WAKIMOTO MODULES, OPERS AND THE CENTER AT THE CRITICAL LEVEL

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Contents

Introduction

Wakimoto modules are representations of affine Kac-Moody algebras in Fock modules over infinite-dimensional Heisenberg algebras. They were introduced in 1986 by M. Wakimoto [W] in the case of $\widehat{\mathfrak{sl}}_2$ and in 1988 by B. Feigin and the author [FF1] in the general case. Wakimoto modules have useful applications in representation theory and conformal field theory. In particular, they have been used to construct chiral correlation functions of the WZW models [ATY, FFR] (reproducing the Schechtman–Varchenko solutions of the KZ equations), in the study of the Drinfeld-Sokolov reduction and $\mathcal{W}$–algebras, and in the description of the center of the completed enveloping algebra of an affine Lie algebra [FF6, F1]. The Wakimoto modules also provide a bridge between representation theory of affine algebras and the geometry of the semi-infinite flag manifold [FF4].

In this paper we present the construction of the Wakimoto modules from the point of view of the vertex algebra theory (see [K2, FB] for an introduction to vertex algebras).

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In the original construction of [FF1]–[FF4] the connection with vertex algebras was not explicitly discussed. Here we make this connection explicit.

Let $\mathfrak{g}$ be a simple Lie algebra and $\widehat{\mathfrak{g}}$ the corresponding (untwisted) affine Kac-Moody algebra. One can associate to $\widehat{\mathfrak{g}}$ and an invariant inner product κ on $\mathfrak{g}$ (equivalently, a level) a vertex algebra, denoted by $V_\kappa(\mathfrak{g})$. The construction of Wakimoto modules amounts to constructing a homomorphism from $V_\kappa(\mathfrak{g})$ to a vertex algebra $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ associated to an infinite-dimensional Heisenberg Lie algebra. In the case of $\mathfrak{g} = \mathfrak{sl}_2$ the construction is spelled out in detail in [FB], Ch. 11–12, where the reader is referred for additional motivation and background. Here we explain it in the case of an arbitrary $\mathfrak{g}$. We follow the original approach of [FF4] and prove the existence of the homomorphism $V_\kappa(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ by cohomological methods. We describe the cohomology class responsible for the obstruction and show that it is equal to the class defining the affine Kac-Moody algebra. We note that explicit formulas for the Wakimoto realization have been given in [FF1] for $\mathfrak{g} = \mathfrak{sl}_n$ and in [dBF] for general $\mathfrak{g}$. Another proof of the existence of this realization has been presented in [FF8]. The construction has been extended to twisted affine algebras in [S].

The Wakimoto modules may be viewed as representations of $\widehat{\mathfrak{g}}$ which are "semi-infinitely induced" from representations of its Heisenberg subalgebra $\widehat{\mathfrak{h}}$. In contrast to the usual induction, however, the level of the $\widehat{\mathfrak{h}}$–module gets shifted by the so-called critical value κ_c. In particular, if we start with an $\widehat{\mathfrak{h}}$–module of level zero (e.g., a one-dimensional module corresponding to a character of the abelian Lie algebra $L\mathfrak{h}$), then the resulting $\widehat{\mathfrak{g}}$–module will be at the critical level. Using these modules, we prove the Kac-Kazhdan conjecture on the characters of irreducible $\widehat{\mathfrak{g}}$–modules of critical level (this proof appeared first in [F1]).

We then extend the construction of the Wakimoto modules to a more general context in which the Lie subalgebra $\widehat{\mathfrak{h}}$ is replaced by a central extension of the loop algebra of the Levi subalgebra of an arbitrary parabolic Lie subalgebra of $\mathfrak{g}$ following the ideas of [FF4]. Thus, we establish a "semi-infinite parabolic induction" pattern for representations of affine Kac-Moody algebras, similar to the parabolic induction for reductive groups over local non-archimedean fields.

Next, we give a uniform construction of intertwining operators between Wakimoto modules, the so-called screening operators (of two kinds) following [FF2, FF3, F1, FFR].

Using the screening operators, we describe the center of the vertex algebra $V_{\kappa_c}(\mathfrak{g})$ at the critical level. We show that it is canonically isomorphic to the classical $\mathcal{W}$–algebra associated to the Langlands dual Lie algebra ${}^L\mathfrak{g}$. The proof presented here is different from the original proof from [FF6, F1] in two respects. First of all, we use here the screening operators of the second kind rather than the first kind; their $\kappa \to \kappa_c$ limits are easier to study. Second, we use the isomorphism between the Verma module of critical level with highest weight 0 and a certain Wakimoto module and the computation of the associated graded of the spaces of singular vectors to estimate the character of the center.

Finally, we show that the above classical $\mathcal{W}$–algebra is isomorphic to the algebra of functions on the space of ${}^LG$–opers on the formal disc. The notion of ${}^LG$–oper (on an arbitrary smooth curve) was introduced by A. Beilinson and V. Drinfeld [BD1],[BD2] following the work of V. Drinfeld and V. Sokolov [DS] on the generalized KdV hierarchies. The algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D}^\times)$ of functions on the space of ^LG–opers on the punctured disc $\mathbb{D}^\times$ carries a canonical Poisson structure defined in [DS] by means of hamiltonian reduction. At the same time, the algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})$ of functions on the space of ^LG–opers on the (unpunctured) disc carries a vertex Poisson structure. We show that the above isomorphism between the center of $V_{\kappa_c}(\mathfrak{g})$ (with its canonical vertex Poisson structure coming from the deformation of the level) and $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})$ preserves vertex Poisson structures. Using this isomorphism, we show that the center of the completed enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level is isomorphic to $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D}^\times)$ as a Poisson algebra. This isomorphism was conjectured by V. Drinfeld.

The paper is organized as follows. We begin in Sect. 1 with the geometric construction of representations of a simple finite-dimensional Lie algebra $\mathfrak{g}$ using an embedding of $\mathfrak{g}$ into a Weyl algebra which is obtained from the infinitesimal action of $\mathfrak{g}$ on the flag manifold. This will serve as a prototype for the construction of Wakimoto modules presented in the subsequent sections. In Sect. 2 we introduce the main ingredients needed for the constructions of Wakimoto modules: the infinite-dimensional Weyl algebra $\mathcal{A}^{\mathfrak{g}}$, the corresponding vertex algebra $M_{\mathfrak{g}}$ and the infinitesimal action of the loop algebra $L\mathfrak{g}$ on the space LU of formal loops to the big cell of the flag manifold of $\mathfrak{g}$. We also introduce the local Lie algebra $\mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathfrak{g}}$ of differential operators on this space of order less than or equal to one. We show that it is a non-trivial extension of the Lie algebra of local vector fields on LU by local functionals on LU and compute the corresponding two-cocycle (most of this material has already been presented in [FB], Ch. 12). In Sect. 3 we prove that the embedding of the loop algebra $L\mathfrak{g}$ into the Lie algebra of local vector fields on LU may be lifted to an embedding of the central extension $\widehat{\mathfrak{g}}$ to $\mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathfrak{g}}$. In order to do that we need to show that the restriction of the above two-cocycle to $L\mathfrak{g}$ is cohomologically equivalent to the two-cocycle corresponding to its Kac-Moody central extension (of level κ_c). This is achieved by replacing the standard cohomological Chevalley complex by a much smaller local subcomplex (where both cocycles belong) and computing the cohomology of the latter.

Having proved the existence of a homomorphism $\widehat{\mathfrak{g}}_{\kappa_c} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathfrak{g}}$, we derive in Sect. 4 the existence of a vertex algebra homomorphism $V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0$, where π_0 is the commutative vertex algebra associated to $L\mathfrak{h}$. Using this homomorphism, we construct a $\widehat{\mathfrak{g}}$–module structure of critical level on the tensor product $M_{\mathfrak{g}} \otimes N$, where N is an arbitrary (smooth) $L\mathfrak{h}$–module. These are the Wakimoto modules of critical level. We show that for a generic one-dimensional module N the corresponding Wakimoto module is irreducible. This enables us to prove the Kac–Kazhdan conjecture on the characters of irreducible representations of critical level.

In Sect. 5 we deform the above construction of Wakimoto modules to other levels. Thus, we construct a homomorphism of vertex algebras $w_\kappa : V_\kappa(\mathfrak{g}) \to M_g \otimes \pi_0^{\kappa-\kappa_c}$, where $\pi_0^{\kappa-\kappa_c}$ is the vertex algebra associated to the Heinseberg Lie algebra $\widehat{\mathfrak{h}}_{\kappa-\kappa_c}$. We describe the (quasi)conformal structure on $M_g \otimes \pi_0^{\kappa-\kappa_c}$ corresponding to the Segal-Sugawara (quasi)conformal structure on $V_\kappa(\mathfrak{g})$. We then use these structures to describe the Wakimoto modules in a coordinate-independent fashion.

In Sect. 6 we generalize the construction to the case of an arbitrary parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. We define the functors of "semi-infinite parabolic induction" from the category of smooth representations of a central extension of $L\mathfrak{m}$, where $\mathfrak{m}$ is the Levi subalgebra of $\mathfrak{p}$, to the category of $\widehat{\mathfrak{g}}$–modules.

Sect. 7 is devoted to a detailed study of the Wakimoto modules over $\widehat{\mathfrak{sl}}_2$. We write down explicit formulas for the homomorphism $V_\kappa(\mathfrak{sl}_2) \to M_{\mathfrak{sl}_2} \otimes \pi_0^{\kappa-\kappa_c}$ and use these formulas to construct intertwining operators between the Wakimoto modules. They are called screening operators, of the first and second kind. We use these operators and the functor of parabolic induction to construct intertwining operators between Wakimoto modules over an arbitrary affine Kac-Moody algebra in Sect. 8.

In Sect. 9 we identify the center of the vertex algebra $V_{\kappa_c}(\mathfrak{g})$ with the intersection of the kernels of certain operators, which we identify in turn with the classical $\mathcal{W}$–algebra associated to the Langlands dual Lie algebra ${}^L\mathfrak{g}$. We also show that the corresponding vertex Poisson algebra structures coincide. In Sect. 10 we define opers and Miura opers. We then show in Sect. 11 that this classical $\mathcal{W}$–algebra (resp., the commutative vertex algebra π_0) is nothing but the algebra of functions on the space of ${}^LG$–opers on the disc (resp., Miura ${}^LG$–opers on the disc). Furthermore, the embedding of the classical $\mathcal{W}$–algebra into π_0 coincides with the Miura map between the two algebras of functions. Finally, in Sect. 12 we consider the center of the completed universal enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level. We identify it with the algebra of functions on the space of ${}^LG$–opers on the punctured disc. We then prove that this identification satisfies various compatibilities. In particular, we show that an affine analogue of the Harish-Chandra homomorphism obtained by evaluating central elements on the Wakimoto modules is nothing but the Miura transformation from Miura ${}^LG$–opers to ${}^LG$–opers on the punctured disc.

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1. Finite-dimensional case

In this section we recall the realization of $\mathfrak{g}$–modules in the space of functions on the big cell of the flag manifold. This will serve as a blueprint for the construction of the Wakimoto modules over affine algebras in the following sections.

1.1. Flag manifold.

Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ. As a vector space, it has a Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-, \tag{1.1}$$

where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n}_\pm$ are the upper and lower nilpotent subalgebras. Let

$$\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$$

be the upper and lower Borel subalgebras. Let $h_i = \check{\alpha}_i$ be the ith coroot of $\mathfrak{g}$. The set $\{h_i\}_{i=1\ldots,\ell}$ is a basis of $\mathfrak{h}$. We choose generators $e_i, i = 1, \ldots, \ell$, and $f_i, i = 1, \ldots, \ell$, of $\mathfrak{n}_+$ and $\mathfrak{n}_-$, respectively, corresponding to the simple roots. We also choose a root basis of $\mathfrak{n}_+$, $\{e^\alpha\}_{\alpha\in\Delta_+}$, where Δ_+ is the set of positive roots of $\mathfrak{g}$, so that $[h, e^\alpha] = \alpha(h)e^\alpha$ for all $h \in \mathfrak{h}$. In particular, $e_i = e_{\alpha_i}, i = 1, \ldots, \ell$.

Let G be the connected simply-connected Lie group corresponding to $\mathfrak{g}$, and $N_\pm$ (respectively, $B_\pm$) the upper and lower unipotent subgroups (respectively, Borel subgroups) of G corresponding to $\mathfrak{n}_\pm$ (respectively, $\mathfrak{b}_\pm$).

Consider the flag manifold G/B_-. It has a unique open N_+–orbit, the so-called big cell $U = N_+ \cdot [1] \subset G/B_-$, which is isomorphic to N_+. Since N_+ is a unipotent Lie group, the exponential map $\mathfrak{n}_+ \to N_+$ is an isomorphism. Therefore N_+ is isomorphic to the vector space $\mathfrak{n}_+$. Thus, N_+ is isomorphic to the affine space $\mathbb{A}^{|\Delta_+|}$, and the space $\mathbb{C}[N_+]$ of regular functions on N_+ is isomorphic to a free polynomial algebra. We will call a system of coordinates $\{y_\alpha\}_{\alpha\in\Delta_+}$ on N_+ *homogeneous* if $h \cdot y_\alpha = -\alpha(h)y_\alpha$ for all $h \in \mathfrak{h}$. In what follows we will consider only homogeneous coordinate systems on N_+.

