WHAT IS THE GEOMETRIC LANGLANDS CORRESPONDENCE ABOUT?

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Context

The (unramified) Geometric Langlands Correspondence (GLC) was established in 2024 by

GLC Team

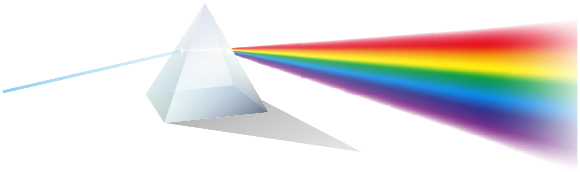
Dennis Gaitsgory and Sam Raskin with
D. Arinkin, D. Beraldo, J. Campbell, L. Chen, J. Faergeman, K. Lin and N. Rozenbylum.

The proof appears in five papers [GLC I – GLC V] totaling over 800 pages, building on a vast body of prior work.

What's this all about?

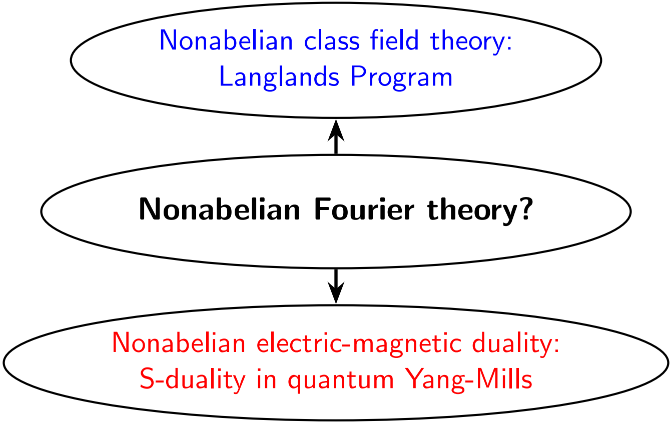
Context

The Fourier transform – or, **the exploitation of abelian symmetry** – is ubiquitous in mathematics and physics.



Context

Nonabelian Fourier theory?


Nonabelian class field theory:
Langlands Program
Nonabelian Fourier theory?
Nonabelian electric-magnetic duality:
S-duality in quantum Yang-Mills

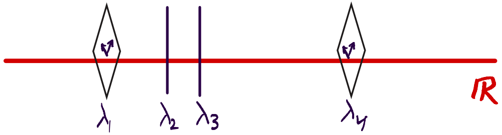
What are we looking at?

Great discovery of GLC: there's an elephant! And we can hear its heartbeat: factorization.

Diagonalizing matrices

T $n \times n$ Hermitian matrix

- Decomposition into eigenspaces $\mathbb{C}^n \simeq \bigoplus_{\lambda \in \mathbb{R}} (\mathbb{C}^n)_\lambda$
- Action of T given by the coordinate t on $\mathbb{R}$.
- All functions on $\mathbb{R}$ act on $\mathbb{C}^n$ by multiplication operators



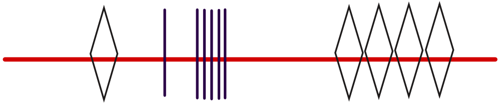
The Spectral Theorem

$T \circlearrowright \mathcal{H}$ self-adjoint operator on Hilbert space:

- Spread $\mathcal{H}$ out over $\mathbb{R}$ as measurable family of Hilbert spaces
- Recover $\mathcal{H}$ as direct integral of Hilbert-space valued measure on $\mathbb{R}$:

$$\mathcal{H} \simeq \int_{\mathbb{R}}^{\oplus} d\mathcal{H}(t)$$

- Functional calculus: Measurable functions act by multiplication operators (t acts by T)



The Spectral Theorem revisited

- The spectral theorem defines an equivalence of categories

$$\{\text{Hilbert spaces with a self-adjoint operator}\} \longleftrightarrow \{\text{measurable families of Hilbert spaces on } \mathbb{R}\}$$

RHS consists of modules for measurable functions on $\mathbb{R}$.

The Algebraic Spectral Theorem

- The "algebraic spectral theorem":

$\{$Vector spaces with a linear operator$\}\longleftrightarrow$ $\{$algebraic families of vector spaces on $\mathbb{C}$ $\}$

RHS: **defined to be** modules for $\mathbb{C}[t]$

Official name: quasicoherent sheaves on the line.

