ANNOTATED BIBLIOGRAPHY

DAVID BEN-ZVI

1. Integrable Systems

[BF1] D. Ben-Zvi and E. Frenkel, Spectral Curves, Opers and Integrable Systems. *Publications Mathèmatiques de l'Institut des Hautes Études Scientifiques* **94** (2001) 87-159. Featured Review MR1896178 (2003j:14047).

We establish a general link between integrable systems in algebraic geometry (expressed as Jacobian flows on spectral curves) and soliton equations (expressed as evolution equations on flat connections). Our main result is a natural isomorphism between a moduli space of spectral data and a moduli space of differential data, each equipped with an infinite collection of commuting flows. The spectral data are principal G-bundles on an algebraic curve, equipped with an abelian reduction near one point. The flows on the spectral side come from the action of a Heisenberg subgroup of the loop group. The differential data are flat connections known as opers. The flows on the differential side come from a generalized Drinfeld-Sokolov hierarchy. Our isomorphism between the two sides provides a geometric description of the entire phase space of the Drinfeld-Sokolov hierarchy. It extends the Krichever construction of special algebro-geometric solutions of the n-th KdV hierarchy, corresponding to $G = SL(n)$. An interesting feature is the appearance of formal spectral curves, replacing the projective spectral curves of the classical approach. The geometry of these (usually singular) curves reflects the fine structure of loop groups, in particular the detailed classification of their Cartan subgroups. To each such curve corresponds a homogeneous space of the loop group and a soliton system. Moreover the flows of the system have interpretations in terms of Jacobians of formal curves.

[BNe2] D. Ben-Zvi and T. Nevins, $\mathcal{D}$-Bundles and Integrable Hierarchies. e-print math.AG/0603070. Subject of workshop at the U. of Michigan, May 2007.

We study the geometry of D-bundles–locally projective D-modules–on algebraic curves, and apply them to the study of integrable hierarchies, specifically the multicomponent Kadomtsev-Petviashvili (KP) and spin Calogero-Moser (CM) hierarchies. We show that KP hierarchies have a geometric description as flows on moduli spaces of D-bundles; in particular, we prove that the local structure of D-bundles is captured by the full Sato Grassmannian. The rational, trigonometric, and elliptic solutions of KP are therefore captured by D-bundles on cubic curves E, that is, irreducible (smooth, nodal, or cuspidal) curves of arithmetic genus 1. We develop a Fourier-Mukai transform describing D-modules on cubic curves E in terms of (complexes of) sheaves on a twisted cotangent bundle over E. We then apply this transform to classify D-bundles on cubic curves, identifying their moduli spaces with phase spaces of general CM particle systems (realized through the geometry of spectral curves in our twisted cotangent bundle). Moreover, it is immediate from the geometric construction that the flows of the KP and CM hierarchies are thereby identified and that the poles of the KP solutions are identified with the positions of the CM particles. This provides a geometric explanation of a much-explored, puzzling phenomenon of the theory of integrable systems: the poles of meromorphic solutions to KP soliton equations move according to CM particle systems.

[BNe3] D. Ben-Zvi and T. Nevins, Flows of Calogero-Moser Systems. e-print math.AG/0603072. To appear, International Math. Research Notices.

The Calogero-Moser (or CM) particle system and its generalizations appear, in a variety of ways, in integrable systems, nonlinear PDE, representation theory, and string theory. Moreover, the partially completed CM systems–in which dynamics of particles are continued through collisions–have been identified as meromorphic Hitchin systems, giving natural "geometric action-angle variables" for the CM system. Motivated by relations of the CM system to nonlinear PDE, we introduce a new class of generalizations of the spin CM particle systems, the framed (rational, trigonometric and elliptic) CM systems. We give two algebro-geometric descriptions of these systems, via meromorphic Hitchin systems with decorations (framing data) on (cuspidal, nodal and smooth) cubic curves and via one-dimensional sheaves on corresponding "twisted" ruled surfaces. We also present a simple geometric formulation of the flows of all meromorphic GL_n Hitchin systems (with no regularity assumptions) as tweaking flows on spectral sheaves. Using this formulation, we show that all spin and framed CM systems are identified with hierarchies of tweaking flows on the corresponding spectral sheaves. This generalizes the well-known description of spinless CM systems in terms of tangential covers.