Remark 1.1. Note that in order to define U it is sufficient to choose only a Borel subgroup B_+ of G. Then $N_+ = [B_+, B_+]$ and U is the open N_+–orbit in the flag manifold defined as the variety of all Borel subgroups of G (so U is an N_+–torsor). All constructions in this paper make sense with the choice of B_+ only, i.e., without making the choice of H and B_-. However, to simplify the exposition we will fix a Cartan subgroup $H \subset B_+$ as well. □

The action of G on G/B_- gives us a map from $\mathfrak{g}$ to the Lie algebra of vector fields on G/B_-, and hence on its open subset $U \simeq N_+$. Thus, we obtain a Lie algebra homomorphism $\mathfrak{g} \to \operatorname{Vect} N_+$.

This homomorphism may be described explicitly as follows. Let G° denote the dense open submanifold of G consisting of elements of the form $g_+g_-, g_+ \in N_+, g_- \in B_-$ (note that such an expression is necessarily unique since $B_- \cap N_+ = 1$). In other words, $G^\circ = p^{-1}(U)$, where p is the projection $G \to G/B_-$. Given $a \in \mathfrak{g}$, consider the one-parameter subgroup $\gamma(\epsilon) = \exp(\epsilon a)$ in G. Since G° is open and dense in G, $\gamma(\epsilon)x \in G^\circ$ for ϵ in the formal neighborhood of 0, so we can write

$$\gamma(\epsilon) = Z_+(\epsilon)Z_-(\epsilon), \qquad Z_+(\epsilon) \in N_+, \quad Z_-(\epsilon) \in B_-.$$

The factor $Z_+(\epsilon)$ just expresses the projection of the subgroup $\gamma(\epsilon)$ onto $N_+ \simeq U \subset G/B_-$ under the map p. Then the vector field ξ_a (equivalently, a derivation of $\mathbb{C}[N_+]$) corresponding to a is given by the formula

$$(\xi_a f)(x) = \left.\left(\frac{d}{d\epsilon} f(Z_+(\epsilon))\right)\right|_{\epsilon=0}. \tag{1.2}$$

To write a formula for ξ_a in more concrete terms, we choose a faithful representation V of $\mathfrak{g}$ (say, the adjoint representation). Since we only need the ϵ–linear term in our calculation, we can and will assume that $\epsilon^2 = 0$. Considering $x \in N_+$ as a matrix whose entries are polynomials in the coordinates $y_\alpha, \alpha \in \Delta_+$, which expresses a generic element of N_+ in $\operatorname{End} V$, we have

$$(1 + \epsilon a)x = Z_+(\epsilon) Z_-(\epsilon). \tag{1.3}$$

We find from this formula that $Z_+(\epsilon) = x + \epsilon Z_+^{(1)}$, where $Z_+^{(1)} \in \mathfrak{n}_+$, and $Z_-(\epsilon) = 1 + \epsilon Z_-^{(1)}$, where $Z_-^{(1)} \in \mathfrak{b}_-$. Therefore we obtain from formulas (1.2) and (1.3) that

$$\xi_a \cdot x = x(x^{-1} a x)_+, \tag{1.4}$$

where z_+ denotes the projection of an element $z \in \mathfrak{g}$ onto $\mathfrak{n}_+$ along $\mathfrak{b}_-$.

1.2. **The algebra of differential operators.** The algebra $\mathcal{D}(U)$ of differential operators on U is isomorphic to the Weyl algebra with generators $\{y_\alpha, \partial/\partial y_\alpha\}_{\alpha \in \Delta_+}$, and the standard relations

$$\left[\frac{\partial}{\partial y_\alpha}, y_\beta \right] = \delta_{\alpha,\beta}, \qquad \left[\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\beta} \right] = [y_\alpha, y_\beta] = 0.$$

The algebra $\mathcal{D}(U)$ has a natural filtration $\{\mathcal{D}_{\leq i}(U)\}$ by the order of the differential operator. In particular, we have an exact sequence

$$0 \to \operatorname{Fun} U \to \mathcal{D}_{\leq 1}(U) \to \operatorname{Vect} U \to 0, \tag{1.5}$$

where $\operatorname{Fun} U \simeq \operatorname{Fun} N_+$ denotes the ring of regular functions on U, and $\operatorname{Vect} U$ denotes the Lie algebra of vector fields on U. This sequence has a canonical splitting: namely, we lift $\xi \in \operatorname{Vect} U$ to the unique first order differential operator D_ξ whose symbol equals ξ and which kills the constant functions, i.e., such that $D_\xi \cdot 1 = 0$. Using this splitting, we obtain an embedding $\mathfrak{g} \to \mathcal{D}_{\leq 1}(N_+)$, and hence the structure of a $\mathfrak{g}$–module on the space of functions $\operatorname{Fun} N_+ = \mathbb{C}[y_\alpha]_{\alpha \in \Delta_+}$.

1.3. **Verma modules and contragredient Verma modules.** By construction, the action of $\mathfrak{n}_+$ on $\operatorname{Fun} N_+$ satisfies $e^\alpha \cdot y_\alpha = 1$ and $e^\alpha \cdot y_\beta = 0$ unless α is less than or equal to β with repsect to the usual partial ordering on the set of positive roots. Therefore, for any $A \in \operatorname{Fun} N_+$ there exists $P \in U(\mathfrak{n}_+)$ such that $P \cdot A = 1$.

Consider the pairing $U(\mathfrak{n}_+) \times \operatorname{Fun} N_+ \to \mathbb{C}$ which maps (P, A) to the value of the function $P \cdot A$ at the identity element of N_+. This pairing is $\mathfrak{n}_+$–invariant. The Poincaré-Birkhoff-Witt basis $\{e^{\alpha(1)} \dots e^{\alpha(k)}\}$ (with respect to some ordering on Δ_+) of $U(\mathfrak{n}_+)$ and the monomial basis $\{y_{\alpha(1)} \dots y_{\alpha(k)}\}$ of $\operatorname{Fun} N_+$ are dual to each other with respect to this pairing. Moreover, both $U(\mathfrak{n}_+)$ and $\operatorname{Fun} N_+$ are graded by the root lattice of $\mathfrak{g}$ (under the action of the Cartan subalgebra $\mathfrak{h}$), and this pairing identifies each graded component of $\operatorname{Fun} N_+$ with the dual of the corresponding graded component of $U(\mathfrak{n}_+)$. Therefore the $\mathfrak{n}_+$–module $\operatorname{Fun} N_+$ is isomorphic to the restricted dual $U(\mathfrak{n}_+)^\vee$ of $U(\mathfrak{n}_+)$, i.e., the direct sum of the dual spaces to the homogeneous components $U(\mathfrak{n}_+)_\gamma$ of $U(\mathfrak{n}_+)$, where γ runs over the root lattice.

Recall the definition of the Verma modules and the contragredient Verma modules. For each $\chi \in \mathfrak{h}^*$, consider the one-dimensional representation $\mathbb{C}_\chi$ of $\mathfrak{b}_\pm$ on which $\mathfrak{h}$ acts according to χ, and $\mathfrak{n}_\pm$ acts by 0. The *Verma module* M_χ with highest weight $\chi \in \mathfrak{h}^*$ is the induced module

$$M_\chi \overset{\text{def}}{=} U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\chi.$$

The Cartan decomposition (1.1) gives us the isomorphisms of vector spaces $U(\mathfrak{g}) \simeq U(\mathfrak{n}_-) \otimes U(\mathfrak{b}_+)$. Therefore, as an $\mathfrak{n}_-$–module, $M_\chi \simeq U(\mathfrak{n}_-)$.

The *contragredient Verma module* M_χ^* with highest weight $\chi \in \mathfrak{h}^*$ is defined as the (restricted) coinduced module

$$M_\chi^* = \operatorname{Hom}^{\text{res}}_{U(\mathfrak{b}_-)}(U(\mathfrak{g}), \mathbb{C}_\chi).$$

Here $U(\mathfrak{g})$ is considered as a $U(\mathfrak{b}_-)$–module with respect to the right action, and the index "res" means that we consider only those linear maps $U(\mathfrak{g}) \to \mathbb{C}_\chi$ which are finite linear combinations of maps supported on the direct summands $U(\mathfrak{n}_+)_\gamma \otimes U(\mathfrak{b}_-)$ of $U(\mathfrak{g})$ with respect to the isomorphism of vector spaces $U(\mathfrak{g}) \simeq U(\mathfrak{n}_+) \otimes U(\mathfrak{b}_-)$. Then, as an $\mathfrak{n}_+$–module, $M_\chi^* \simeq U(\mathfrak{n}_+)^\vee$.

1.4. **Identification of** $\operatorname{Fun} N_+$ **with** M_χ^***.** The module M_0^* is isomorphic to $\operatorname{Fun} N_+$ with its $\mathfrak{g}$–module structure defined above. Indeed, the vector $1 \in \operatorname{Fun} N_+$ is annihilated by $\mathfrak{n}_+$ and has weight 0 with respect to h. Hence there is a non-zero homomorphism $\operatorname{Fun} N_+ \to M_0^*$ sending $1 \in \operatorname{Fun} N_+$ to a non-zero vector $v_0^* \in M_\chi^*$ of weight 0. Since both $\operatorname{Fun} N_+$ and M_χ^* are isomorphic to $U(\mathfrak{n}_+)^\vee$ as $\mathfrak{n}_+$–modules, this homomorphism is an isomorphism.

Now we identify the module M_χ^* with an arbitrary weight χ with $\operatorname{Fun} N_+$, where the latter is equipped with a modified action of $\mathfrak{g}$.

Recall that we have a canonical lifting of $\mathfrak{g}$ to $\mathcal{D}_{\leq 1}(N_+)$, $a \mapsto \xi_a$. But this lifting is not unique. We can modify it by adding to each ξ_a a function $\phi_a \in \operatorname{Fun} N_+$ so that $\phi_{a+b} = \phi_a + \phi_b$. One readily checks that the modified differential operators $\xi_a + \phi_a$ satisfy the commutation relations of $\mathfrak{g}$ if and only if the linear map $\mathfrak{g} \to \operatorname{Fun} N_+$ given by $a \mapsto \phi_a$ is a one-cocycle of $\mathfrak{g}$ with coefficients in $\operatorname{Fun} N_+$.

Each such lifting gives $\operatorname{Fun} N_+$ the structure of a $\mathfrak{g}$–module. Let us impose the extra condition that the modified action of $\mathfrak{h}$ on V remains diagonalizable. This means that ϕ_h is a constant function on N_+ for each $h \in \mathfrak{h}$, and therefore our one-cocycle should be $\mathfrak{h}$–invariant: $\phi_{[h,a]} = \xi_h \cdot \phi_a$, for all $h \in \mathfrak{h}, a \in \mathfrak{g}$. It is easy to see that the space of $\mathfrak{h}$–invariant one-cocycles of $\mathfrak{g}$ with coefficients in $\operatorname{Fun} N_+$ is canonically isomorphic to the first cohomology of $\mathfrak{g}$ with coefficients in $\operatorname{Fun} N_+$, i.e., $H^1(\mathfrak{g}, \operatorname{Fun} N_+)$ (see [FB], § 11.2.6). By Shapiro's lemma (see [Fu], § 5.4)

$$H^1(\mathfrak{g}, \operatorname{Fun} N_+) = H^1(\mathfrak{g}, M_0^*) \simeq H^1(\mathfrak{b}_-, \mathbb{C}_0) = (\mathfrak{b}_-/[\mathfrak{b}_-, \mathfrak{b}_-])^* \simeq \mathfrak{h}^*.$$

Thus, for each $\chi \in \mathfrak{h}^*$ we obtain an embedding $\rho_\chi : \mathfrak{g} \hookrightarrow \mathcal{D}_{\leq 1}(N_+)$ and hence the structure of an $\mathfrak{h}^*$–graded $\mathfrak{g}$–module on $\operatorname{Fun} N_+$. Let us analyze this $\mathfrak{g}$–module in more detail.

We have $\xi_h \cdot y_\alpha = -\alpha(h) y_\alpha, \alpha \in \Delta_+$, so the weight of any monomial in $\operatorname{Fun} N_+$ is equal to a sum of negative roots. Since our one-cocycle is $\mathfrak{h}$–invariant, we obtain $\xi_h \cdot \phi_{e^\alpha} = \phi_{[h,e^\alpha]} = \alpha(h) \phi_{e^\alpha}$, $\alpha \in \Delta_+$, so the weight of ϕ_{e^α} has to be equal to the positive root α. Therefore $\phi_{e^\alpha} = 0$ for all $\alpha \in \Delta_+$. Thus, the action of $\mathfrak{n}_+$ on $\operatorname{Fun} N_+$ is not modified. On the other hand, by construction, the action of $h \in \mathfrak{h}$ is modified by

$h \mapsto h+\chi(h)$. Therefore the vector $1 \in \operatorname{Fun} N_+$ is still annihilated by $\mathfrak{n}_+$, but now it has weight χ with respect to h. Hence there is a non-zero homomorphism $\operatorname{Fun} N_+ \to M_\chi^*$ sending $1 \in \operatorname{Fun} N_+$ to a non-zero vector $v_\chi^* \in M_\chi^*$ of weight χ. Since both $\operatorname{Fun} N_+$ and M_χ^* are isomorphic to $U(\mathfrak{n}_+)^\vee$ as $\mathfrak{n}_+$–modules, this homomorphism is an isomorphism. Thus, under the modified action obtained via the lifting ρ_χ, the $\mathfrak{g}$–module $\operatorname{Fun} N_+$ is isomorphic to the contragredient Verma module M_χ^*.