The Algebraic Spectral Theorem

- The "algebraic spectral theorem":

$\{$Vector spaces with a linear operator$\} \longleftrightarrow QC(\mathbb{C})$

RHS: **defined to be** modules for $\mathbb{C}[t]$

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Symbiosis: Fourier Transform via the Spectral Theorem

The Fourier transform

$$L^2(\mathbb{R}_x) \overset{\sim}{\longleftrightarrow} L^2(\mathbb{R}_t), \qquad f(x) = \int_{\mathbb{R}_t} \widehat{f}(t) e^{2\pi i x t} dt$$

performs spectral decomposition for $\mathcal{H} = L^2(\mathbb{R}_x)$:

- $\left\{ \begin{array}{c} \textit{differentiation} \\ \textit{translation} \\ \textit{convolution} \end{array} \right\}$ operators in $x \Longrightarrow$ multiplication operators in t
- Functions of $x \rightsquigarrow$ superpositions of eigenfunctions $e^{2\pi i x t}$: monochromatic waves parametrized by colors t.

Symbiosis: the Spectral Theorem as a Fourier transform

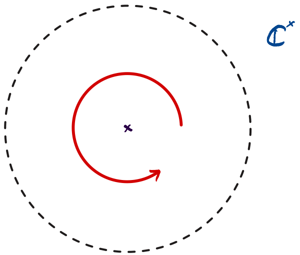
OTOH: the spectral theorem is itself a "categorified" Fourier transform:

Fourier Transform	Spectral Theorem
$e^{i\lambda x}$	$\mathbb{C}_\lambda$
$f(x)$	$T \circlearrowright \mathcal{H}$
integral of exponentials	direct integral of eigenspaces
numbers	vector spaces
Hilbert space $L^2(\mathbb{R})$	category of self-adjoint operators

Mellin transform

Multiplicative spectral theorem:

$\{$Vector spaces with invertible linear operator$\} \longleftrightarrow \{$algebraic families of vector spaces on $\mathbb{C}^\times\}$



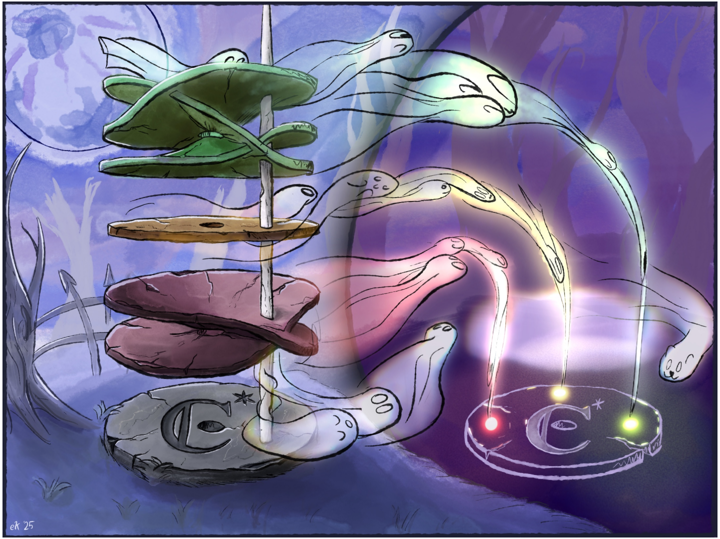
- LHS: monodromy of diff. eqn. on $\mathbb{C}^\times$ (multivaluedness of solutions)
- Spectrally decompose diff. eqns. into