[BNe5] D. Ben-Zvi and T. Nevins, Toda Lattice Hierarchy and Noncommutative Geometry. In preparation.

We apply the algebraic geometry of additive, multiplicative and elliptic difference operators using generalizations of the Fourier-Mukai transform. Namely we construct moduli spaces of difference modules on (cuspidal, nodal or smooth) cubic curves and identify them with moduli spaces of spectral sheaves on $C^{\times}$ bundles over the same curves. We then apply this description to a problem of integrable systems explored by Krichever and Zabrodin: the relation between rational, trigonometric and elliptic solutions of the Toda lattice hierarchy (and its nonabelian generalizations) and the Ruijsenaars-Schneiders (RS) particle systems (and their spin variants). We show that the Toda lattice has a natural interpretation as flows on difference modules and that the Fourier-Mukai transform puts the Toda hierarchy in action-angle form, identifying the difference modules with the spectral curves of the RS systems and the poles of the Toda solutions with the positions of the RS particles.

[BNe6] D. Ben-Zvi and T. Nevins, W_{∞}–algebras and D–moduli spaces. In preparation.

We give a geometric construction of the $\mathcal{W}_{1+\infty}$-algebra and of the Backlund transformations for the KP hierarchy in terms of Wilson's adelic Grassmannian. Namely we show the adelic Grassmannian has a natural structure of factorization space, and the $\mathcal{W}_{1+\infty}$-algebra is obtained by linearizing it, while the Backlund transformations appear as Hecke modifications for D-bundles. These spaces give the natural symmetries for the moduli space of D-bundles on a curve, which plays the same role for the $\mathcal{W}_{1+\infty}$-algebra as the Virasoro algebra plays for the moduli space of curves.

2. $\mathcal{D}$-modules and derived categories

[BNe1] D. Ben-Zvi and T. Nevins, Cusps and $\mathcal{D}$-modules. e-print math.AG/0212094. *Journal of the American Math. Society* **17** no. 1 (2004) 155-179.

We study interactions between the categories of $\mathcal{D}$-modules on smooth and singular varieties. For a large class of singular varieties Y, we use an extension of the Grothendieck-Sato formula to show that D_Y-modules are equivalent to stratifications on Y, and as a consequence are unaffected by a class of homeomorphisms, the *cuspidal quotients*. In particular, when Y has a smooth bijective normalization X, we obtain a Morita equivalence of $\mathcal{D}_Y$ and $\mathcal{D}_X$ and a Kashiwara theorem for $\mathcal{D}_Y$, thereby solving conjectures of Hart-Smith and Berest-Etingof-Ginzburg (generalizing results for complex curves and surfaces and rational Cherednik algebras). We also use this equivalence to enlarge the category of induced $\mathcal{D}$-modules on a smooth variety X by collecting induced $\mathcal{D}_X$-modules on varying cuspidal quotients. The resulting *cusp-induced* $\mathcal{D}_X$-modules possess both the good properties of induced $\mathcal{D}$-modules (in particular, a Riemann-Hilbert description) and, when X is a curve, a simple characterization as the generically torsion-free $\mathcal{D}_X$-modules.

[BNe4] D. Ben-Zvi and T. Nevins, Perverse Bundles and Calogero-Moser Spaces. e-print math.AG/0610097.

