Rademacher Lectures

IV. Duality

Riemann on $\Gamma\,\xi(s)$

$$\int_0^\infty y^{s/2}\,\theta(y)\,dy = \Gamma \prod_p \frac{1}{1-p^{-s}}$$

A / automorphic $\longleftrightarrow$ B / Galois

Class field theory:

χ character of $[GL_1]_F$ $\longleftrightarrow$ ρ character of $\mathrm{Gal}\,\bar{F}/F$

$$\int_{[GL_1]_F} \chi\, \Theta \qquad\qquad \prod_{p} \frac{1}{1-N(p)^{-s}\rho(Fr_p)}$$

period = L-function

A / automorphic $\qquad\qquad$ B / Galois

Class field theory:

$$\begin{tikzcd} H^{\bullet}([GL_1]_F) \arrow[r, phantom, "\simeq"] \arrow[dr, "\text{count sections of } \mathbb{A}^1\text{-bundle}"'] & \mathbb{C}[Loc_{GL_1} F] \arrow[d, "\text{volume of fixed points on } \mathbb{A}^1"] \\ & \mathbb{C} \end{tikzcd}$$

A / automorphic

B / Galois

Class field theory:

$$\begin{tikzcd}
\mathcal{A}_{GL_1}(F) \arrow[dr, "H_{/A'}(F)"'] & \simeq & \mathcal{B}_{GL_1}(F) \arrow[dl, "\mathcal{L}_{/A'}(F)"] \\
& \mathbb{C} &
\end{tikzcd}$$

A-type character of $GL_1(\mathbb{C})/A'$

B-type character of $GL_1(\mathbb{C})/A'$

A / automorphic

B / Galois

Recall: Polarized Geometric Quantization

$$\begin{tikzcd}
 & & \text{(G-) manifolds} \arrow[dll, "T^*"'] \arrow[ddl, "\substack{\text{linearization} \\ \text{(e.g. } L^2\text{)}}"] \\
\text{(Hamiltonian G-) Symplectic manifolds} \arrow[dr, dashed] & & \\
 & \text{(G-) quantum theories} &
\end{tikzcd}$$

Natural domain of quantization is symplectic geometry:

- More examples:
  lots more than cotangent bundles . .

- More symmetry:
  T^*X has many more symplectomorphisms than X has automorphisms

hallmark of geometric quantization: independent of polarizatn

$\Theta_{/A'}$ function of T^*/A' (Fourier transform)

$\Downarrow$

Functional equation $\xi(s)=\xi(1-s)$

$\Uparrow$

$L_{/A'}$ function of T^*/A' (Poincaré duality)

Relative Langlands:
general study of matching of measurements

$$
\begin{tikzcd}
\text{automorphic forms} \arrow[dr, "\text{period integrals}"'] & \simeq & \text{Galois representations} \arrow[dl, "L\text{-functions}"] \\
& \mathbb{C} &
\end{tikzcd}
$$

BZ-Sakellaridis-Venkatesh:

- Period integrals

$\Uparrow$

geometric quantization $\Theta_M \in \mathcal{A}_G$
of $G \circlearrowright M$ hamiltonian
(eg $M = T^*X$, $G \circlearrowright X$)

- L-functions

$\Uparrow$

geometric quantization $\mathcal{L}_{\check{M}} \in \mathcal{B}_{\check{G}}$
of $\check{G} \circlearrowright \check{M}$ hamiltonian
(eg $\check{M} = T^*\check{X}$, $\check{G} \circlearrowright \check{X}$)

- Duality: describe matching of boundary theories

$$H_M \longleftrightarrow \mathcal{L}_{\check{M}}$$

(under <u>restrictive</u> hypotheses: $M, \check{M}$ hyperspherical)

- Symmetry: predict same pairs match in reverse,

$$H_{\check{M}} \longleftrightarrow \mathcal{L}_M$$

Some dual pairs

Tate $\quad GL_1 \curvearrowright T^*\mathbb{A}^1 \qquad T^*\mathbb{A}^1 \circlearrowleft GL_1 \quad$ Tate

Hecke $\quad PGL_2 \curvearrowright T^* PGL_2/T \qquad T^*\mathbb{A}^2 \circlearrowleft SL_2 \quad$ standard