$$z\frac{d}{dz}f = sf$$

- solved by $f = z^s$, monodromy $t = exp(s)$

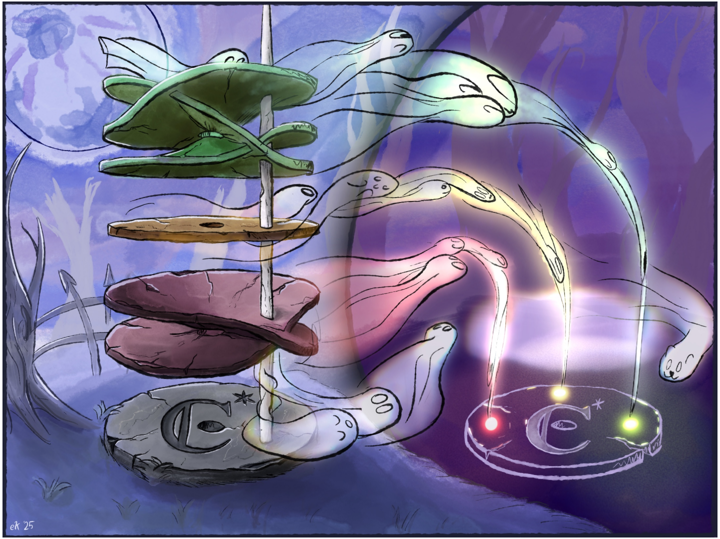
Mellin transform, by Elliot Kienzle

$$\{\text{Systems of diff.eqs. on } \mathbb{C}_z^\times\} \longleftrightarrow \{\text{families of vector spaces on } \mathbb{C}_t^\times\}$$



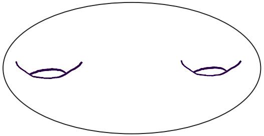
Mellin transform, by Elliot Kienzle

$$\mathcal{D}iff(\mathbb{C}_z^\times) \longleftrightarrow QC(\mathbb{C}_t^\times)$$



GLC: the setting

- We fix a complex[1] Lie group G
  – **today:** $GL_n\mathbb{C}$, invertible matrices.
- It comes with a "dual" group $\check{G}$ (also $GL_n\mathbb{C}$).
- We also fix a Riemann surface $\leftrightarrow$ complex projective curve C
  ($\rightsquigarrow$ spacetime manifolds in QFT and global fields in number theory)



[1] reductive group $\leftrightarrow$ complexification of compact Lie group

What does GLC say?

Unramified Geometric Langlands Correspondence

$$\{\text{Systems of diff.eqs. on } Bun_G(C)\ \} \longleftrightarrow \{\text{families of vector spaces on } Loc_{\check{G}}(C)\ \}$$

- A spectral theorem for a class of differential equations[2] (differentiation) or their topological monodromy data (translation): **automorphic sheaves** (LHS).
- GLC asserts they can be decomposed into monochromatic objects.
- The corresponding "colors" are $\check{G}$-**local systems** on C (RHS).

[2] Linear system of PDE

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Unramified Geometric Langlands Correspondence

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[2] Linear system of PDE

“New Analysis”

Like the spectral theorem, GLC is also a kind of Fourier transform

but with vector spaces rather than numbers,

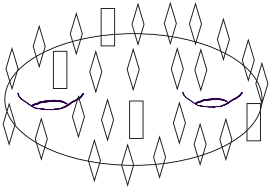
diff. eqns. ($z\frac{d}{dz} - s$) rather than their solutions (z^s).

Major theme: development of

- **functional analysis** for derived categories
  (playing role of topological vector spaces), and
- **harmonic analysis** for categories of sheaves
  (playing role of function spaces)

The Moduli Space of Bundles

$Bun_G(C)$: moduli space of rank n vector bundles[3] on C:
algebraic families of n-dim vector spaces



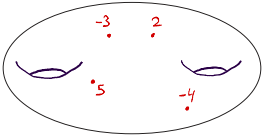
- Variant of arithmetic locally symmetric spaces (eg moduli of elliptic curves),
- and of moduli of instantons (solutions of Yang-Mills eqns)

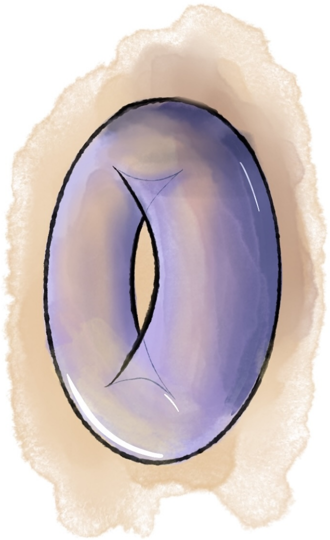
[3]- or principal G-bundles

The Jacobian

Line bundles ($n = 1$):

- $Bun_{GL_1} \leftrightarrow Jac(C)$ the Jacobian[a] of C, a $dim_{\mathbb{R}} = 2g$ torus.
- It is an abelian group!
- quotient of group of **divisors** on C (integer combos of points: ways to prescribe zeros and poles)