1.5. **Explicit formulas.** Choose a basis $\{J^a\}_{a=1,\dots,\dim\mathfrak{g}}$ of $\mathfrak{g}$. Under the homomorphism ρ_χ, we have

$$J^a \mapsto P_a\left(y_\alpha, \frac{\partial}{\partial y_\alpha}\right) + f_a(y_\alpha), \tag{1.6}$$

where P_a is a polynomial in the y_α's and $\partial/\partial y_\alpha$'s of degree one in the $\partial/\partial y_\alpha$'s, which is independent of χ, and f_a is a polynomial in the y_α's only which depends on χ.

Let $e_i, h_i, f_i, i = 1\dots,\ell$, be the generators of $\mathfrak{g}$. Using formula (1.4) we find the following explicit formulas:

$$\rho_\chi(e_i) = \frac{\partial}{\partial y_{\alpha_i}} + \sum_{\beta\in\Delta_+} P^i_\beta(y_\alpha)\frac{\partial}{\partial y_\beta}, \tag{1.7}$$

$$\rho_\chi(h_i) = -\sum_{\beta\in\Delta_+} \beta(h_i) y_\beta \frac{\partial}{\partial y_\beta} + \chi(h_i), \tag{1.8}$$

$$\rho_\chi(f_i) = \sum_{\beta\in\Delta_+} Q^i_\beta(y_\alpha)\frac{\partial}{\partial y_\beta} + \chi(h_i) y_{\alpha_i}, \tag{1.9}$$

for some polynomials P^i_β, Q^i_β in $y_\alpha, \alpha \in \Delta_+$.

In addition, we have a Lie algebra anti-homomorphism $\rho^R : \mathfrak{n}_+ \to \mathcal{D}_{\leq 1}(N_+)$ which corresponds to the *right* action of $\mathfrak{n}_+$ on N_+. The differential operators $\rho^R(x), x \in \mathfrak{n}_+$, commute with the differential operators $\rho_\chi(x'), x' \in \mathfrak{n}_+$ (though their commutation relations with $\rho_\chi(x'), x' \notin \mathfrak{n}_+$, are complicated in general). We have

$$\rho^R(e_i) = \frac{\partial}{\partial y_{\alpha_i}} + \sum_{\beta\in\Delta_+} P^{R,i}_\beta(y_\alpha)\frac{\partial}{\partial y_\beta}$$

for some polynomials $P^{R,i}_\beta, Q^i_\beta$ in $y_\alpha, \alpha \in \Delta_+$.

2. The case of affine algebras

2.1. **The infinite-dimensional Weyl algebra.** Our goal is to generalize the above construction to the case of affine Kac-Moody algebras. Let again U be the open N_+–orbit of the flag manifold of G, which we identify with the group N_+ and hence with the Lie algebra $\mathfrak{n}_+$. Consider the formal loop space $LU = U((t))$ as a complete topological vector space with the basis of open neighborhoods of zero formed by the subspaces $U \otimes t^N\mathbb{C}[[t]] \subset LU, N \in \mathbb{Z}$. Thus, LU is an affine ind-scheme

$$LU = \varinjlim U \otimes t^N \mathbb{C}[[t]], \qquad N < 0.$$

Using the coordinates $y_\alpha, \alpha \in \Delta_+$, on U, we can write

$$U \otimes t^N \mathbb{C}[[t]] \simeq \left\{ \sum_{n \geq N} y_{\alpha,n} t^n \right\}_{\alpha \in \Delta_+} = \operatorname{Spec} \mathbb{C}[y_{\alpha,n}]_{n \geq N}.$$

Therefore we obtain that the ring of functions on LU, denoted by $\operatorname{Fun} LU$, is the inverse limit of the rings $\mathbb{C}[y_{\alpha,n}]_{\alpha \in \Delta_+, n \geq N}, N < 0$, with respect to the natural surjective homomorphisms

$$s_{N,M} : \mathbb{C}[y_{\alpha,n}]_{\alpha \in \Delta_+, n \geq N} \to \mathbb{C}[y_{\alpha,n}]_{\alpha \in \Delta_+, n \geq M}, \qquad N < M,$$

such that $y_{\alpha,n} \mapsto 0$ for $N \leq n < M$ and $y_{\alpha,n} \mapsto y_{\alpha,n}, n \geq M$. This is a complete topological ring, with the basis of open neighborhoods of 0 given by the ideals generated by $y_{\alpha,n}, n < N$, i.e., the kernels of the homomorphisms $s_{\infty,M} : \operatorname{Fun} LU \to \operatorname{Fun} U \otimes t^N \mathbb{C}[[t]]$.

A vector field on LU is by definition a continuous linear endomorphism ξ of $\operatorname{Fun} LU$ which satisfies the Leibniz rule: $\xi(fg) = \xi(f)g + f\xi(g)$. In other words, a vector field is a linear endomorphism ξ of $\operatorname{Fun} LU$ such that for any $M < 0$ there exist $N \leq M$ and a derivation

$$\xi_{N,M} : \mathbb{C}[y_{\alpha,n}]_{\alpha \in \Delta_+, n \geq N} \to \mathbb{C}[y_{\alpha,n}]_{\alpha \in \Delta_+, n \geq M}$$

which satisfies

$$s_{\infty,M}(\xi \cdot f) = \xi_{N,M} \cdot s_{\infty,N}(f)$$

for all $f \in \operatorname{Fun} LU$. The space of vector fields is naturally a topological Lie algebra, which we denote by $\operatorname{Vect} LU$.

Remark 2.1. More concretely, an element of $\operatorname{Fun} LU$ may be represented as a (possibly infinite) series

$$P_0 + \sum_{n<0} P_n y_{\alpha,n},$$

where $P_0 \in \mathbb{C}[y_{\alpha,n}]_{\alpha \in \Delta_+, n \geq 0}$, and the P_n's are arbitrary (finite) polynomials in $y_{\alpha,m}, m \in \mathbb{Z}$.

The Lie algebra $\operatorname{Vect} LU$ may also be described as follows. Identify the tangent space $T_0 LU$ to the origin in LU with LU, equipped with the structure of a complete topological vector space. Then $\operatorname{Vect} LU$ is isomorphic to the completed tensor product of $\operatorname{Fun} LU$ and LU. This means that vector fields on LU can be described more concretely as series

$$\sum_{n \in \mathbb{Z}} P_n \frac{\partial}{\partial y_{\alpha,n}},$$

where $P_n \in \operatorname{Fun} LU$ satisfies the following property: for each $M \geq 0$, there exists $N \leq M$ such that each $P_n, n \leq N$, lies in the ideal generated by the $y_{\alpha,m}, \alpha \in \Delta_+, m \leq M$. The commutator between two such series is computed in the standard way (term by term), and it is again a series of the above form. □

2.2. **Action of $L\mathfrak{g}$ on LU.** We have a natural Lie algebra homomorphism

$$\widehat{\rho} : L\mathfrak{g} \to \operatorname{Vect} LU,$$

which may be described explicitly by the formulas that we obtained in the finite-dimensional case, in which we replace the ordinary variables y_α with the "loop variables" $y_{\alpha,n}$. More precisely, we have the following analogue of formula (1.3),

$$(1 + \epsilon A \otimes t^m) x(t) = Z_+(\epsilon) Z_-(\epsilon)$$

where $x \in N_+((t)), Z_+(\epsilon) = x(t) + \epsilon Z_+^{(1)}$, $Z_+^{(1)} \in \mathfrak{n}_+((t))$, and $Z_-(\epsilon) = 1 + \epsilon Z_-^{(1)}$, $Z_-^{(1)} \in \mathfrak{b}_-((t))$. As before, we choose a faithful finite-dimensional representation V of $\mathfrak{g}$ and consider $x(t)$ as a matrix whose entries are Laurent power series in t with coefficients in the ring of polynomials in the coordinates $y_{\alpha,n}, \alpha \in \Delta_+, n \in \mathbb{Z}$, expressing a generic element of $N_+((t))$ in $\operatorname{End} V((t))$. We define $\widehat{\rho}$ by the formula

$$\widehat{\rho}(a \otimes t^m) \cdot x(t) = Z_+^{(1)}.$$

Then we have the following analogue of formula (1.4),

$$\widehat{\rho}(a \otimes t^m) \cdot x(t) = x(t) \left(x(t)^{-1} (a \otimes t^m) x(t) \right)_+ , \tag{2.1}$$

where z_+ denotes the projection of an element $z \in \mathfrak{g}((t))$ onto $\mathfrak{n}_+((t))$ along $\mathfrak{b}_-((t))$.

This formula implies that for any $a \in \mathfrak{g}$ the series

$$\widehat{\rho}(a(z)) = \sum_{n \in \mathbb{Z}} \widehat{\rho}(a \otimes t^n) z^{-n-1}$$

may be obtained from the formula for $\rho_0(a) = \xi_a$ by the substitution

$$y_\alpha \mapsto \sum_{n \in \mathbb{Z}} y_{\alpha,n} z^n,$$

$$\frac{\partial}{\partial y_\alpha} \mapsto \sum_{n \in \mathbb{Z}} \frac{\partial}{y_{\alpha,n}} z^{-n-1}.$$

2.3. **The Weyl algebra.** Let $\mathcal{A}^{\mathfrak{g}}$ be the Weyl algebra with generators

$$a_{\alpha,n} = \frac{\partial}{\partial y_{\alpha,n}}, \quad a^*_{\alpha,n} = y_{\alpha,-n}, \qquad \alpha \in \Delta_+, n \in \mathbb{Z},$$

and relations

$$[a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta} \delta_{n,-m}, \qquad [a_{\alpha,n}, a_{\beta,m}] = [a^*_{\alpha,n}, a^*_{\beta,m}] = 0. \tag{2.2}$$

Introduce the generating functions

$$a_\alpha(z) = \sum_{n \in \mathbb{Z}} a_{\alpha,n} z^{-n-1}, \tag{2.3}$$

$$a^*_\alpha(z) = \sum_{n \in \mathbb{Z}} a^*_{\alpha,n} z^{-n}. \tag{2.4}$$

Consider a topology on $\mathcal{A}^{\mathfrak{g}}$ in which the basis of open neighborhoods of 0 is formed by the left ideals $I_{N,M}, N, M \in \mathbb{Z}$, generated by $a_{\alpha,n}, \alpha \in \Delta_+, n \geq N$, and $a^*_{\alpha,m}, \alpha \in \Delta_+, m \geq M$. The *completed Weyl algebra* $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ is by definition the completion of $\mathcal{A}^{\mathfrak{g}}$ with respect to this topology. The algebra $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ should be thought of as an analogue of the algebra of differential operators on LU (see [FB], Ch. 12, for more details).

In concrete terms, elements of $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ may be viewed as arbitrary series of the form

$$\sum_{n>0} P_n a_n + \sum_{m>0} Q_m a_m^* + R, \qquad P_n, Q_m, R \in \mathcal{A}^{\mathfrak{g}}. \tag{2.5}$$

Let $\mathcal{A}_0^{\mathfrak{g}}$ be the (commutative) subalgebra of $\mathcal{A}^{\mathfrak{g}}$ generated by $a^*_{\alpha,n}, \alpha \in \Delta_+, n \in \mathbb{Z}$, and $\widetilde{\mathcal{A}}_0$ its completion in $\widetilde{\mathcal{A}}$. Next, let $\mathcal{A}^{\mathfrak{g}}_{\leq 1}$ be the subspace of $\mathcal{A}^{\mathfrak{g}}$ spanned by the products of elements of $\mathcal{A}_0^{\mathfrak{g}}$ and the generators $a_{\alpha,n}$. Denote by $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$ its completion in $\widetilde{\mathcal{A}}^{\mathfrak{g}}$. Thus, $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$ consists of all elements P of $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ with the property that

$$P \mod I_{N,M} \in \mathcal{A}^{\mathfrak{g}}_{\leq 1} \mod I_{N,M}, \qquad \forall N, M \in \mathbb{Z}.$$

Here is a more concrete description of $\widetilde{\mathcal{A}}_0^{\mathfrak{g}}$ and $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$ using the realization of $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ by the series of the form (2.5). The space $\widetilde{\mathcal{A}}_0^{\mathfrak{g}}$ consists of series of the form (2.5), where all $P_n = 0$, and $Q_m, R \in \mathcal{A}_0^{\mathfrak{g}}$. In other words, these are the series which do not contain $a_n, n \in \mathbb{Z}$. The space $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$ consists of elements of the form (2.5), where $P_n \in \mathcal{A}_0^{\mathfrak{g}}$, and $Q_m, R \in \mathcal{A}^{\mathfrak{g}}_{\leq 1}$. The following assertion is proved in Proposition 12.1.6 of [FB].