We present a simple description of moduli spaces of torsion-free $\mathcal{D}$-modules ("$\mathcal{D}$-bundles") on general smooth complex curves X, generalizing the identification of the space of ideals in the Weyl algebra with Calogero-Moser quiver varieties. Namely we show that the moduli of $\mathcal{D}$-bundles form twisted cotangent bundles to stacks of torsion sheaves on X, answering a question of Ginzburg. The corresponding (untwisted) cotangent bundles are identified with moduli of "perverse vector bundles" on T^*X, which contain as open subsets the moduli of framed torsion-free sheaves (the Hilbert schemes $(T^*X)^{[n]}$ in the rank one case). The proof is based on the description of the derived category of $\mathcal{D}$-modules on X by a noncommutative version of the Beilinson transform on the projective line.

[BFN] D. Ben-Zvi, J. Francis and D. Nadler, Centers of Stable Categories and Derived Loop Spaces. In preparation.

We define the center of a monoidal stable category, generalizing the notion of Drinfeld double for tensor categories to the homotopical setting, and relate it to categorified notions of Hochschild homology and cohomology. The main geometric result states that the center of the derived category of quasicoherent sheaves on a derived stack is the category of sheaves on its loop stack (generalizing an underived result of Hinich for

Deligne-Mumford stacks). More generally, we calculate the centers of categories defined by convolution of sheaves. The center of a monoidal stable category has a natural E_2 (braided) structure, which we interpret in the geometric setting as associating a 2d topological field theory (valued in stable categories) to any derived stack.

3. Vertex Algebras

[FB] E. Frenkel and D. Ben-Zvi, Vertex Algebras and Algebraic Curves. *Mathematical Surveys and Monographs* **88**, American Mathematical Society (First edition: 2001, Second edition: 2004). Reviewed by Y.-Z. Huang in Bull. AMS Vol. 39 No. 4 (2002) 585-591. Featured Review MR1849359 (2003f:17036).

Vertex algebras are algebraic objects that encapsulate the concepts of vertex operators and operator product expansion from two-dimensional conformal field theory. This book is an introduction to the theory of vertex algebras with a particular emphasis on the relationship between vertex algebras and the geometry of algebraic curves. The notion of a vertex algebra is introduced in the book in a coordinate-independent way, allowing the authors to give global geometric meaning to vertex operators on arbitrary smooth algebraic curves. From this perspective, vertex algebras appear as the algebraic objects that encode the geometric structure of various moduli spaces associated with algebraic curves. Numerous examples and applications of vertex algebras are included, The authors also explain the connections between vertex algebras and the chiral and factorization algebras introduced by A. Beilinson and V. Drinfeld.

[BF2] D. Ben-Zvi and E. Frenkel, Geometric Realization of the Segal–Sugawara Construction. e-print math.AG/0301206. *London Math Society Lecture Note Series* **308**, 2004.

We apply the technique of localization for vertex algebras to the Segal-Sugawara construction of an “internal” action of the Virasoro algebra on affine Kac-Moody algebras. The result is a lifting of twisted differential operators from the moduli of curves to the moduli of curves with bundles, with arbitrary decorations and complex twistings. This construction gives a uniform approach to a collection of phenomena describing the geometry of the moduli spaces of bundles over varying curves: the KZB equations and heat kernels on non-abelian theta functions, their critical level limit giving the quadratic parts of the Beilinson-Drinfeld quantization of the Hitchin system, and their infinite level limit giving a Hamiltonian description of the isomonodromy equations.

[BHS] D. Ben-Zvi, R. Heluani and M. Szczesny, Supersymmetry of the Chiral de Rham Complex. e-print math.QA/0601532. To appear, *Compositio Math.*

We present a superfield formulation of the chiral de Rham complex (CDR) of Malikov-Schechtman-Vaintrob in the setting of a general smooth manifold, and use it to endow CDR with superconformal structures of geometric origin. Given a Riemannian metric, we construct an N=1 structure on CDR (action of the N=1 super–Virasoro, or Neveu–Schwarz, algebra). If the metric is Kähler, and the manifold Ricci-flat, this is augmented to an N=2 structure. Finally, if the manifold is hyperkähler, we obtain an N=4 structure. The superconformal structures are constructed directly from the

Levi-Civita connection. These structures provide an analog for CDR of the extended supersymmetries of nonlinear sigma-models.