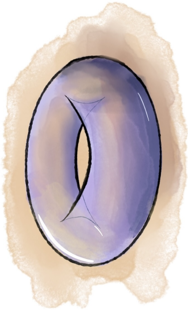




[a] Abel, Jacobi, Riemann

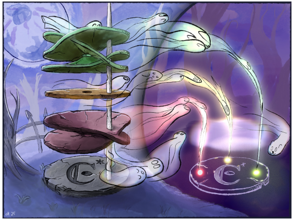
Geometric Class Field Theory

GLC in rank 1: repeat the Mellin story with $\mathbb{C}_z^\times \rightsquigarrow Jac(C)$:

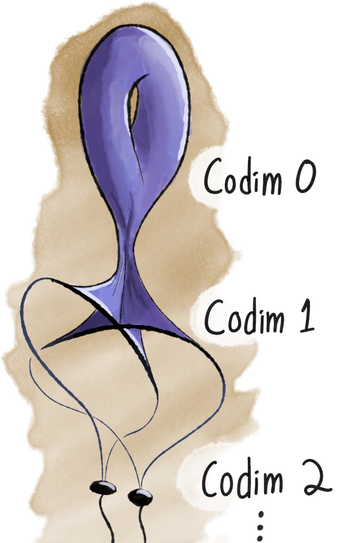


- Study diff.eqs. on $Jac(C)$
- Now have $2g$ directions
- $\rightsquigarrow$ $2g$ commuting monodromies A_i, B_i
- $\rightsquigarrow$ spectral side given by $(\mathbb{C}^\times)^{2g}$

$$\mathcal{D}iff(Jac(C)) \longleftrightarrow QC((\mathbb{C}^\times)^{2g})$$



What about higher rank bundles?



General case:

- Bun_G noncompact!
- open "core" looks like projective variety
- long "tail" of strata:
  bundles with many automorphisms
- Bun_G not a group!

The Beating Heart

What is so special about $Bun_G(C)$??

The essence of geometric Langlands:

Bun_G behaves surprisingly like an abelian group.

– need to look at Bun_G not through points but harmonic analysis.

Two deep manifestations:

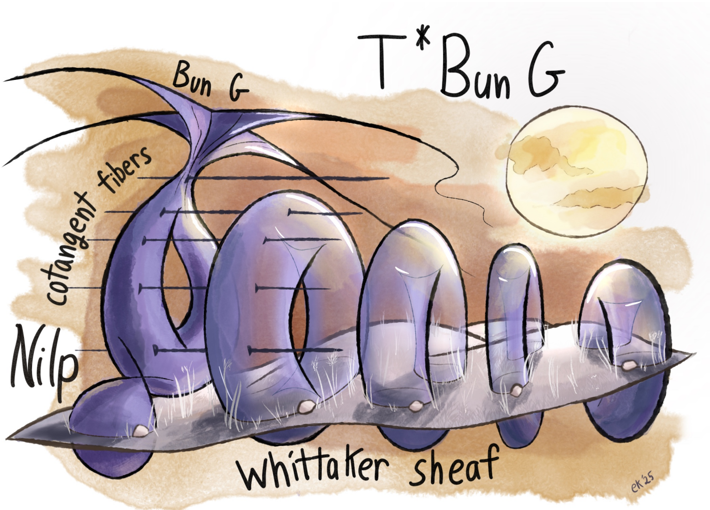
- The Hitchin System - microlocalization
- Hecke Operators - convolution

Microlocalization

- Differential equations closer to geometry of **phase space** than position space (characteristics, wavefronts,. . .)

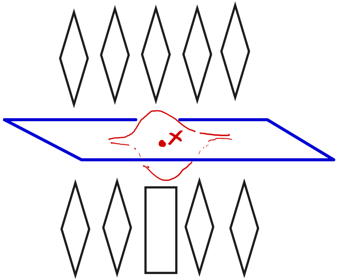
- So look not at Bun_G but at its cotangent T^*Bun_G.

Action-angle coordinates: $T^* Bun_G$ fibers over a vector space, generic fibers tori (Jacobians) – realization of Higgs mechanism.