Proposition 2.2.

(1) $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$ *is a Lie algebra, and* $\widetilde{\mathcal{A}}_0^{\mathfrak{g}}$ *is its ideal.*
(2) $\widetilde{\mathcal{A}}_0^{\mathfrak{g}} \simeq \operatorname{Fun} LU$.
(3) $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}/\widetilde{\mathcal{A}}_0^{\mathfrak{g}} \simeq \operatorname{Vect} LU$ *as Lie algebras.*

According to Proposition 2.2, $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$ is an extension of the Lie algebra $\operatorname{Vect} LU$ by its module $\widetilde{\mathcal{A}}_0^{\mathfrak{g}}$,

$$0 \to \operatorname{Fun} LU \to \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1} \to \operatorname{Vect} LU \to 0. \tag{2.6}$$

This extension is however different from the standard (split) extension defining the Lie algebra of the usual differential operators on LU of order less than or equal to 1. In particular, our extension (2.6) is non-split (see Lemma 12.1.9 of [FB]). Therefore we cannot expect to lift the homomorphism $\widehat{\rho} : L\mathfrak{g} \to \operatorname{Vect} LU$ to a homomorphism $L\mathfrak{g} \to \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$, as in the finite-dimensional case. Nevertheless, we will show that the homomorphism $\widehat{\rho}$ may be lifted to a homomorphism $\widehat{\mathfrak{g}}_\kappa \to \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1}$, where $\widehat{\mathfrak{g}}_\kappa$ is the universal one-dimensional central extension of $L\mathfrak{g}$ defined as follows.

Fix an invariant inner product κ on $\mathfrak{g}$ and let $\widehat{\mathfrak{g}}_\kappa$ denote the central extension of $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$ with the commutation relations

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\kappa(A, B) \operatorname{Res} f dg)K, \tag{2.7}$$

where K is the central element. The Lie algebra $\widehat{\mathfrak{g}}_\kappa$ is the *affine Kac-Moody algebra* associated to κ.

We will begin by showing that the image of the embedding $L\mathfrak{g} \to \operatorname{Vect} LU$ belongs to a Lie subalgebra of "local" vector fields $\mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}} \subset \operatorname{Vect} LU$. This observation will allow us to replace the extension (2.6) by its "local" part.

2.4. **Heisenberg vertex algebra.** Let $M_{\mathfrak{g}}$ be the Fock representation of $\mathcal{A}^{\mathfrak{g}}$ generated by a vector $|0\rangle$ such that

$$a_{\alpha,n}|0\rangle = 0, \quad n \geq 0; \qquad a^*_{\alpha,n}|0\rangle = 0, \quad n > 0.$$

It is clear that the action of $\mathcal{A}^{\mathfrak{g}}$ on $M_{\mathfrak{g}}$ extends to a continuous action of the topological algebra $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ (here we equip $M_{\mathfrak{g}}$ with the discrete topology). Moreover, $M_{\mathfrak{g}}$ carries a vertex algebra structure defined as follows (see Definition 1.3.1 of [FB] for the definition of vertex algebra).

- Gradation: $\deg a_{\alpha,n} = \deg a^*_{\alpha,n} = -n, \deg|0\rangle = 0$;
- Vacuum vector: $|0\rangle$;
- Translation operator: $T|0\rangle = 0, [T, a_{\alpha,n}] = -na_{\alpha,n-1}, [T, a^*_{\alpha,n}] = -(n-1)a^*_{\alpha,n-1}$.
- Vertex operators:

$$Y(a_{\alpha,-1}|0\rangle, z) = a_\alpha(z), \qquad Y(a^*_{\alpha,0}|0\rangle, z) = a^*_\alpha(z),$$

$$Y(a_{\alpha_1,n_1} \dots a_{\alpha_k,n_k} a^*_{\beta_1,m_1} \dots a^*_{\beta_l,m_l}|0\rangle, z) = \prod_{i=1}^{k} \frac{1}{(-n_i-1)!} \prod_{j=1}^{l} \frac{1}{(-m_j)!} \cdot$$
$$:\partial_z^{-n_1-1} a_{\alpha_1}(z) \dots \partial_z^{-n_k-1} a_{\alpha_k}(z) \partial_z^{-m_1} a^*_{\beta_1}(z) \dots \partial_z^{-m_l} a^*_{\beta_l}(z):.$$

(see Lemma 11.3.8 of [FB]).

Here we use the normal ordering operation (denoted by the columns). Let us recall the definition. We call the generators $a_{\alpha,n}, n \geq 0$, and $a^*_{\alpha,m}, m > 0$, annihilation operators, and the generators $a_{\alpha,n}, n < 0$, and $a^*_{\alpha,m}, m \leq 0$, creation operators. A monomial P in $a_{\alpha,n}$ and $a^*_{\alpha,n}$ is called normally ordered if all factors of P which are annihilation operators stand to the right of all factors of P which are creation operators. Given any monomial P, we define the normally ordered monomial $:P:$ as the monomial obtained by moving all factors of P which are annihilation operators to the right, and all factors of P which are creation operators to the left. Note that since the annihilation operators commute with each other, it does not matter how we order them among themselves. The same is true for the creation operators. This shows that $:P:$ is well-defined by the above conditions.

Given two monomials P and Q, their normally ordered product is by definition the normally ordered monomial $:PQ:$. By linearity, we define the normally ordered product of any number of vertex operators from the vertex algebra $M_{\mathfrak{g}}$ by applying the above definition to each monomial appearing in each Fourier coefficient.

Set

$$U(M_{\mathfrak{g}}) \overset{\mathrm{def}}{=} (M_{\mathfrak{g}} \otimes \mathbb{C}((t)))/\operatorname{Im}(T \otimes 1 + \operatorname{Id} \otimes \partial_t).$$

According to [FB], § 4.1–4.2, this is a Lie algebra, which may be viewed as the completion of the span of all Fourier coefficients of vertex operators from $M_{\mathfrak{g}}$. Moreover, the map $U(M_{\mathfrak{g}}) \to \widetilde{\mathcal{A}}^{\mathfrak{g}}$ sending $A \otimes f(t)$ to $\operatorname{Res} Y(A, z) f(z) dz.$ is a homomorphism of Lie algebras (actually, an embedding in this case).

Note that $U(M_{\mathfrak{g}})$ is not an algebra. For instance, it contains the generators $a_{\alpha,n}, a^*_{\alpha,n}$ of the Heisenberg algebra, but does not contain monomials in these generators of degree greater than one. However, we will only need the Lie algebra structure on $U(M_{\mathfrak{g}})$.

The elements of $\widetilde{\mathcal{A}}_0^{\mathfrak{g}} = \operatorname{Fun} LU$ which lie in the image of $U(M_{\mathfrak{g}})$ are called "local functionals" on LU. The elements of $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ which belong to $U(M_{\mathfrak{g}})$ are given by (possibly infinite) linear combinations of Fourier coefficients of normally ordered polynomials in $a_\alpha(z), a^*_\alpha(z)$ and their derivatives. We refer to them as "local" elements of $\widetilde{\mathcal{A}}^{\mathfrak{g}}$.

2.5. **Coordinate independent definition of $M_{\mathfrak{g}}$.** The above definition of the vertex algebra $M_{\mathfrak{g}}$ referred to a particular system of coordinates $y_\alpha, \alpha \in \Delta_+$, on the group N_+. If we choose a different coordinate system $y'_\alpha, \alpha \in \Delta_+$, on N_+, we obtain another Heisenberg algebra with generators $a'_{\alpha,n}$ and ${a^*_{\alpha,n}}'$ and a vertex algebra $M'_{\mathfrak{g}}$. However, the vertex algebras $M'_{\mathfrak{g}}$ and $M_{\mathfrak{g}}$ are canonically isomorphic to each other. In particular, it is easy to express the vertex operators $a'_\alpha(z)$ and ${a^*_\alpha}'(z)$ in terms of $a_\alpha(z)$ and $a^*_\alpha(z)$. Namely, if $y'_\alpha = F_\alpha(y_\beta)$, then

$$\begin{aligned} a'_\alpha(z) &\mapsto \sum_{\gamma\in\Delta_+} {:}\partial_{y_\gamma} F_\alpha(a^*_\beta(z))\, a_\gamma(z){:}, \\ {a^*_\alpha}'(z) &\mapsto F_\alpha(a^*_\beta(z)). \end{aligned}$$

It is also possible to define $M_{\mathfrak{g}}$ without any reference to a coordinate system on N_+. Namely, we set

$$M_{\mathfrak{g}} = \operatorname{Ind}_{\mathfrak{n}_+[[t]]}^{\mathfrak{n}_+((t))} \operatorname{Fun}(N_+[[t]]) \simeq U(\mathfrak{n}_+ \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \operatorname{Fun}(N_+[[t]]),$$

where $\operatorname{Fun}(N_+[[t]])$ is the ring of regular functions on the proalgebraic group $N_+[[t]]$, considered as an $\mathfrak{n}_+[[t]]$–module. If we choose a coordinate system $\{y_\alpha\}_{\alpha\in\Delta_+}$ on N_+, then we obtain a coordinate system $\{y_{\alpha,n}\}_{\alpha\in\Delta_+,n\geq 0}$ on $N_+[[t]]$. Then $u \otimes P(y_{\alpha,n}) \in M_{\mathfrak{g}}$, where $u \in U(\mathfrak{n}_+ \otimes t^{-1}\mathbb{C}[t^{-1}])$ and $P(y_{\alpha,n}) \in \operatorname{Fun}(N_+[[t]]) = \mathbb{C}[y_{\alpha,n}]_{\alpha\in\Delta_+,n\geq 0}$, corresponds to $u \cdot P(a^*_{\alpha,n})$ in our previous description of $M_{\mathfrak{g}}$.

It is straightforward to define a vertex algebra structure on $M_{\mathfrak{g}}$ (see [FF8], § 2). Namely, the vacuum vector of $M_{\mathfrak{g}}$ is the vector $1 \otimes 1 \in M_{\mathfrak{g}}$. The translation operator T is defined as the operator $-\partial_t$, which naturally acts on $\operatorname{Fun}(N_+[[t]])$ as well as on $\mathfrak{n}_+((t))$ preserving $\mathfrak{n}_+[[t]]$. Next, we define the vertex operators corresponding to the elements of $M_{\mathfrak{g}}$ of the form $x_{-1}|0\rangle$, where $x \in \mathfrak{n}_+$, by the formula

$$Y(x_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} x_n z^{-n-1},$$

where $x_n = x \otimes t^n$, and we consider its action on $M_{\mathfrak{g}}$ viewed as the induced representation of $\mathfrak{n}_+((t))$. We also need to define the vertex operators

$$Y(P|0\rangle, z) = \sum_{n\in\mathbb{Z}} P_{(n)} z^{-n-1}$$

for $P \in \operatorname{Fun}(N_+[[t]])$. The corresponding linear operators $P_{(n)}$ are completely determined by their action on $|0\rangle$:

$$\begin{aligned} P_{(n)}|0\rangle &= 0, \qquad n \geq 0, \\ P_{(n)}|0\rangle &= \frac{1}{(-n-1)!} T^{-n-1} P|0\rangle, \end{aligned}$$

their mutual commutativity and the following commutation relations with $\mathfrak{n}_+((t))$:

$$[x_m, P_{(k)}] = \sum_{n \geq 0} \begin{pmatrix} m \\ n \end{pmatrix} (x_n \cdot P)_{(m+k-n)}.$$

Using the Reconstruction Theorem 4.4.1 of [FB], it is easy to prove that these formulas define a vertex algebra structure on $M_{\mathfrak{g}}$.

In fact, the same definition works if we replace N_+ by any algebraic group G. In the general case, it is natural to consider the central extension $\widehat{\mathfrak{g}}_\kappa$ of the loop algebra $\mathfrak{g}((t))$ corresponding to an invariant inner product κ on $\mathfrak{g}$ defined as in Section 2.3. Then we have the induced module

$$\operatorname{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}_\kappa} \operatorname{Fun}(G[[t]]),$$

where the central element K acts on $\operatorname{Fun}(G[[t]])$ as the identity. The corresponding vertex algebra is the algebra of chiral differential operators on G, considered in [GMS, AG]. As shown in [GMS, AG], in addition to the natural (left) action of $\widehat{\mathfrak{g}}_\kappa$ on this vertex algebra, there is another (right) action of $\widehat{\mathfrak{g}}_{-\kappa-\kappa_K}$, which commutes with the left action. Here κ_K is the Killing form on $\mathfrak{g}$, defined by the formula $\kappa_K(x,y) = \operatorname{Tr}_{\mathfrak{g}}(\operatorname{ad} x \operatorname{ad} y)$. In the case when $\mathfrak{g} = \mathfrak{n}_+$, there are no non-zero invariant inner products (in particular, $\kappa_K = 0$), and so we obtain a commuting right action of $\mathfrak{n}_+((t))$ on $M_{\mathfrak{g}}$. We will use this right action below (see Remark 4.4).

