s}}$$

$$L(\chi, s) = \prod \frac{1}{1-p^{-s}\chi(p)}$$

L-functions of Hecke characters

B. Galois

$\rightsquigarrow$ Artin L-functions

- $\check{G}_{/\mathbb{C}}$ reductive group

$$\mathrm{Gal}(\bar{F}/F) \xrightarrow{\rho} \check{G} \xrightarrow{V} GL(V)$$

$$\boxed{L(\rho, V, s) = \prod_{\mathfrak{p}} \frac{1}{\det\left(I - N(\mathfrak{p})^{-s} \rho_V(\mathrm{Fr}_{\mathfrak{p}})\right)}}$$

(up to ramified places)

- product of local factors: inverse characteristic polynomials

What is L a fnctn of?

$$\mathrm{Loc}_{\check{G}}(F) = \left\{ \mathrm{Gal}\, \bar{F}/F \longrightarrow \check{G} \right\} / \check{G}$$

character variety of Galois reps.

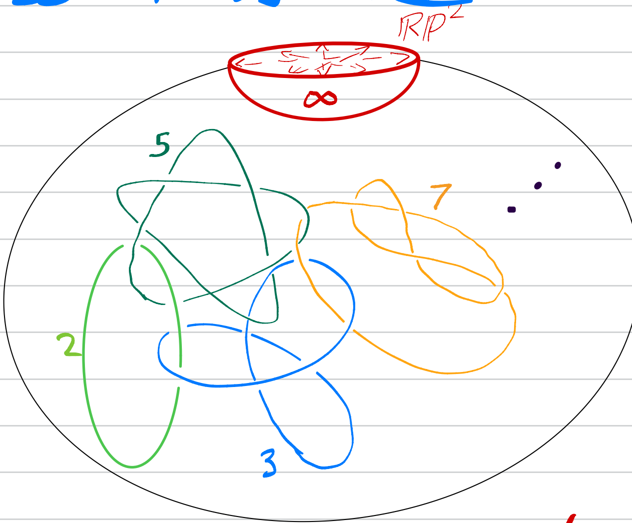
$$\Rightarrow \boxed{\begin{array}{c} L_V \text{ meromorphic}^{*} \text{ function on } \mathrm{Loc}_{\check{G}}: \\ \rho \longmapsto L(\rho, V, 0) \end{array}}$$

[s variable is not independent:

$$L(\rho, V, s) = L(\rho \otimes \|\,\|^{s}, V, 0)$$

norm / cyclotomic character]

Problem: Number fields are complicated!



- archimedean places: noncompact

  $\rightsquigarrow$ need to specify boundary data

  ($\Rightarrow$ ξ <u>distribution</u> on $\mathbb{R}$)

- primes all different

  (circles of length $\log p$)