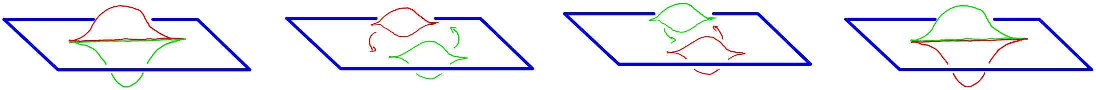
Hecke Operators

- ~~Adding divisors~~
  Relation on bundles: modification localized at points $x \in C$.
- Averaging over modifications $\Rightarrow$ convolution operators on Bun_G.



Factorization

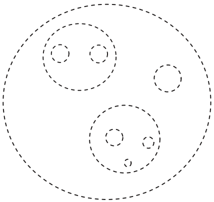
- **Crucial Idea:**[4] Factorization ($\Leftarrow$ OPE in QFT).
- To compose Hecke operators:
  Slide modifications to different points of the curve
- $\Longrightarrow$ automatically commute!
- Same picture as commutativity of π_2 in topology.



[4] Beilinson-Drinfeld

Uses of Factorization

- New notion: **factorization algebra –** algebraic structure encoded in collision of points.
- Unifies vertex algebras, iterated loop spaces, braided tensor categories,. . .
- Applications in QFT, representation theory, number theory: fundamental mechanism behind Langlands and electric-magnetic duality.



Spectral Theorem for Hecke Operators

Factorization $\Longrightarrow$ Hecke operators form commutative algebra.
What is its **spectrum**, i.e., possible eigenvalues?

Geometric Satake Correspondence

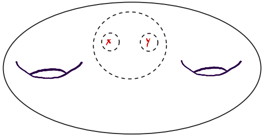
Possible eigenvalues of Hecke operators at a point of $C \Longleftrightarrow$ n-dim vector spaces

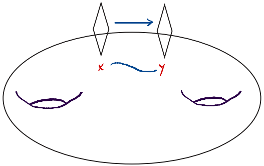
- For general G, get vector spaces with extra structure: birth of the **Langlands dual group**[5] $\check{G}$
- e.g. $G = SO(2n+1)$ orthogonal group $\Longleftrightarrow$ $\check{G} = Sp(2n)$: spectrum is symplectic vector spaces!

[5] Lusztig, Ginzburg, Mirković-Vilonen

Global Hecke Algebra as Factorization Homology

- Factorization homology: put together Hecke operators at all $x \in C$ ($\sim$ **relations** among divisors $Div(C) \twoheadrightarrow Jac(C)$)
- Prototype of passage from local to global observables in QFT.[6]





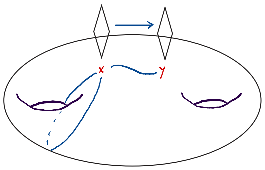
[6] Costello-Gwilliam

The spectral action[7] identifies colors in GLC:

The Spectral Action

The possible eigenvalues of the global Hecke action are given by rank n local systems on C

- Each local system ρ attaches $\{x \in C\} \mapsto \{\rho_x \in Vect_n\}$, and identify $\rho_x \xrightarrow{\sim} \rho_y$ for nearby points.



[7] Drinfeld-Gaitsgory

The Spectral Side: The Rainbow

- Local systems $\leftrightarrow$ points of the **character variety** $Loc_{\check{G}}(C)$:
  n-dimensional (or $\check{G}$) representations ρ of the fundamental group,

  $$Loc_{\check{G}}(C) = \{\rho : \pi_1(C) \longrightarrow \check{G}\}/\check{G},$$

  i.e., by $2g$ $n \times n$ matrices satisfying

  $$\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = 1$$

  up to conjugacy.
- Solves "automorphic-to-spectral" direction of the Langlands correspondence in many settings[8].

[8] V. Lafforgue, Nadler-Yun, GLC team+Kazhdan-Varshavsky, Fargues-Scholze,. . .

Outline of the Proof

How does the GLC team establish the conjecture?

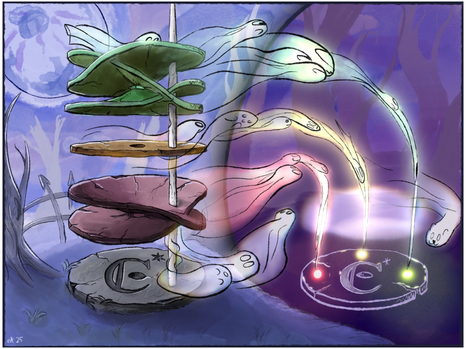
Rough idea:

- Want to show the Hecke algebra has a **simple spectrum** on $\mathcal{D}iff(Bun_G)$:
- multiplicity one for every possible eigenvalue –
  "quantum complete integrability"
- (up to getting growth conditions right..)