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Group $\quad G\times G \curvearrowright T^*G \qquad T^*\check{G} \circlearrowleft \check{G}\times\check{G} \quad$ Group

Whittaker $\quad G \curvearrowright T^* G/\!/_{\psi} U \qquad T^* pt \circlearrowleft \check{G} \quad$ Point

Eisenstein $\quad G\times T \curvearrowright T^* G/U \qquad T^*\check{G}/\check{U} \circlearrowleft \check{G}\times\check{T} \quad$ Eisenstein

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$GL_n\times GL_n \curvearrowright T^*(GL_n\times\mathbb{A}^n) \qquad T^*\mathfrak{gl}_n \circlearrowleft GL_n\times GL_n$

Rankin-Selberg $\qquad$ Godement-Jacquet

$SO_{2n}\times SO_{2n+1} \curvearrowright T^* SO_{2n+1} \qquad \mathrm{std}\otimes\mathrm{std} \circlearrowleft SO_{2n}\times Sp_{2n}$

Gan-Gross-Prasad $\qquad$ Theta Correspondence

Q: How does the dual $\check{G} \curvearrowright \check{M}$ arise from geometric quantization of $G \curvearrowright T^*X$?

A: The geometric Plancherel formula!

Synopsis: Relative Langlands

- Langlands Duality matches automorphic spaces $\mathcal{A}_G(F)$ for every input "manifold" with spectral spaces $\mathcal{B}_{\check{G}}(F)$ given simply by linearizing

  $\mathrm{Loc}_{\check{G}} F$: spaces of Langlands parameters

- Our goal: to describe the data coming from quantization of $G \circlearrowright X$ spectrally

Synopsis of automorphic quantization

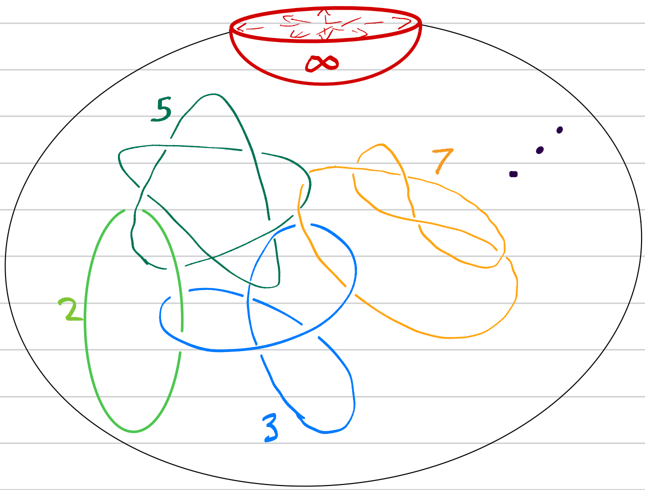
$\Theta_X \in \mathcal{A}_G$ of $T^*X \circlearrowleft G$

3d

X-period on automorphic forms

F number field

$H^*([G]_F)$



or curve

$C / \mathbb{F}_q$

$\Theta_X \in \mathbb{C}[\mathrm{Bun}_G C(\mathbb{F}_q)]$

count of sections of associated X-bundle

Global duality question:

describe period functional

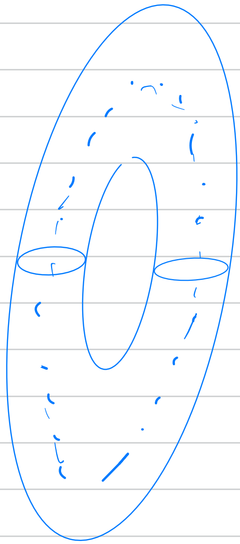
spectrally, as functional on

$$\mathcal{B}_{G^\vee}(F) = \mathbb{C}[\mathrm{Loc}_{G^\vee} F]$$

– eg as L-function . .

2d $\qquad C^\infty(X(K)) \in$ Smooth reps of $G(K)$

K local field (nonarch.)