Remark 2.3. The above formulas in fact define a canonical vertex algebra structure on

$$\operatorname{Ind}_{\mathfrak{n}_+[[t]]}^{\mathfrak{n}_+((t))} \operatorname{Fun}(U[[t]]),$$

which is independent of the choice of identification $N_+ \simeq U$. Recall that in order to define U we only need to fix a Borel subgroup B_+ of G. Then U is defined as the open B_+–orbit of the flag manifold and so it is naturally an N_+–torsor. In order to identify U with N_+ we need to choose a point in U, i.e., an opposite Borel subgroup B_-, or equivalently, a Cartan subgroup $H = B_+ \cap B_-$ of B_+. But in the above formulas we never used an identification of N_+ and U, only the action of $L\mathfrak{n}_+$ on LU, which is determined by the canonical $\mathfrak{n}_+$–action on $\mathbb{C}[U]$.

If we do not fix an identification $N_+ \simeq U$, then the right action of $\mathfrak{n}_+((t))$ discussed above becomes an action of the "twisted" Lie algebra $\mathfrak{n}_{+,U}((t))$, where $\mathfrak{n}_{+,U} = U \underset{N_+}{\times} \mathfrak{n}_+$. It is interesting to observe that unlike $\mathfrak{n}_+$, this twisted Lie algebra $\mathfrak{n}_{+,U}$ has a canonical decomposition into the one-dimensional subspaces $\mathfrak{n}_{\alpha,U}$ corresponding to the roots Δ_+. Therefore there exist canonical (up to a scalar) generators e_i^R of the right action of $\mathfrak{n}_+$ on U (this observation is due to D. Gaitsgory). We use these operators below to define the screening operators, and their independence of the choice of the Cartan subgroup in B_+ implies the independence of the kernel of the screening operator from any additional choices. $\square$

2.6. **Local extension.** For our purposes we may replace $\widetilde{\mathcal{A}}^{\mathfrak{g}}$, which is a very large topological algebra, by a relatively small "local part" $U(M_{\mathfrak{g}})$. Accordingly, we replace $\widetilde{\mathcal{A}}_0^{\mathfrak{g}}$ and $\widetilde{\mathcal{A}}_{\leq 1}^{\mathfrak{g}}$ by their local versions $\mathcal{A}_{0,\mathrm{loc}}^{\mathfrak{g}} = \widetilde{\mathcal{A}}_0^{\mathfrak{g}} \cap U(M_{\mathfrak{g}})$ and $\mathcal{A}_{\leq 1,\mathrm{loc}}^{\mathfrak{g}} = \widetilde{\mathcal{A}}_{\leq 1}^{\mathfrak{g}} \cap U(M_{\mathfrak{g}})$.

Let us describe $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ and $\mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$ more explicitly. The space $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ is spanned (topologically) by the Fourier coefficients of all polynomials in the $\partial_z^n a^*_\alpha(z), n \geq 0$. Note that because the $a^*_{\alpha,n}$'s commute among themselves, these polynomials are automatically normally ordered. The space $\mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$ is spanned by the Fourier coefficients of the fields of the form $:P(a^*_\alpha(z), \partial_z a^*_\alpha(z), \ldots) a_\beta(z):$ (the normally ordered product of $P(a^*_\alpha(z), \partial_z a^*_\alpha(z), \ldots)$ and $a_\beta(z)$). Observe that the Fourier coefficients of all fields of the form

$$:P(a^*_\alpha(z), \partial_z a^*_\alpha(z), \ldots) \partial_z^m a_\beta(z): , \qquad m > 0,$$

may be expressed as linear combinations of the fields of the form

$$:P(a^*_\alpha(z), \partial_z a^*_\alpha(z), \ldots) a_\beta(z): . \tag{2.8}$$

Further, we define a local version $\mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}}$ of $\operatorname{Vect} LU$ as the subspace, which consists of finite linear combinations of Fourier coefficients of the formal power series

$$P(a^*_\alpha(z), \partial_z a^*_\alpha(z), \ldots) a_\beta(z), \tag{2.9}$$

where $a_\alpha(z)$ and $a^*_\alpha(z)$ are given by formulas (2.3), (2.4).

Since $\mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$ is the intersection of Lie subalgebras of $\widetilde{\mathcal{A}}^{\mathfrak{g}}$, it is also a Lie subalgebra of $\widetilde{\mathcal{A}}$. By construction, its image in $\operatorname{Vect} LU$ under the homomorphism $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\leq 1} \to \operatorname{Vect} LU$ equals $\mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}}$. Finally, the kernel of the resulting surjective Lie algebra homomorphism $\mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}} \to \mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}}$ equals $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$. Hence we obtain that the extension (2.6) restricts to the "local" extension

$$0 \to \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}} \to \mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}} \to 0. \tag{2.10}$$

This sequence is non-split (see Lemma 12.1.9 of [FB]). The corresponding two-cocycle will be computed explicitly in Lemma 2.5 using the Wick formula (it comes from the "double contractions" of the corresponding vertex operators).

According to Section 2.2, the image of $L\mathfrak{g}$ in $\operatorname{Vect} LU$ belongs to $\mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}}$. We will show that the homomorphism $L\mathfrak{g} \to \mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}}$ may be lifted to a homomorphism $\widehat{\mathfrak{g}}_\kappa \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$, where $\widehat{\mathfrak{g}}_\kappa$ is the central extension of $L\mathfrak{g}$ defined in Section 2.3.

2.7. Computation of the two-cocycle of the extension (2.10).

Recall that an exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0,$$

where $\mathfrak{h}$ is an abelian ideal, with prescribed $\mathfrak{g}$–module structure, gives rise to a two-cocycle of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$. It is constructed as follows. Choose a splitting $\imath : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ of this sequence (considered as a vector space), and define $\sigma : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}$ by the formula

$$\sigma(a, b) = \imath([a, b]) - [\imath(a), \imath(b)].$$

One checks that σ is a two-cocycle in the Chevalley complex of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$, and that changing the splitting $\imath$ amounts to changing σ by a coboundary.

Conversely, suppose we are given a linear functional $\sigma : \bigwedge^2 \mathfrak{g} \to \mathfrak{h}$. Then we associate to it a Lie algebra structure on the direct sum $\mathfrak{g} \oplus \mathfrak{h}$. Namely, the Lie bracket of any two elements of $\mathfrak{h}$ is equal to 0, $[X,h] = X \cdot h$ for all $X \in \mathfrak{g}, h \in \mathfrak{h}$, and

$$[X,Y] = [X,Y]_{\mathfrak{g}} + \omega(X,Y), \qquad X,Y \in \mathfrak{g}.$$

These formulas define a Lie algebra structure on $\widetilde{\mathfrak{g}}$ if and only if σ is a two-cocycle in the standard Chevalley complex of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$. Therefore we obtain a bijection between the set of isomorphism classes of extensions of $\mathfrak{g}$ by $\mathfrak{h}$ and the cohomology group $H^2(\mathfrak{g},\mathfrak{h})$.

Consider the extension (2.10). The operation of normal ordering gives us a splitting $\imath$ of this extension as vector space. Namely, $\imath$ maps the nth Fourier coefficient of the series (2.9) to the nth Fourier coefficient of the series (2.8). To compute the corresponding two-cocycle we have to learn how to compute commutators of Fourier coefficients of generating functions of the form (2.8) and (2.9). Those may be computed from the operator product expansion (OPE) of the corresponding vertex operators. We now explain how to compute the OPEs of vertex operators using the Wick formula following [FB], §§ 12.2.4–12.2.10.

2.8. **Contractions of fields.** In order to state the Wick formula, we have to introduce the notion of contraction of two fields. In order to simplify notation, we will assume that $\mathfrak{g} = \mathfrak{sl}_2$ and suppress the index α in $a_{\alpha,n}$ and $a^*_{\alpha,n}$. The general case is treated in the same way.

From the commutation relations, we obtain the following OPEs:

$$a(z)a^*(w) = \frac{1}{z-w} + {:}a(z)a^*(w){:}\ ,$$
$$a^*(z)a(w) = -\frac{1}{z-w} + {:}a^*(z)a(w){:}\ .$$

We view them now as identities on formal power series, in which by $1/(z-w)$ we understand its expansion in positive powers of w/z. Differentiating several times, we obtain

$$\partial_z^n a(z)\partial_w^m a^*(w) = (-1)^n \frac{(n+m)!}{(z-w)^{n+m+1}} + {:}\partial_z^n a(z)\partial_w^m a^*(w){:}\ , \tag{2.11}$$

$$\partial_z^m a^*(z)\partial_w^n a(w) = (-1)^{m+1} \frac{(n+m)!}{(z-w)^{n+m+1}} + {:}\partial_z^m a^*(z)\partial_w^n a(w){:} \tag{2.12}$$

(here again by $1/(z-w)^n$ we understand its expansion in positive powers of w/z).

Suppose that we are given two normally ordered monomials in $a(z), a^*(z)$ and their derivatives. Denote them by $P(z)$ and $Q(z)$. A single *pairing* between $P(z)$ and $Q(w)$ is by definition either the pairing $(\partial_z^n a(z), \partial_w^m a^*(w))$ of $\partial_z^n a(z)$ occurring in $P(z)$ and $\partial_w^m a^*(w)$ occurring in $Q(w)$, or $(\partial_z^m a^*(z), \partial_w^n a(w))$ of $\partial_z^m a^*(z)$ occurring in $P(z)$ and $\partial_w^n a(w)$ occurring in $Q(w)$. We attach to it the functions

$$(-1)^n \frac{(n+m)!}{(z-w)^{n+m+1}} \qquad \text{and} \qquad (-1)^{m+1} \frac{(n+m)!}{(z-w)^{n+m+1}},$$

respectively. A multiple pairing B is by definition a disjoint union of single pairings. We attach to it the function $f_B(z,w)$, which is the product of the functions corresponding to the single pairings in B.

Note that the monomials $P(z)$ and $Q(z)$ may well have multiple pairings of the same type. For example, the monomials $:a^*(z)^2\partial_z a(z):$ and $:a(w)\partial_z^2 a^*(w):$ have two different pairings of type $(a^*(z), a(w))$; the corresponding function is $-1/(z-w)$. In such a case we say that the multiplicity of this pairing is 2. Note that these two monomials also have a unique pairing $(\partial_z a(z), \partial_w^2 a^*(w))$, and the corresponding function is $-6/(z-w)^4$.

Given a multiple pairing B between $P(z)$ and $Q(w)$, we define $(P(z)Q(w))_B$ as the product of all factors of $P(z)$ and $Q(w)$ which do not belong to the pairing (if there are no factors left, we set $(P(z)Q(w))_B = 1$). The *contraction* of $P(z)Q(w)$ with respect to the pairing B, denoted $:P(z)Q(w):_B$, is by definition the normally ordered formal power series $:(P(z)Q(w)_B):$ multiplied by the function $f_B(z,w)$. We extend this definition to the case when B is the empty set by stipulating that

$$:P(z)Q(w):_\emptyset = :P(z)Q(w): .$$

Now we are in a position to state the *Wick formula*, which gives the OPE of two arbitrary normally ordered monomial vertex operators. The proof of this formula is straightforward and is left to the reader.

Lemma 2.4. *Let $P(z)$ and $Q(w)$ be two monomials as above. Then $P(z)Q(w)$ equals the sum of terms $:P(z)Q(w):_B$ over all pairings B between P and Q including the empty one, counted with multiplicity.*

Now we can compute our two–cocycle. For that, we need to apply the Wick formula to the fields of the form

$$:R(a^*(z), \partial_z a^*(z), \ldots)a(z): ,$$

whose Fourier coefficients span the pre-image of $\mathcal{T}_{\text{loc}}$ in $A_{\leq 1,\text{loc}}$ under our splitting $\imath$. Two fields of this form may have only single or double pairings, and therefore their OPE can be written quite explicitly.

A field of the above form may be written as $Y(P(a_n^*)a_{-1}, z)$ (or $Y(Pa_{-1}, z)$ for short), where P is a polynomial in the $a_n^*, n \leq 0$ (recall that $a_n^*, n \leq 0$, corresponds to $\partial_z^{-n} a^*(z)/(-n)!$). Applying the Wick formula, we obtain

Lemma 2.5.

$$\begin{aligned} Y(Pa_{-1}, z)Y(Qa_{-1}, w) = {} & :Y(Pa_{-1}, z)Y(Qa_{-1}, w): \\ & + \sum_{n \geq 0} \frac{1}{(z-w)^{n+1}} :Y(P, z) Y\left(\frac{\partial Q}{\partial a_{-n}^*} a_{-1}, w\right): \\ & - \sum_{n \geq 0} \frac{1}{(z-w)^{n+1}} :Y\left(\frac{\partial P}{\partial a_{-n}^*} a_{-1}, z\right) Y(Q, w): \\ & - \sum_{n,m \geq 0} \frac{1}{(z-w)^{n+m+2}} Y\left(\frac{\partial P}{\partial a_{-n}^*}, z\right) Y\left(\frac{\partial Q}{\partial a_{-m}^*}, w\right) \end{aligned}$$

Note that we do not need to put normal ordering in the last summand.