White light

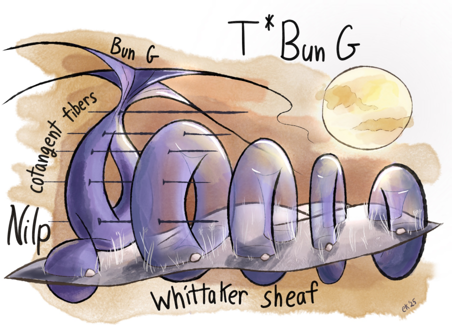
Simple spectrum $\iff$ have a cyclic vector:
white light

- Identify candidate object
- Show its constituents span (cyclic)
- Show every color appears (white)



Whittaker lore [GLC I, GLC IV]

- Representation theory suggests Whittaker equation
- (Spectral action) + (Whittaker normalization) uniquely characterize GLC.
- Cyclicity $\sim$ meets every fiber transversely (!)

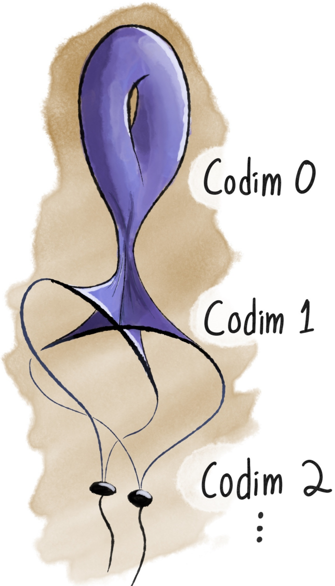


Divide and conquer [GLC III]

"Hard but standard" part:
the continuous spectrum.

- Classical inductive strategy[a]:
- Theory of **Eisenstein series** $\implies$ Contribution of cusps reduces to lower rank groups

[a] Harish-Chandra, Gelfand–Graev–Piateskii-Shapiro, Langlands



The Ace up the Sleeve [GLC II]

Finally, need to find cusp forms:
the "rare gems" that do not arise from induction.

- Write down explicit differential equations[9] on Bun_G
  (analogs of $z\frac{d}{dz} - s$)
  from local data near a point $x \in C$:
  representations of ∞-dim. (affine Kac-Moody) Lie algebras
- Conformal field theory $\Longrightarrow$ treasure chest of monochromatic reps[10].
- Local-to-global compatibility $\rightsquigarrow$ every color is realized.

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$\rightsquigarrow$ **QED** **[GLC V]**

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Why do we care? What's next?

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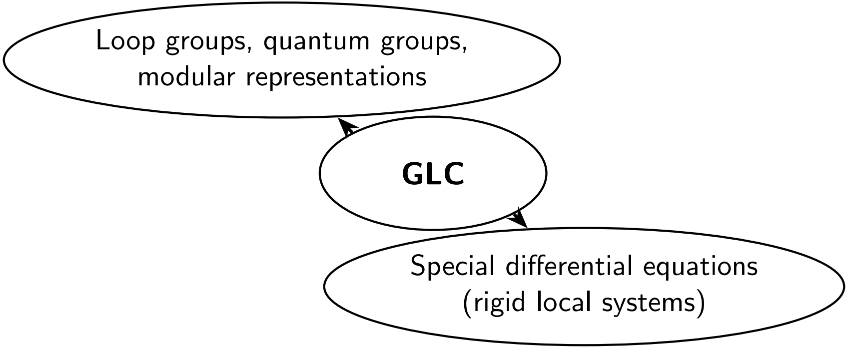

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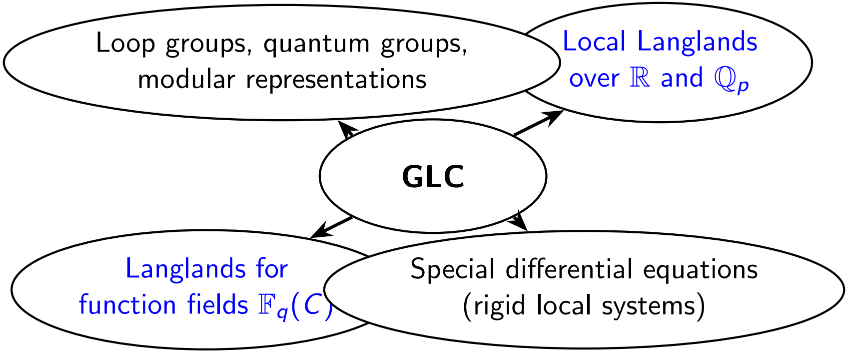