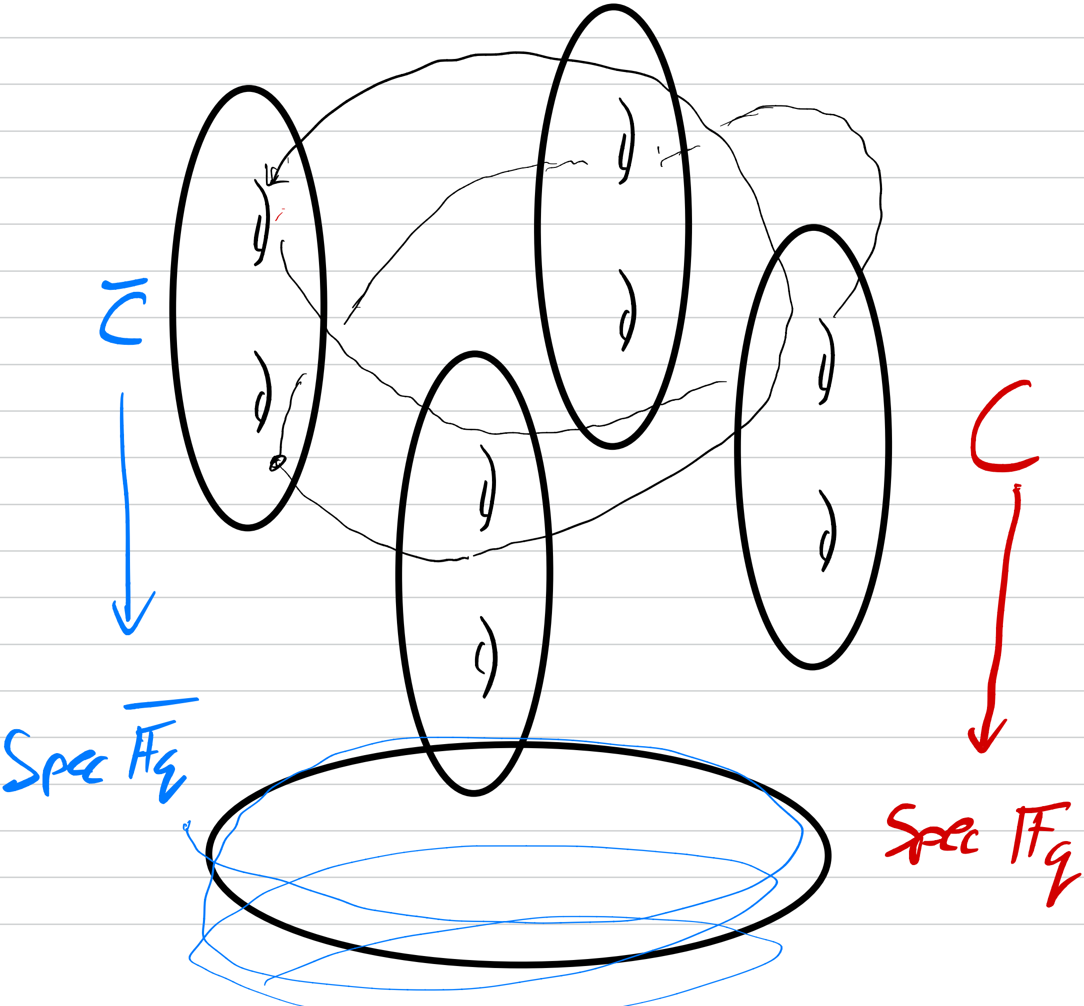
$\rightsquigarrow$ <u>Function fields</u>, good simplified model

Function Fields

C smooth projective curve $/\mathbb{F}_q$

"compact 3-manifold"

$\bar{C} = C \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ "compact 2-manifold"



$$F \rightsquigarrow \mathbb{F}_q(C)$$

$$\mathrm{Gal}(\bar{F}^{ur}/F) \rightsquigarrow \pi_1(C)$$

unramified quotient

Etale fundamental group

$\Rightarrow$ L-functions L_V on

$$\mathrm{Loc}_{\check{G}}(C) = \{\pi_1 C \longrightarrow \check{G}\}/\check{G}$$

given by

$$L(\rho, V, t) = \prod_{x \in |C|} \frac{1}{\det\left(I - t^{\deg x} \rho(Fr_x)\right)}$$

$$t \sim q^s$$

Grothendieck - Lefschetz f.p.p.:
evaluate Euler product globally
in terms of étale cohomology $H^\bullet(\bar{C})$ $\circlearrowleft$ Fr

("integrate over C")

$$L(\rho, V, t) = \frac{\det(I - t\rho_v(Fr)|_{H^1})}{\det(I - t\rho_v(Fr)|_{H^0})\det(I - t\rho_v(Fr)|_{H^2})}$$

$$= \frac{1}{\det(I - t\rho_v(Fr)|_{H^\bullet})}$$

= inverse <u>super</u> - characteristic polynomial
of $\rho_v(Fr) \circlearrowright H^\bullet(\bar{C}, \rho_v)$

What *is* a characteristic polynomial?