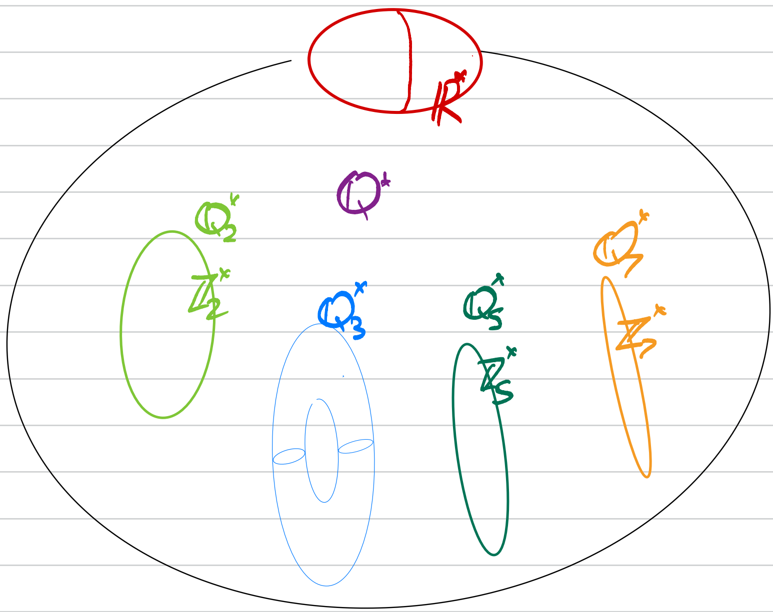


$K = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$

Local-to-global:

reps give input for Θ-series operators

$$\bigotimes_{\text{ramified places}} C^\infty(X(K_i)) \xrightarrow{\Theta} H^*([G]_F)$$



(TQFT on 3-manifold w/ boundary)

Local duality question:

describe decomposition of $C^\infty(X(F))$

as $G(F)$-representation –

via local Langlands correspondence

as family of multiplicity spaces

over $\mathcal{B}_{\check{G}}(k) = \mathrm{Coh}(\mathrm{Loc}_{\check{G}} F)$

space of Langlands parameters

... want <u>finite</u> multiplicities

$G \circlearrowright X$ spherical :

strong finiteness condition

$\iff$ all multiplicities in $X(F)$ are finite.

$\iff$ $X(F)/G(O)$ countable

$\iff$ Borel B has finitely many orbits on X $(/k = \bar{k})$

Spherical varieties include

- toric varieties $(G = T)$
- symmetric spaces
- flag varieties
- $G \times G \circlearrowright G$
- $GL_n \times GL_{n-1} \circlearrowright GL_n$
- $SO_{2n+1} \times SO_{2n} \circlearrowright SO_{2n}$
- "very small" representations

Filling in the loop:

2d

K local field (nonarch.)

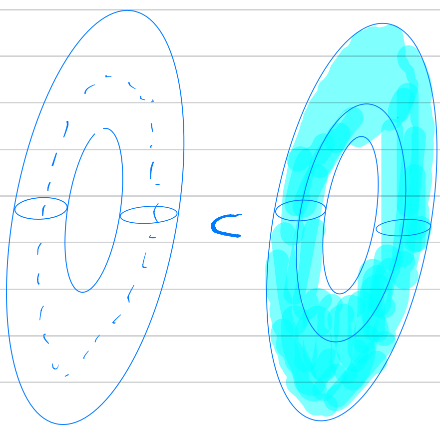
$\cup$

$\mathcal{O}$ ring of integers

$$\begin{array}{ccc} C^\infty(X(K)) & \in & \text{Smooth reps of } G(K) \\ \Big\downarrow (\ )^{G(\mathcal{O})} & & \Big\downarrow (\ )^{G(\mathcal{O})} \\ C^\infty(X(K)/G(\mathcal{O})) & \in & H_{sph}\text{-mod} \end{array}$$