4. Geometric Langlands Program

[BNa1] D. Ben-Zvi and D. Nadler, Loop Spaces and Langlands Parameters. e-print arXiv:0706.0322.

We apply the technique of S^1-equivariant localization to sheaves on loop spaces in derived algebraic geometry, and obtain a fundametnal link between two families of categories at the heart of geometric representation theory. Namely, we categorify the well known relationship between free loop spaces, cyclic homology and de Rham cohomology to recover the category of $\mathcal{D}$-modules on a smooth stack X as a localization of the category of S^1-equivariant coherent sheaves on its loop spaces LX. The main observation is that this procedure connects categories of equivariant $\mathcal{D}$-modules on flag varieties with categories of equivariant coheernt sheaves on the Steinberg variety and its relatives. This provides a direct connetion between the geometry of finite and affine Hecke algebras and braid groups, and a uniform geometric construction of all of the categorical parameters for representations of real and complex reductive groups. This paper forms the first step in a project to apply the geometric Langlands program to the complex and real local Langlands programs, which we describe.

[BNa2] D. Ben-Zvi and D. Nadler, Langlands Duality for Character Sheaves. In preparation.

We establish a Langlands duality for the representation theory of complex reductive groups. By applying S^1 equivariant localization techniques developed in our paper [BNa1] to the tamely ramified local geometric Langlands theorem of Bezrukavnikov, we obtain a canonical monoidal equivalence between the category of Harish-Chandra modules for a complex group G and the finite Hecke category for the Langlands dual group $G^\vee$ (generalizing results of Beilinson, Ginzburg and Soergel). We also prove a result describing the derived centers (Drinfeld doubles or Hochschild categories) of Hecke categories in terms of Lusztig's theory of character sheaves. Combining these results we obtain an equivalence (as braided or ribbon categories) between the categories of character sheaves associated to the groups G and $G^\vee$. This result is interpreted as a Langlands duality for three dimensional topological gauge theories, obtained by dimensional reduction from the four dimensional gauge theories behind geometric Langlands à la Kapustin-Witten.

[BNa3] D. Ben-Zvi and D. Nadler, Affine Hecke Algebras and Vogan Duality. In preparation.

We apply ideas from the geometric Langlands program to study the representation theory of real Lie groups. Our main result may be interpreted as an affine version of Vogan's character duality for representations of real Lie groups, or as a real version of Kazhdan and Lusztig's construction of the affine Hecke algebra. More precisely, we give a geometric description of the K-group of sheaves on a real form of the moduli space of bundles on $\mathbb{P}^1$ (or of Harish-Chandra modules for a real loop group), as a module for the affine Hecke algebra. This result, combined with equivariant localization, implies Vogan's duality and provides it with a conceptual proof, linking the real local Langlands program (as studied by Adams-Barbasch-Vogan) and the geometric Langlands program. This provides strong evidence for a program to prove Soergel's categorical real local Langlands conjecture, of which this is the K-theoretic shadow.

[BNa4] D. Ben-Zvi and D. Nadler, Geometric Base Change. In preparation.

We propose a geometric version of Langlands' base change principle. This describes the behavior of categories of sheaves on moduli of bundles under the passage from a curve to its covering spaces. As evidence for the conjecture we prove its spectral version (describing coherent sheaves on moduli of local systems on a curve and its coverings) and the special case of abelian groups. We then show how a twisted (or "orientifold") version of geometric base change for the covering $\mathbb{P}^1 \to \mathbb{RP}^2$ implies Soergel's conjecture, a categorical form of the real local Langlands program. We also discuss other applications and relations with topological field theory.