$A \in \operatorname{End} W$

$$\det(I - tA) = \sum (-1)^i t^i \operatorname{Tr}(\Lambda^i A)$$
$$= \operatorname{Tr}^\bullet(A, \Lambda^\bullet W)$$

$$\frac{1}{\det(I - tA)} = \sum t^i \operatorname{Tr}(\operatorname{Sym}^i A)$$
$$= \operatorname{Tr}^\bullet(A, \operatorname{Sym}^\bullet W)$$

character of symmetric algebra on W

$$\Longrightarrow L(\rho, V, t) \text{ is}$$

$$\mathrm{Tr}^{\bullet}\big(Fr, \underbrace{\mathrm{Sym}^{\bullet} H^{\bullet}(\bar{C}, \rho_V)}\big)$$

DAG: interpret as ring of functions

$$\mathcal{O}\left((V)^{\pi_1 \bar{C}}\right)$$

derived fixed point locus of

$$\pi_1 \bar{C} \underset{\rho_V}{\circlearrowright} V$$

... or, using Atiyah-Bott

Tr <=> fixed points,

& identification

$$\left((V)^{\pi_1, \overline{C}}\right)^{Fr} = (V)^{\pi_1 C}$$

$\Longrightarrow$

$$L(\rho, V, t)\Big|_{t=1} = \text{Volume}\left((V)^{\pi_1 C}\right)$$

More geometrically:

$(V)^{\pi_1 C}$ = ρ - twisted

locally constant maps $C \longrightarrow V$,

$$\begin{tikzcd} (V)^{\pi_1 C, \rho} \arrow[d] \arrow[r, phantom, "\subset"] & \mathrm{Loc}^{V}_{\check{G}}(C) \arrow[d, "\pi_C"] \\ \rho \arrow[r, phantom, "\in"] & \mathrm{Loc}_{\check{G}}(C) \end{tikzcd}$$

$$\boxed{L(V) = \pi_{C*}\, \mathrm{dvol} \in \mathbb{C}[\mathrm{Loc}_{\check{G}} C]}$$

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This is now in the form of a boundary state / character in higher geometric quantization

$$\begin{array}{c} \mathrm{Loc}^{V}_{\check{G}}\, C \\ \big\downarrow \pi_C \\ \mathrm{Loc}_{\check{G}}\, C \end{array}$$

$$\mathcal{L}_V(C) \in \mathcal{B}_{\check{G}}(C) = \mathbb{C}[\mathrm{Loc}_{\check{G}} C]$$

$$\boxed{L(V) = \pi_{\Xi *}\, \mathrm{dvol} \in \mathbb{C}[\mathrm{Loc}_{\check{G}}\, \Xi]}$$

$C \rightsquigarrow \Xi$

(arithmetic)
3-manifolds

$$\begin{array}{c} \mathrm{Loc}^{V}_{\check{G}}\, \Xi \\ \Big\downarrow \pi_{\Xi} \\ \mathrm{Loc}_{\check{G}}\, \Xi \end{array}$$

$$\mathcal{L}_V(\Xi) \in \mathcal{B}_{\check{G}}(\Xi) = \mathbb{C}[\mathrm{Loc}_{\check{G}}\, \Xi]$$

$$\boxed{L(\check{X}) = \pi_{\Xi *}\, d\mathrm{vol} \in \mathbb{C}[\mathrm{Loc}_{\check{G}}\,\Xi]}$$

$$\begin{array}{c} \mathrm{Loc}_{\check{G}}^{\check{X}}\,\Xi \\ \downarrow {\scriptstyle \pi_\Xi} \\ \mathrm{Loc}_{\check{G}}\,\Xi \end{array}$$

$C \rightsquigarrow \Xi$

(arithmetic) 3-manifolds

$V \rightsquigarrow \check{X}$

(smooth affine) $\check{G}$-variety

$$\mathcal{L}_{\check{X}}(\Xi) \in \mathcal{B}_{\check{G}}(\Xi) = \mathbb{C}[\mathrm{Loc}_{\check{G}}\,\Xi]$$

A. Automorphic

How should we understand Riemann's integral

$$(\Gamma)\,\xi(s) = \int_0^\infty \Big(\sum_{-\infty}^{\infty} \varphi(nx) - 1\Big)\, x^s \frac{dx}{x} \;?$$

$$\left(\varphi(x) = e^{-\pi x^2} \right)$$

Tate, Iwasawa:
It's about $GL_1 \circlearrowright \mathbb{A}^1$!