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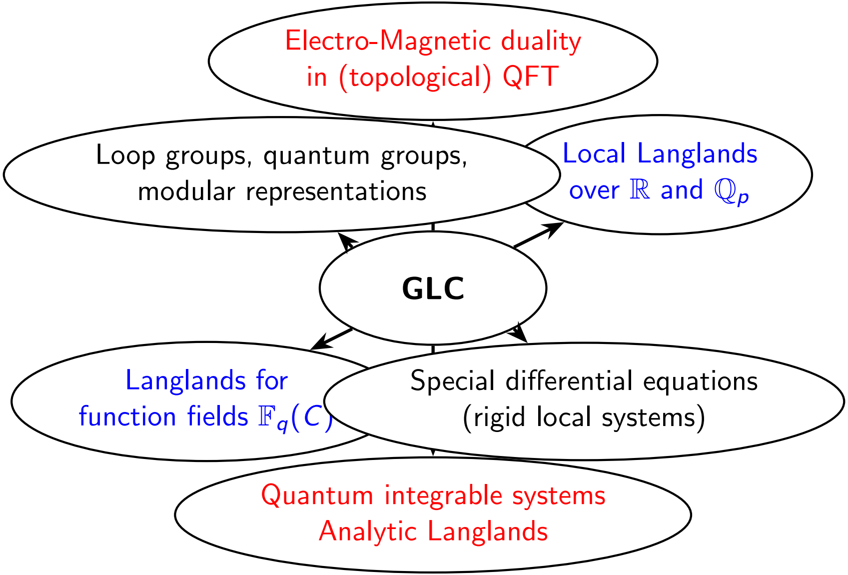

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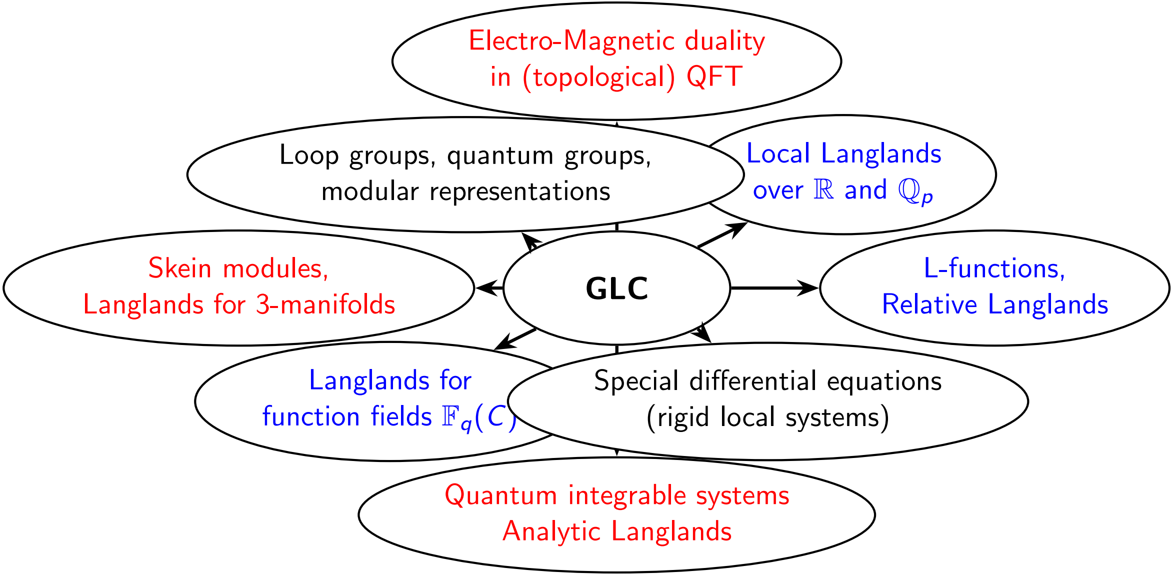

Thank You!

Artwork by Elliot Kienzle
https://chessapig.github.io/