2.9. **Double contractions.** Using this formula, we can now easily obtain the commutators of the Fourier coefficients of the fields $Y(Pa_{-1},z)$ and $Y(Qa_{-1},w)$, using the residue calculus (see [FB], § 3.3).

The first two terms in the right hand side of the formula in Lemma 2.5 correspond to single contractions between $Y(Pa_{-1},z)$ and $Y(Qa_{-1},w)$. The part in the commutator of the Fourier coefficients induced by these terms will be exactly the same as the commutator of the corresponding vector fields, computed in $\mathcal{T}_{\mathrm{loc}}$. Thus, we see that the discrepancy between the commutators in $A_{\leq 1,\mathrm{loc}}$ and in $\mathcal{T}_{\mathrm{loc}}$ (as measured by our two–cocycle) is due to the last term in the formula from Lemma 2.5, which comes from the *double contractions* between $Y(Pa_{-1},z)$ and $Y(Qa_{-1},w)$.

Explicitly, we obtain the following formula for our two-cocycle

$$\omega((Pa_{-1})_{(k)},(Qa_{-1})_{(s)}) \tag{2.13}$$

$$= - \sum_{n,m\geq 0} \int \frac{1}{(n+m+1)!} \partial_z^{n+m+1} Y\left(\frac{\partial P}{\partial a^*_{-n}},z\right) Y\left(\frac{\partial Q}{\partial a^*_{-m}},w\right) z^k w^s \bigg|_{z=w} dw.$$

Here and below, for a formal power series $A(z)=\sum_{n\in\mathbb{Z}} A_n z^n$ we set

$$\int A(z)dz = A_{-1}.$$

2.10. **A reminder on cohomology.** Let again

$$0 \to \mathfrak{h} \to \widetilde{\mathfrak{l}} \to \mathfrak{l} \to 0$$

be an extension of Lie algebras, where $\mathfrak{h}$ is an abelian Lie subalgebra and an ideal in $\widetilde{\mathfrak{l}}$. Choosing a splitting $\imath$ of this sequence considered as a vector space we define a two-cocycle of $\mathfrak{l}$ with coefficients in $\mathfrak{h}$ as in Section 2.7. Suppose that we are given a Lie algebra homomorphism $\alpha : \mathfrak{g} \to \mathfrak{l}$ for some Lie algebra $\mathfrak{g}$. Pulling back our two-cocycle under α we obtain a two-cocycle of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$.

On the other hand, given a homomorphism $\mathfrak{h}' \xrightarrow{i} \mathfrak{h}$ of $\mathfrak{g}$–modules we obtain a map i_* between the spaces of two-cocycles of $\mathfrak{g}$ with coefficients in $\mathfrak{h}$ and $\mathfrak{h}'$. The corresponding map of the cohomology groups $H^2(\mathfrak{g},\mathfrak{h}') \to H^2(\mathfrak{g},\mathfrak{h})$ will also be denoted by i_*.

Lemma 2.6. *Suppose that we are given a two-cocycle σ of $\mathfrak{g}$ with coefficients in $\mathfrak{h}'$ such that the cohomology classes of $i_*(\sigma)$ and ω are equal in $H^2(\mathfrak{g},\mathfrak{h}')$. Denote by $\widetilde{\mathfrak{g}}$ the extension of $\mathfrak{g}$ by $\mathfrak{h}'$ corresponding to σ defined as above. Then the map $\mathfrak{g} \to \mathfrak{l}$ may be augmented to a map of commutative diagrams*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] & \widetilde{\mathfrak{l}} \arrow[r] & \mathfrak{l} \arrow[r] & 0 \\ 0 \arrow[r] & \mathfrak{h}' \arrow[r] \arrow[u] & \widetilde{\mathfrak{g}} \arrow[r] \arrow[u] & \mathfrak{g} \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{2.14}$$

Moreover, the set of isomorphism classes of such diagrams is a torsor over $H^1(\mathfrak{g},\mathfrak{h})$.

Proof. If the cohomology classes of $i_*(\sigma)$ and ω coincide, then $i_*(\sigma)+d\gamma=\omega$, where γ is a one-cochain, i.e., a linear functional $\mathfrak{g} \to \mathfrak{h}$. Define a linear map $\beta : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{l}}$ as follows. By definition, we have a splitting $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}'$ as a vector space. We set

$\beta(X) = \imath(\alpha(X)) + \gamma(X)$ for all $X \in \mathfrak{g}$ and $\beta(h) = i(h)$ for all $h \in \mathfrak{h}$. Then the above equality of cocycles implies that β is a Lie algebra homomorphism which makes the diagram (2.14) commutative. However, the choice of γ is not unique as we may modify it by adding to it an arbitrary one-cocycle γ'. But the homomorphisms corresponding to γ and to $\gamma + \gamma'$, where γ' is a coboundary, lead to isomorphic diagrams. This implies that the set of isomorphism classes of such diagrams is a torsor over $H^1(\mathfrak{g}, \mathfrak{h})$. $\square$

2.11. **Two cocycles.** Restricting the two-cocycle ω of $\mathcal{T}_{\mathrm{loc}}$ with coefficients in $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ corresponding to the extension (2.10) to $L\mathfrak{g} \subset \mathcal{T}_{\mathrm{loc}}$ we obtain a two-cocycle of $L\mathfrak{g}$ with coefficients in $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$. We also denote it by ω. The $L\mathfrak{g}$–module $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ contains the trivial subrepresentation $\mathbb{C}$, i.e., the span of $\int Y(|0\rangle, z) dz/z$ (which we view as the constant function on LU), and the inclusion $\mathbb{C} \xrightarrow{i} \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ induces a map i_* of the corresponding spaces of two-cocycles and the cohomology groups $H^2(L\mathfrak{g}, \mathbb{C}) \to H^2(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$.

It is well known that $H^2(L\mathfrak{g}, \mathbb{C})$ is one-dimensional and is isomorphic to the space of invariant inner products on $\mathfrak{g}$ (the corresponding Kac-Moody two-cocyles are described in Section 2.3). We denote by σ the class corresponding to the inner product $\kappa_c = -\frac{1}{2}\kappa_K$, where κ_K denotes the Killing form on $\mathfrak{g}$. Thus, by definition,

$$\kappa_c(x, y) = -\frac{1}{2} \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y).$$

We will show that the cohomology classes of $i_*(\sigma)$ and ω are equal. Lemma 2.6 will then imply that there exists a family of Lie algebra homomorphisms $\widehat{\mathfrak{g}}_{\kappa_c} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$ such that $K \mapsto 1$.

Unfortunately, the Chevalley complex that calculates $H^2(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ is unmanageably large. So as the first step we will show in the next section that ω and σ actually both belong to a much smaller subcomplex, where they are more easily compared.

3. Comparison of cohomology classes

3.1. **Clifford Algebras.** Choose a basis $\{J^a\}_{a=1,\dots,\dim \mathfrak{g}}$ of $\mathfrak{g}$, and set $J^a_n = J^a \otimes t^n$. Introduce the Clifford algebra with generators $\psi_{a,n}, \psi^*_{a,m}, a = 1, \dots, \dim \mathfrak{g}; m, n \in \mathbb{Z}$, with anti-commutation relations

$$[\psi_{a,n}, \psi_{b,n}]_+ = [\psi^*_{a,n}, \psi^*_{b,m}]_+ = 0, \qquad [\psi_{a,n}, \psi^*_{b,m}]_+ = \delta_{a,b}\delta_{n,-m}.$$

Let $\bigwedge_{\mathfrak{g}}$ be the module over this Clifford algebra generated by a vector $|0\rangle$ such that

$$\psi_{a,n}|0\rangle = 0, \quad n > 0, \qquad \psi^*_{a,n}|0\rangle = 0, \quad n \geq 0.$$

Then $\bigwedge_{\mathfrak{g}}$ carries the following structure of a vertex superalgebra (see [FB], § 15.1.1):

- Gradation: $\deg \psi_{a,n} = \deg \psi^*_{a,n} = -n, \deg |0\rangle = 0$;
- Vacuum vector: $|0\rangle$;
- Translation operator: $T|0\rangle = 0, [T, \psi_{a,n}] = -n\psi_{a,n-1}, [T, \psi^*_{a,n}] = -(n-1)\psi^*_{a,n-1}$.

- Vertex operators:

$$
\begin{aligned}
Y(\psi_{a,-1}|0\rangle, z) &= \psi_a(z) = \sum_{n\in\mathbb{Z}} \psi_{a,n} z^{-n-1},\\
Y(\psi^*_{a,0}|0\rangle, z) &= \psi^*_a(z) = \sum_{n\in\mathbb{Z}} \psi^*_{a,n} z^{-n},
\end{aligned}
$$

$$
\begin{aligned}
Y(\psi_{a_1,n_1}\dots\psi_{a_k,n_k}\psi^*_{b_1,m_1}\dots\psi^*_{b_l,m_l}|0\rangle, z) = \prod_{i=1}^{k}\frac{1}{(-n_i-1)!}\prod_{j=1}^{l}\frac{1}{(-m_j)!}\cdot\\
{:}\partial_z^{-n_1-1}\psi_{a_1}(z)\dots\partial_z^{-n_k-1}\psi_{a_k}(z)\partial_z^{-m_1}\psi^*_{b_1}(z)\dots\partial_z^{-m_l}\psi^*_{b_l}(z){:}\,.
\end{aligned}
$$

The tensor product of two vertex superalgebras is naturally a vertex superalgebra (see Lemma 1.3.6 of [FB]), and so $M_{\mathfrak{g}}\otimes\bigwedge_{\mathfrak{g}}$ is a vertex superalgebra.

3.2. **The local Chevalley complex.** The ordinary Chevalley complex computing the Lie algebra cohomology $H^\bullet(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ is $C^\bullet(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) = \bigoplus_{i\geq 0} C^i(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$,

$$C^i(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) = \mathrm{Hom}_{\mathrm{cont}}\Big(\bigwedge{}^i(L\mathfrak{g}),\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}\Big),$$

where $\bigwedge^i(L\mathfrak{g})$ stands for the natural completion of the ordinary ith exterior power of the topological vector space $L\mathfrak{g}$, and we consider all continuous linear maps. For $f\in\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ we denote by $\psi^*_{b_1,-k_1}\dots\psi^*_{b_i,-k_i}f$ the linear functional $\phi\in\mathrm{Hom}_{\mathrm{cont}}(\bigwedge^i(L\mathfrak{g}),\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ defined by the formula

$$
\phi(J^{a_1}_{m_1}\wedge\ldots\wedge J^{a_i}_{m_i}) = \begin{cases}(-1)^{l(\tau)}f, & \text{if}\quad ((a_1,m_1),\dots,(a_i,m_i)) = \tau((b_1,k_1),\dots,(b_i,k_i)),\\ 0, & \text{if}\quad ((a_1,m_1),\dots,(a_i,m_i)) \neq \tau((b_1,k_1),\dots,(b_i,k_i)),\end{cases}
$$

where τ runs over the symmetric group on i letters and $l(\tau)$ is the length of τ. Then any element of the space $C^i(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ may be written as a (possibly infinite) linear combination of terms of this form.

The differential $d: C^i(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})\to C^{i+1}(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ is given by the formula

$$
\begin{aligned}
(d\phi)(X_1,\dots,X_{i+1}) &= \sum_{j=1}^{i+1}(-1)^{j+1}X_j\phi(X_1,\dots,\widehat{X_j},\dots,X_{i+1})\\
&\quad+\sum_{j<k}(-1)^{j+k+1}\phi([X_j,X_k],X_1,\dots,\widehat{X_j},\dots,\widehat{X_k},\dots,X_{i+1}).
\end{aligned}
$$

It follows from the definition of the vertex operators that the linear maps

$$\int Y(\psi^*_{a_1,n_1}\cdots\psi^*_{a_i,n_i}a^*_{\alpha_1,m_1}\cdots a^*_{\alpha_j,m_j}|0\rangle, z)\,dz, \qquad n_p\leq 0, m_p\leq 0. \tag{3.1}$$

from $\bigwedge^i(L\mathfrak{g})$ to $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ are continuous. Let $C^i_{\mathrm{loc}}(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ be the subspace of the space $\mathrm{Hom}_{\mathrm{cont}}(\bigwedge^i(L\mathfrak{g}),\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$, spanned by all linear maps of the form (3.1).

Lemma 3.1. *$C^\bullet_{\mathrm{loc}}(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ is preserved by the differential d, and so it forms a subcomplex of $C^\bullet(L\mathfrak{g},\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$.*

Proof. The action of the differential d on $\int Y(A,z)dz$, where A is of the form (3.1), may be written as the commutator

$$\left[\int Q(z)dz, \int Y(A,z)dz\right],$$

where $Q(z)$ is the field

$$Q(z) = \sum_a J^a(z)\psi_a^*(z) - \frac{1}{2}\sum_{a,b,c} \mu_c^{ab} {:}\psi_a^*(z)\psi_b^*(z)\psi_c(z){:}\,. \tag{3.2}$$

Therefore it is equal to $Y\left(\int Q(z)dz \cdot A, z\right)$, which is of the form (3.1). ☐

We call $C^i_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ the *local* subcomplex of $C^\bullet(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$.