$L(\chi, \varphi)$:

- χ automorphic form for GL_1/F

$\Longleftrightarrow$

character of idele class group

$[GL_1]_F = F^\times \backslash \mathbb{A}^\times_F$ = "line bundles on Spec F"

- $\varphi \in \mathcal{S}(\mathbb{A}^1_{\text{ramified places}})$

test function $\Longrightarrow$

$$(\Gamma) \, L(\chi, \varphi) = \int_{GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A})} \Theta(\varphi) \, \chi(x) \, d^*x$$

$$\Theta(\varphi)(x) = \sum_{\gamma \in \mathbb{A}^1(\mathbb{Q})} \varphi(\gamma x)$$

$$\boxed{\text{Analytic L-function} = \text{pairing with } \Theta \text{ series}}$$

What does this mean geometrically?

When no ramification

($\rightsquigarrow$ function field $C/\mathbb{F}_q$)

$$x \in [GL_1]_F = \mathrm{Bun}_{GL_1}(C) = \mathrm{Pic}\, C$$

[$\mathbb{F}_q$-points]

$$\boxed{\begin{gathered} \Theta(\mathcal{L}) = \# \Gamma(C, \mathcal{L}) \\ \text{counts sections of } \mathcal{L} \end{gathered}}$$

[more generally, count sections with poles weighted by test functions φ on pole data]

Geometrically

$$\begin{tikzcd}
(\mathcal{L}, s \in \Gamma(\mathcal{L})) \arrow[d] & \in & \mathrm{Bun}^{\mathbb{A}^1}_{GL_1}(C) \arrow[d, "\pi_{/\mathbb{A}^1}"] \\
\mathcal{L} & \in & \mathrm{Bun}_{GL_1}(C)
\end{tikzcd}$$

[This is the Abel-Jacobi map

$\coprod \mathrm{Sym}^n C \longrightarrow \mathrm{Pic}\, C$]

$$\boxed{\Theta = \pi_{/\mathbb{A}^1 !} \underline{1} \in H^*([GL_1]_C)}$$

Prototype for many many

period integrals

of automorphic forms

$G \circlearrowright X$ smooth, affine

$$\Rightarrow \mathcal{P}_X : H^*([G]_F) \longrightarrow \mathbb{C}$$

$$\boxed{\mathcal{P}_X(\chi) = \int_{[G]_F} \chi \cdot \Theta_X}$$

pairing with X-Θ-series

$$\left\{ \begin{array}{l} G\text{-bundles} \\ + \text{ section of} \\ \text{assoc. } X\text{-bundle} \end{array} \right\} = \mathrm{Bun}_G^X C$$

$$\begin{tikzcd} \mathrm{Bun}_G^X C \arrow[d, "\pi_X"] \\ \mathrm{Bun}_G C \end{tikzcd}$$

$$\Theta_X = \pi_{X*} 1 \in H^*(\mathrm{Bun}_G)$$

- again has form of a boundary state / geometric quantization

$$\mathcal{P}_X(\Xi) \in A_G(\Xi)$$

e.g $X = G/H$

$\Rightarrow \mathrm{Bun}_G^H C = \mathrm{Bun}_H$:

$$P_H(\chi) = \int_{[G]} \chi \, \Theta_{G/H}$$

$$= \int_{[H]} \chi|_{[H]}$$

integrating automorphic forms along $[H]_F \hookrightarrow [G]_F$

e.g. $G = PGL_2 \curvearrowright PGL_2/T = \begin{pmatrix} * & \\ & * \end{pmatrix}$

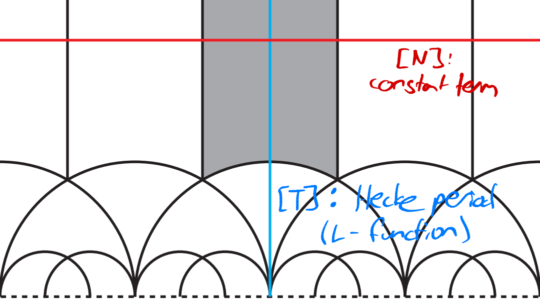
$\rightsquigarrow$ Hecke period:

L-functions of modular forms

$G = PGL_2 \curvearrowright PGL_2/N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

$\rightsquigarrow$ Eisenstein period:

constant term of modular forms



[N]:
constant term
[T]: Hecke period
(L-function)