Simpler "unramified" question

$K = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$



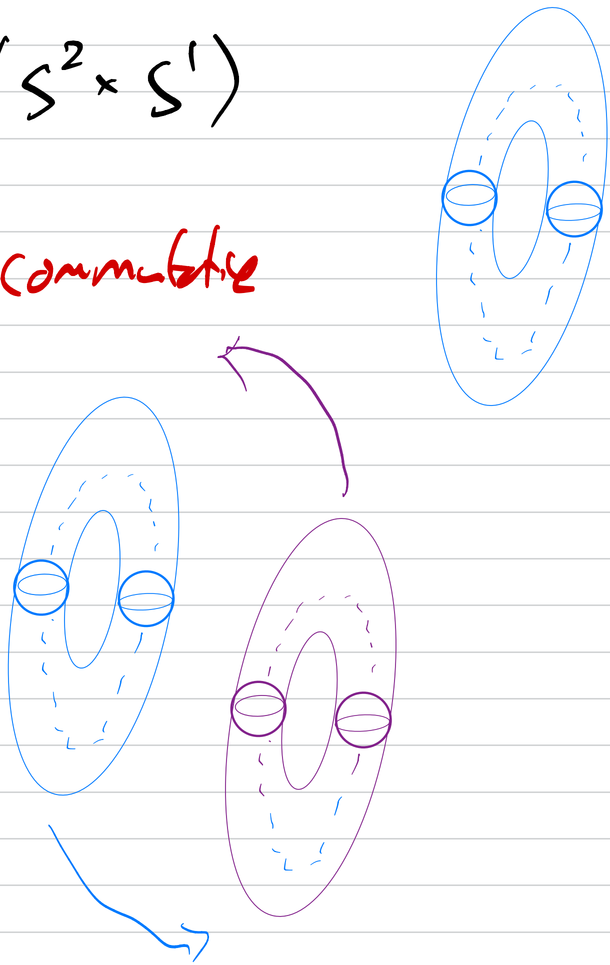
$\mathcal{O} = \mathbb{Z}_p$ or $\mathbb{F}_q[[t]]$

Spherical Hecke algebra

$$H_{sph} = C_c^{\infty}\left({}_{G(O)}\backslash G(K)/G(O)\right)$$

$= \underline{\text{loop operators}}$

$$A_G(S^2 \times S^1)$$



form a commutative algebra by TFT magic

Satake isomorphism

$$H_{sph} = A_G(S^2 \times S^1) \underline{\simeq}$$

$$B_{\check{G}}(S^2 \times S^1) = \mathbb{C}\left[\begin{array}{c} Loc_{\check{G}}(S^2 \times S^1) \simeq \\ \frac{\check{G}}{\check{G}} \sim \check{H}/W \\ \simeq \mathbb{C}^n \text{ for } SL_{n+1} \end{array} \right]$$

so H_{sph}-modules spectrally decompose over $\check{H}/W$

$$C^\infty(X(F))^{G(O)} \ni 1_{X(O)}$$

canonical "unramified" test vector

$\Longrightarrow$ get Plancherel density

$$\langle \_ * 1_{X(O)}, 1_{X(O)} \rangle \in H_{sph}^*$$

(or more general

$$\langle \_ * \_, \_ \rangle : V \otimes V^* \longrightarrow H_{sph}^*$$

cf. Hilbert C^*-modules)

Spectrally: Plancherel measure gives a functional on $\mathbb{C}\left[\frac{\check{G}}{\check{G}}\right]$

- Sakellaridis: for $G \circlearrowright X$ spherical this is integrating against an L-factor (~ characteristic polynomial of a representation) on a subgroup
- S-Venkatesh: this gives local factor in Euler product for X-period.

Source of answers

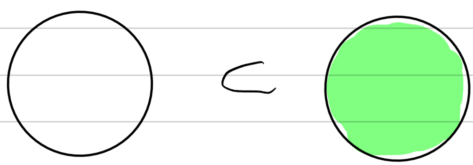
$$\begin{array}{lcl}
\mathrm{Id} & \mathrm{Shv}(X(\bar{K})) \in & \text{Categorical reps of } G(\bar{K}) \\
& \Big\downarrow {\scriptstyle (\ )^{G(\mathcal{O})}} & \Big\downarrow {\scriptstyle (\ )^{G(\mathcal{O})}} \\
& \mathrm{Shv}(X(K)/G(\mathcal{O})) \in & \mathcal{H}_{sph}\text{-mod}
\end{array}$$

$\bar{K}$ local geometric

$\cup$

$\bar{\mathcal{O}}$

$\bar{K} = \bar{\mathbb{F}}_q((t))$ or $\mathbb{C}((t))$



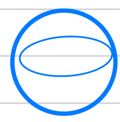
$\bar{\mathcal{O}} = \bar{\mathbb{F}}_q[[t]]$ or $\mathbb{C}[[t]]$