Lemma 3.2. *Both ω and $i_*(\sigma)$ belong to $C^2_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$.*

Proof. We begin with σ given by the formula

$$\sigma(J^a_n, J^b_m) = n\delta_{n,-m}\kappa_c(J^a, J^b) \tag{3.3}$$

(see Section 2.3). Therefore the cocycle $i_*(\sigma)$ is equal to the following element of $C^2_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$,

$$\begin{aligned} i_*(\sigma) &= \sum_{a \leq b}\sum_{n\in\mathbb{Z}} \kappa_c(J^a, J^b) n \psi^*_{a,-n}\psi^*_{b,n} \\ &= \sum_{a\leq b} \kappa_c(J^a,J^b) \int Y(\psi^*_{a,-1}\psi^*_{b,0}|0\rangle, z)\, dz. \end{aligned} \tag{3.4}$$

Next we consider ω. Combining the discussion of Section 2.2 with formulas of Section 1.5, we obtain that

$$\imath(J^a_n) = \sum_{\beta\in\Delta_+} \int Y(R^\beta_a(a^*_{\alpha,0})a_{\beta,-1}|0\rangle, z)\, z^n dz,$$

where R^β_a is a polynomial. Then formula (2.13) implies that

$$\omega(J^a_n, J^b_m) =$$
$$- \sum_{\alpha,\beta\in\Delta_+} \int \left(Y\left(T\frac{\partial R^\alpha_a}{\partial a^*_{\beta,0}} \cdot \frac{\partial R^\beta_b}{\partial a^*_{\alpha,0}}, w\right) w^{n+m} + nY\left(\frac{\partial R^\alpha_a}{\partial a^*_{\beta,0}}\frac{\partial R^\beta_b}{\partial a^*_{\alpha,0}}, w\right) w^{n+m-1}\right) dw.$$

Therefore

$$\omega = \tag{3.5}$$
$$- \sum_{a\leq b;\alpha,\beta\in\Delta_+} \int \left(Y\left(\psi^*_{a,0}\psi^*_{b,0} T\frac{\partial R^\alpha_a}{\partial a^*_{\beta,0}} \cdot \frac{\partial R^\beta_b}{\partial a^*_{\alpha,0}}, z\right) + Y\left(\psi^*_{a,-1}\psi^*_{b,0}\frac{\partial R^\alpha_a}{\partial a^*_{\beta,0}}\frac{\partial R^\beta_b}{\partial a^*_{\alpha,0}}, z\right)\right) dz$$

(in the last two formulas we have omitted $|0\rangle$). Hence it belongs to $C^2_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$. ☐

We need to show that the cocycles $i_*(\sigma)$ and ω represent the same cohomology class in the local complex $C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$. The idea is that this is equivalent to checking that the restrictions of these cocycles to the Lie subalgebra $L\mathfrak{h} \subset L\mathfrak{g}$ are the same. These restrictions can be easily computed and we indeed obtain that they coincide. The passage to $L\mathfrak{h}$ is achieved by a version of the "Schapiro lemma", as we explain in the next section.

3.3. **Another complex.** Given a Lie algebra $\mathfrak{l}$, we denote the Lie subalgebra $\mathfrak{l}[[t]]$ of $L\mathfrak{l} = \mathfrak{l}((t))$ by $L_+\mathfrak{l}$.

Observe that the Lie algebra $L_+\mathfrak{n}_+$ acts naturally on the space

$$M_{\mathfrak{g},+} = \mathbb{C}[a^*_{\alpha,n}]_{\alpha\in\Delta_+, n\leq 0} \simeq \mathbb{C}[y_{\alpha,n}]_{\alpha\in\Delta_+, n\geq 0},$$

which is identified with the ring of functions on the space $U[[t]] \simeq N_+[[t]]$. We identify the standard Chevalley complex $C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) = \mathrm{Hom}_{\mathrm{cont}}(\bigwedge^\bullet L_+\mathfrak{g}, M_{\mathfrak{g},+})$ with the tensor product $M_{\mathfrak{g},+} \otimes \bigwedge^\bullet_{\mathfrak{g},+}$, where

$$\bigwedge\nolimits^\bullet_{\mathfrak{g},+} = \bigwedge(\psi^*_{a,n})_{n\leq 0}.$$

Introduce a superderivation T on $C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+})$ acting by the formulas

$$T \cdot a^*_{a,n} = -(n-1)a^*_{a,n-1}, \qquad T \cdot \psi^*_{a,n} = -(n-1)\psi^*_{a,n-1}.$$

We have a linear map

$$\int : C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \to C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$$

sending $A \in C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+})$ to $\int Y(A,z)dz$ (recall that $\int$ picks out the (-1)st Fourier coefficient of a formal power series).

Recall that in any vertex algebra V we have the identity $Y(TA,z) = \partial_z Y(A,z)$. Hence if $A \in \mathrm{Im}\, T$, then $\int Y(A,z)dz = 0$.

Lemma 3.3. *The map $\int$ defines an isomorphism*

$$C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) \simeq C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+})/(\mathrm{Im}\, T + \mathbb{C}),$$

and $\mathrm{Ker}\, T = \mathbb{C}$. *Moreover, the following diagram is commutative*

$$\begin{tikzcd}
C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r, "T"] & C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r, "\int"] & C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) \\
C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r, "T"] \arrow[u, "d"] & C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r, "\int"] \arrow[u, "d"] & C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) \arrow[u, "d"]
\end{tikzcd}$$

Proof. It is easy to see that the differential of the Chevalley complex $C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+})$ acts by the formula $A \mapsto \int Q(z)dz \cdot A$, where $Q(z)$ is given by formula (3.2). The lemma now follows from the argument used in the proof of Lemma 3.1. □

Consider the double complex

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r, "T"] & C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r] & \mathbb{C} \\
\mathbb{C} \arrow[r] & C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r, "T"] \arrow[u] & C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \arrow[r] \arrow[u] & \mathbb{C}
\end{tikzcd}$$

According to the lemma, the cohomology of the complex $C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ is given by the second term of the spectral sequence, in which the zeroth differential is vertical.

We start by computing the first term of this spectral sequence. Observe that the module $M_{\mathfrak{g},+}$ is isomorphic to the coinduced module $\mathrm{Coind}_{L_+\mathfrak{b}_-}^{L_+\mathfrak{g}} \mathbb{C}$. Define a map of complexes

$$\mu' : C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \to C^\bullet(L_+\mathfrak{b}_-, \mathbb{C})$$

as follows. If γ is an i–cochain in the complex $C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) = \mathrm{Hom}_{\mathrm{cont}}(\bigwedge^i L_+\mathfrak{g}, M_{\mathfrak{g},+})$, then $\mu'(\gamma)$ is by definition the restriction of γ to $\bigwedge^i L_+\mathfrak{b}_-$ composed with the natural projection $M_{\mathfrak{g},+} = \mathrm{Coind}_{L_+\mathfrak{b}_-}^{L_+\mathfrak{g}} \mathbb{C} \to \mathbb{C}$. It is clear that μ' is a morphism of complexes. The following is an example of the Shapiro lemma (see [Fu], § 5.4, for the proof).

Lemma 3.4. *The map μ' induces an isomorphism at the level of cohomologies, i.e.,*

$$H^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \simeq H^\bullet(L_+\mathfrak{b}_-, \mathbb{C}). \tag{3.6}$$

Now we compute the right hand side of (3.6). Since $L_+\mathfrak{n}_- \subset L_+\mathfrak{b}_-$ is an ideal, and $L_+\mathfrak{b}_-/L_+\mathfrak{n}_- \simeq L_+\mathfrak{h}$, we obtain from the Serre-Hochshield spectral sequence (see [Fu], § 5.1) that

$$H^n(L_+\mathfrak{b}_-, \mathbb{C}) = \bigoplus_{p+q=n} H^p(L_+\mathfrak{h}, H^q(L_+\mathfrak{n}_-, \mathbb{C})).$$

But $\mathfrak{h}\otimes 1 \in L_+\mathfrak{h}$ acts diagonally on $H^\bullet(L_+\mathfrak{n}_-, \mathbb{C})$, inducing an inner grading. According to [Fu], § 5.2,

$$H^p(L_+\mathfrak{h}, H^q(L_+\mathfrak{n}_-, \mathbb{C})) = H^p(L_+\mathfrak{h}, H^q(L_+\mathfrak{n}_-, \mathbb{C})_0),$$

where $H^q(L_+\mathfrak{n}_-, \mathbb{C})_0$ is the subspace where $\mathfrak{h} \otimes 1$ acts by 0. Clearly, $H^0(L_+\mathfrak{n}_-, \mathbb{C})_0$ is the one-dimensional subspace of the scalars $\mathbb{C} \subset H^0(L_+\mathfrak{n}_-, \mathbb{C})$, and $H^q(L_+\mathfrak{n}_-, \mathbb{C})_0 = 0$ for $q \neq 0$. Thus, we find that

$$H^p(L_+\mathfrak{h}, H^q(L_+\mathfrak{n}_-, \mathbb{C})) = H^p(L_+\mathfrak{h}, \mathbb{C}).$$

Furthermore, we have the following result. Define a map of complexes

$$\mu : C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \to C^\bullet(L_+\mathfrak{h}, \mathbb{C}) \tag{3.7}$$

as follows. If γ is an i–cochain in the complex $C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) = \mathrm{Hom}_{\mathrm{cont}}(\bigwedge^i L_+\mathfrak{g}, M_{\mathfrak{g},+})$, then $\mu(\gamma)$ is by definition the restriction of γ to $\bigwedge^i L_+\mathfrak{h}$ composed with the natural projection

$$p : M_{\mathfrak{g},+} = \mathrm{Coind}_{L_+\mathfrak{b}_-}^{L_+\mathfrak{g}} \mathbb{C} \to \mathbb{C}. \tag{3.8}$$

Lemma 3.5. *The map μ induces an isomorphism at the level of cohomologies, i.e.,*

$$H^\bullet(L_+, \mathfrak{g}, M_{\mathfrak{g},+}) \simeq H^\bullet(L_+\mathfrak{h}, \mathbb{C}).$$

In particular, the cohomology class of a cocycle in the complex $C^i(L_+\mathfrak{g}, M_{\mathfrak{g},+})$ is uniquely determined by its restriction to $\bigwedge^i L_+\mathfrak{h}$.

Proof. It follows from the construction of the Serre-Hochshield spectral sequence (see [Fu], § 5.1) that the restriction map $C^\bullet(L_+\mathfrak{b}_-, \mathbb{C}) \to C^\bullet(L_+\mathfrak{h}, \mathbb{C})$ induces an isomorphism at the level of cohomologies. The statement of the lemma now follows by combining this with Lemma 3.4. □

Since $L_+\mathfrak{h}$ is abelian, we have

$$H^\bullet(L_+\mathfrak{h}, \mathbb{C}) = \bigwedge{}^\bullet(L_+\mathfrak{h}),$$

and so

$$H^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+}) \simeq \bigwedge{}^\bullet(L_+\mathfrak{h}).$$

It is clear that this isomorphism is compatible with the action of T on both sides. The kernel of T acting on the right hand side is equal to the subspace of the scalars. Therefore we obtain

$$H^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) \simeq \bigwedge{}^\bullet(L_+\mathfrak{h})/(\mathrm{Im}\, T + \mathbb{C}).$$

3.4. **Restricting the cocycles.** Any cocycle in $C^i_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ is a cocycle in

$$C^i(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) = \mathrm{Hom}_{\mathrm{cont}}(\bigwedge{}^i(L\mathfrak{g}), \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}),$$

and as such it may be restricted to $\bigwedge^i(L\mathfrak{h})$.

Lemma 3.6. *Any two cocycles in $C^i_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ whose restrictions to $\bigwedge^i(L\mathfrak{h})$ coincide represent the same cohomology class.*

Proof. We need to show that any cocycle ϕ in $C^i_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$, whose restriction to $\bigwedge^i(L\mathfrak{h})$ is equal to zero, is equivalent to the zero cocycle. According to the above computation, ϕ may be written as $\int Y(A,z)dz$, where A is a cocycle in $C^i(L_+\mathfrak{g}, M_{\mathfrak{g},+}) = \mathrm{Hom}_{\mathrm{cont}}(\bigwedge^i L_+\mathfrak{g}, M_{\mathfrak{g},+})$. But then the restriction of A to $\bigwedge^i L_+\mathfrak{h} \subset \bigwedge^i L_+\mathfrak{g}$, denoted by $\overline{A}$, must be in the image of the operator T, and hence so is $p(\overline{A}) = \mu(A) \in \bigwedge^i(L_+\mathfrak{h})$ (here p is the projection defined in (3.8) and μ is the map defined in (3.7)). Thus, $\mu(A) = T(h)$ for some $h \in \bigwedge^i(L_+\mathfrak{h})$. Since T commutes with the differential and the kernel of T on $\bigwedge^i(L_+\mathfrak{h})$ consists of the scalars, we obtain that h is also a cocycle in $C^i(L_+\mathfrak{h}, \mathbb{C}) = \bigwedge^i(L_+\mathfrak{h})$.