[BZ] D. Ben-Zvi, Lectures on the Geometric Langlands Program. In Preparation. Under contract with Cambridge University Press, *London Math Society Lecture Note Series.*

This book provides an informal introduction to the geometric Langlands program, based on the 2007 London Math Society Invited Lecture Series and a course at the University of Texas. Chapter I presents a general introduction to the geometry of derived categories of sheaves and Fourier-Mukai transforms, while Chapter II provides a survey of the moduli spaces of bundles on curves, Hitchin's integrable system and its quantization. Chapter III describes the role of Hecke operators and the origin of the Langlands dual group. In Chapter IV, the geometric Langlands correspondence is recast in the language of topological field theory, as an aspect of four-dimensional electric-magnetic duality, following the exciting recent work of Kapustin and Witten. Finally in Chapter V the role of the geometric Langlands program in representation theory is surveyed, with emphasis on the work of Bezrukavnikov and Ben-Zvi–Nadler.

5. Riemann Surfaces and Theta Functions

[BB1] D. Ben-Zvi and I. Biswas, A Quantization on Riemann Surfaces with Projective Structure. *Letters in Mathematical Physics* **54** (2000) 73–82.

The bundles of half-forms, or theta characteristics, on Riemann surfaces carry canonical symplectic forms on the complement of the zero section, obtained by pullback from the cotangent bundle. In this paper we construct canonical deformation quantizations for these symplectic structures starting from a projective structure on the Riemann surface. The construction uses the relation between projective structure and connections on jets of half forms, and can be viewed as a drastically simplified form of the Fedosov quantization in this setting.

[BB2] D. Ben-Zvi and I. Biswas, Theta Functions and Szegö Kernels. *International Math. Research Notices* **24** (2003) 1305–1340.

We study relations between two fundamental constructions associated to vector bundles on a smooth complex projective curve: the theta function (a section of a line bundle

on the moduli space of vector bundles) and the Szegö kernel (a section of a vector bundle on the square of the curve). Two types of relations are demonstrated. First, we establish a higher–rank version of the prime form, describing the pullback of determinant line bundles by difference maps, and show the theta function pulls back to the determinant of the Szegö kernel. Next, we prove that the expansion of the Szegö kernel at the diagonal gives the logarithmic derivative of the theta function over the moduli space of bundles for a fixed, or moving, curve. In particular, we recover the identification of the space of connections on the theta line bundle with moduli space of flat vector bundles, when the curve is fixed. When the curve varies, we identify this space of connections with the moduli space of *extended connections*, which we introduce.

[BB3] D. Ben-Zvi and I. Biswas, Opers and Theta Functions. *Advances in Mathematics* **181** (2004) no.2, 368–395.

We construct natural maps (the Klein and Wirtinger maps) from moduli spaces of vector bundles on an algebraic curve X to affine spaces, as quotients of the nonabelian theta linear series. We prove a finiteness result for these maps over generalized Kummer varieties (moduli of torus bundles), leading us to conjecture that the maps are finite in general. The conjecture provides canonical explicit coordinates on the moduli space. The finiteness results give low–dimensional parametrizations of Jacobians (in $\mathbb{P}^{3g-3}$ for generic curves), described by 2Θ functions or second logarithmic derivatives of theta. We interpret the Klein and Wirtinger maps in terms of opers on X. Opers are generalizations of projective structures, and can be considered as differential operators, kernel functions or special bundles with connection. The matrix opers (analogues of opers for matrix differential operators) combine the structures of flat vector bundle and projective connection, and map to opers via generalized Hitchin maps. For vector bundles off the theta divisor, the Szegö kernel gives a natural construction of matrix oper. The Wirtinger map from bundles off the theta divisor to the affine space of opers is then defined as the determinant of the Szegö kernel. This generalizes the Wirtinger projective connections associated to theta characteristics, and the assoicated Klein bidifferentials.

6. Other.

[BNe] From Solitons to Many–Body Systems (with T. Nevins). e-print math.AG/0310490. To appear, *Pure and Applied Math Quarterly.*

Survey of the approach to integrable systems developed in the papers [BNe2] and [BNe3].

[BZ2] Moduli Spaces. To appear, *Princeton Companion to Mathematics*, ed. T. Gowers, 2007.

Expository article on the notion of moduli spaces and their construction.

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