Spherical Hecke category

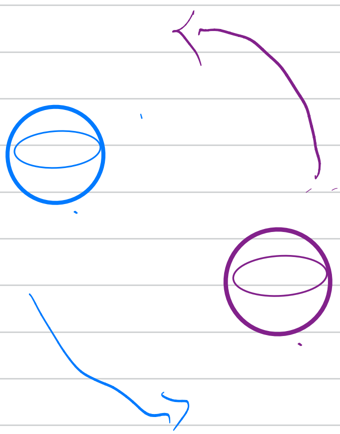
$$\mathcal{H}_{sph} = Shv\left(G(O)\backslash G(K)/G(O)\right)$$

$= \underline{\text{line operators}}$

$$\mathcal{A}_G(S^2)$$



form a commutative tensor category by TFT magic



Geometric Satake isomorphism

$\mathcal{H}_{sph} = \mathcal{A}_G(S^2) \simeq$

$$\mathcal{B}_{\check{G}}(S^2) = \mathrm{Coh}\left[\begin{array}{c} \mathrm{Loc}_{\check{G}}(S^2) \simeq \\ \check{\mathfrak{g}}^*[2]/\check{G} \\ \text{coadjoint quotient} \end{array}\right]$$

- Tannakian reconstruction:

this is the birth of $\check{G}$

& Langlands duality

- BZSV: the $\check{\mathfrak{g}}^*$

$\implies$ adjoint L-factor

in Macdonald's formula

(Plancherel measure on p-adic groups)

- More geometrically, $\check{\mathfrak{g}}^*/\check{G}$

is target for (equivariant)

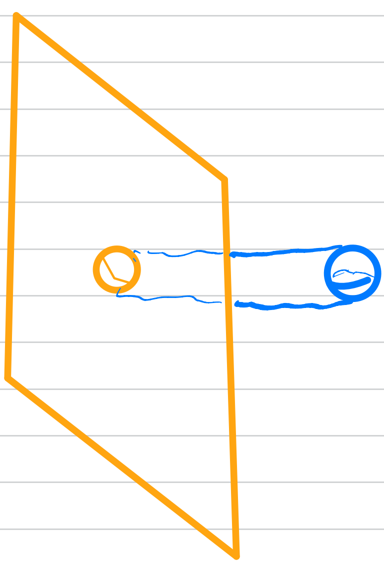
moment maps

$$\mathrm{Shv}(X(\bar{F}))^{G(\bar{O})} \ni \underline{1}_{X(\bar{O})}$$

$\Longrightarrow$ geometric Plancherel density

$$\mathbb{PL}_X = \mathrm{Hom}(- * \underline{1}_{X(\bar{O})}, \underline{1}_{X(\bar{O})}) \in \mathrm{Alg}(\mathcal{H}_{sph})$$

commutative algebra object in Hecke category



$$\Theta_X(S^2) \in \mathcal{A}_G(S^2)$$

local operators on boundary

$\Longrightarrow$ Discover affine Poisson variety $\check{M} = \mathrm{Spec}(\mathbb{PL}_X)$ equipped with $\check{G}$-action & equivariant moment map

$$\begin{array}{ccc} & \circlearrowright & \check{M} \\ \check{G} & & \downarrow \\ & \circlearrowright & \check{\mathfrak{g}}^* \end{array}$$

(relative Coulomb branch, Braverman-Finkelberg-Nakajima)

So : geometric Plancherel formula

for $G \circlearrowright X$

$\rightsquigarrow$

Hamiltonian $\check{G}$ space $\check{M}$

(sometimes polarized $\check{M} = T^*\check{X}$)

- Explicit conjectural description
- Conjecturally

M hyperspherical $\Longleftrightarrow$ $\check{M}$ hyperspherical

(~ T^*(spherical), smooth, affine)

- In every setting of Langlands correspondence

get conjectural matching

$$\begin{array}{ccc} \mathcal{A}_G & \simeq & \mathcal{B}_{\check{G}} \\ \cup & & \cup \\ \Theta_M & \hookrightarrow & \mathcal{L}_{\check{M}} \end{array}$$