According to Lemma 3.5, the map μ induces an isomorphism on the cohomologies. Hence h is equal to $\mu(B)$ for some cocycle B in $C^i(L_+\mathfrak{g}, M_{\mathfrak{g},+})$. It is clear from the definition that μ commutes with the action of the translation operator T. Therefore it follows that the cocycles A and $T(B)$ are equivalent in $C^i(L_+\mathfrak{g}, M_{\mathfrak{g},+})$. But then ϕ is equivalent to the zero cocycle. □

Now we obtain

Theorem 3.7. *The cocycles* ω *and* $i_*(\sigma)$ *represent the same cohomology class in* $H^2_{\rm loc}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\rm loc})$*. Therefore there exists a lifting of the homomorphism* $L\mathfrak{g} \to \mathcal{T}^{\mathfrak{g}}_{\rm loc}$ *to a homomorphism* $\widehat{\mathfrak{g}}_{\kappa_c} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\rm loc}$ *such that* $K \mapsto 1$*. Moreover, this homomorphism may be chosen in such a way that*

$$J^a(z) \mapsto Y(P_a(a^*_{\alpha,0}, a_{\beta,-1})|0\rangle, z) + Y(B_a, z), \tag{3.9}$$

where P_a *is a polynomial introduced in formula* (1.6) *and* B_a *is a polynomial in the* $a^*_{\alpha,n}$*'s of degree one.*

Proof. By Lemma 3.6, it suffices to check that the restrictions of ω and $i_*(\sigma)$ to $\bigwedge^2(L\mathfrak{h})$ coincide. We have

$$i_*(\sigma)(h_n, h'_m) = n\kappa_c(h, h')\delta_{n,-m}$$

for all $h, h' \in \mathfrak{h}$. Now let us compute the restriction of ω. We find from the formulas given in Section 1.5 and Section 2.2 that

$$\imath(h(z)) = -\sum_{\alpha\in\Delta_+} \alpha(h) :a^*_\alpha(z)a_\alpha(z):$$

(recall that $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}$). Therefore we find that

$$\omega(h_n, h'_m) = -n\delta_{n,-m} \sum_{\alpha\in\Delta_+} \alpha(h)\alpha(h') = n\kappa_c(h, h')\delta_{n,-m},$$

because by definition $\kappa_c(\cdot,\cdot) = -\frac{1}{2}\kappa_K(\cdot,\cdot)$ and for the Killing form κ_K we have

$$\kappa_K(h, h') = 2\sum_{\alpha\in\Delta_+} \alpha(h)\alpha(h').$$

Therefore the cocycles ω and $i_*(\sigma)$ represent the same class in $H^2_{\rm loc}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\rm loc})$.

Hence there exists $\gamma \in C^1_{\rm loc}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\rm loc})$ such that $\omega + d\gamma = i_*(\sigma)$. We may write γ as

$$\gamma = \sum_a \int \psi^*_a(z)Y(B_a, z)\, dz, \qquad B_a \in M_{\mathfrak{g},+}.$$

It follows from the computations made in the proof of Lemma 3.2 that both ω and $i_*(\sigma)$ are homogeneous elements of $C^2_{\rm loc}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\rm loc})$ of degree 0. Therefore γ may be chosen to be of degree 0, i.e., B_a may be chosen to be of degree 1.

Then by Lemma 2.6 formulas (3.9) define a homomorphism of Lie algebras $\widehat{\mathfrak{g}}_{\kappa_c} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\rm loc}$ such that $K \mapsto 1$. This completes the proof. $\square$

In the next section we will interpret the above homomorphism in the vertex algebra language.

4. Wakimoto modules of critical level

4.1. **Homomorphism of vertex algebras.** Recall the definition of the vertex algebra associated to the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa$ (see [FB], § 2.4). Define the vacuum $\widehat{\mathfrak{g}}_\kappa$–module $V_\kappa(\mathfrak{g})$ corresponding to κ as the induced representation

$$V_\kappa(\mathfrak{g}) \overset{\rm def}{=} \operatorname{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{L_+\mathfrak{g}\oplus\mathbb{C}K} \mathbb{C}_1 = U(\widehat{\mathfrak{g}}_\kappa) \underset{U(L_+\mathfrak{g}\oplus\mathbb{C}K)}{\otimes} \mathbb{C}_1,$$

where $\mathbb{C}_1$ is the one-dimensional representation on which $L_+\mathfrak{g}$ acts by 0 and K acts as the identity. We denote the vector $1 \otimes 1 \in V_\kappa(\mathfrak{g})$ where the first 1 is the unit in $U(\widehat{\mathfrak{g}}_\kappa)$ and the second 1 is the generator of the one-dimensional module $\mathbb{C}_1$, by v_κ.

Recall that κ is unique up to a scalar. Therefore it is often convenient to fix a particular invariant inner product κ_0 and write an arbitrary one as $\kappa = k\kappa_0, k \in \mathbb{C}$. This is the point of view taken in [FB], where as κ_0 we take the inner product with respect to which the square of the maximal root is equal to 2, and denote the corresponding vacuum module by $V_k(\mathfrak{g})$. In particular, since the Killing form $\kappa_K(\cdot,\cdot)$ is equal to $2h^\vee$ times κ_0 (see [K1]), where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, we obtain that for the inner product $\kappa_c(\cdot,\cdot) = -\frac{1}{2}\kappa_K(\cdot,\cdot)$ introduced above we have $k = -h^\vee$. Thus, $V_{\kappa_c}(\mathfrak{g})$ is $V_{-h^\vee}(\mathfrak{g})$ in the notation of [FB]. We will call this module the vacuum module of *critical level.*

The vertex algebra structure on $V_\kappa(\mathfrak{g})$ is defined as follows:

- Gradation: $\deg J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_\kappa = -\sum_{i=1}^m n_i$.
- Vacuum vector: $|0\rangle = v_\kappa$.
- Translation operator: $Tv_\kappa = 0$, $[T, J^a_n] = -nJ^a_{n-1}$.
- Vertex operators:

$$Y(J^a_{-1}v_\kappa, z) = J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1},$$

$$Y(J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_\kappa, z) = \prod_{i=1}^m \frac{1}{(-n_i-1)!} {:}\partial_z^{-n_1-1} J^{a_1}(z) \dots \partial_z^{-n_m-1} J^{a_m}(z){:}$$

(see Theorem 2.4.4 of [FB]).

Lemma 4.1. *Defining a homomorphism of vertex algebras $V_\kappa(\mathfrak{g}) \to V$ is equivalent to choosing vertex operators $\widetilde{J}^a(z), a = 1, \dots, \dim \mathfrak{g}$, of conformal dimension 1 in the vertex algebra V, whose Fourier coefficients satisfy the relations (2.7) with $K = 1$.*

Proof. Given a homomorphism $\rho : V_\kappa(\mathfrak{g}) \to V$, take the images of the generating vertex operators $J^a(z)$ of $V_\kappa(\mathfrak{g})$ under ρ. The fact that ρ is a homomorphism of vertex algebras implies that the OPEs between these fields, and hence the commutation relations between their Fourier coefficients, are preserved.

Conversely, suppose that we are given vertex operators $\widetilde{J}^a(z)$ satisfying the condition of the lemma. Denote by $\widetilde{J}^a_n$ the corresponding Fourier coefficients. Define a linear map $\rho : V_\kappa(\mathfrak{g}) \to V$ by the formula

$$J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_\kappa \mapsto \widetilde{J}^{a_1}_{n_1} \dots \widetilde{J}^{a_m}_{n_m}|0\rangle.$$

It is easy to check that this map is a vertex algebra homomorphism. □

The complexes $C^\bullet(L_+\mathfrak{g}, M_{\mathfrak{g},+})$ and $C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ carry natural $\mathbb{Z}$–gradations defined by the formulas $\deg a^*_{\alpha,n} = \deg \psi^*_{a,n} = -n$, $\deg |0\rangle = 0$, and $\deg \int Y(A,z)dz = \deg A - 1$. The differentials preserve these gradations. The following is a corollary of Theorem 3.7.

Corollary 4.2. *There exists a homomorphism of vertex algebras $V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}}$.*

Proof. According to Theorem 3.7, there exists a homomorphism $\widehat{\mathfrak{g}}_{\kappa_c} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$ such that $K \mapsto 1$ and

$$J^a(z) \mapsto Y(P_a(a^*_{\alpha,0}, a_{\beta,-1})|0\rangle, z) + Y(B_a, z),$$

where the vectors P_a and B_a have degree one. Therefore by Lemma 4.1 we obtain a homomorphism of vertex algebras $V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}}$ such that

$$J^a_{-1} v_\kappa \mapsto P_a(a^*_{\alpha,0}, a_{\beta,-1})|0\rangle + B_a.$$

□

In addition to the $\mathbb{Z}$–gradations described above, the complex $C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ also carries a weight gradation with respect to the root lattice of $\mathfrak{g}$ such that $\mathrm{wt}\, a^*_{\alpha,n} = -\alpha$, $\mathrm{wt}\, \psi^*_{a,n} = -\mathrm{wt}\, J^a$, $\mathrm{wt}\, |0\rangle = 0$, and $\mathrm{wt} \int Y(A,z) dz = \mathrm{wt}\, A$. The differentials preserve this gradation, and it is clear that the cocycles ω and $i_*(\sigma)$ are homogeneous. Therefore the element γ introduced in the proof of Corollary 4.2 may be chosen in such a way that it is also homogeneous with respect to the weight gradation.

For such γ we necessarily have $B_a = 0$ for all $J^a \in \mathfrak{h} \oplus \mathfrak{n}_+$. Furthermore, the term B_a corresponding to $J^a = f_i$ must be proportional to $a^*_{\alpha_i,-1}|0\rangle$. Using the formulas of Section 1.5 and the discussion of Section 2.2, we therefore obtain a more explicit description of the homomorphism $V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}}$:

Theorem 4.3. *There exist constants $c_i \in \mathbb{C}$ such that the Fourier coefficients of the vertex operators*

$$e_i(z) = a_{\alpha_i}(z) + \sum_{\beta \in \Delta_+} {:}P^i_\beta(a^*_\alpha(z)) a_\beta(z){:},$$

$$h_i(z) = -\sum_{\beta \in \Delta_+} \beta(h_i) {:}a^*_\beta(z) a_\beta(z){:}\,,$$

$$f_i(z) = \sum_{\beta \in \Delta_+} {:}Q^i_\beta(a^*_\alpha(z)) a_\beta(z){:} + c_i \partial_z a^*_{\alpha_i}(z),$$

where the polynomials P^i_β, Q^i_β are introduced in formulas (1.7)–(1.9)*, generate an action of $\widehat{\mathfrak{g}}_{\kappa_c}$ on $M_{\mathfrak{g}}$.*

Remark 4.4. In addition to the above homomorphism of Lie algebras $w_{\kappa_c} : \widehat{\mathfrak{g}}_{\kappa_c} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$, there is also a Lie algebra anti-homomorphism

$$w^R : L\mathfrak{n}_+ \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$$

which is induced by the right action of $\mathfrak{n}_+$ on N_+ (see Section 1.5). By construction, the images of $L\mathfrak{n}_+$ under w_{κ_c} and w^R commute. We have

$$w^R(e_i(z)) = a_{\alpha_i}(z) + \sum_{\beta \in \Delta_+} P^{R,i}_\beta(a^*_\alpha(z)) a_\beta(z), \tag{4.1}$$

where the polynomials $P^{R,i}_\beta$ are defined in Section 1.5. More generally, we have

$$w^R(e_\alpha(z)) = a_\alpha(z) + \sum_{\beta \in \Delta_+} P^{R,\alpha}_\beta(a^*_\gamma(z)) a_\beta(z), \tag{4.2}$$

for some polynomials $P_\beta^{R,\alpha}$. $\square$

4.2. **Other $\widehat{\mathfrak{g}}$–module structures on $M_{\mathfrak{g}}$.** In Theorem 4.3 we constructed the structure of a $\widehat{\mathfrak{g}}_{\kappa_c}$–module on $M_{\mathfrak{g}}$. To obtain other $\widehat{\mathfrak{g}}_{\kappa_c}$–module structures of critical level on $M_{\mathfrak{g}}$ we need to consider other homomorphisms $\widehat{\mathfrak{g}}_{\kappa_c} \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$ lifting the homomorphism $L\mathfrak{g} \to \mathcal{T}^{\mathfrak{g}}_{\mathrm{loc}}$. According to Lemma 2.6, the set of isomorphism classes of such liftings is a torsor over $H^1(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$. Since we have chosen a particular lifting in Theorem 4.3, we may identify this set with $H^1(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$.

Recall that our complex $C^\bullet_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ has a weight gradation, and our cocycle ω has weight zero. Therefore among all liftings we consider those which have weight zero. The set of such liftings is in bijection with the weight zero homogeneous component of $H^1(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$.

Lemma 4.5. *The weight* 0 *component of* $H^1(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ *is isomorphic to the (topological) dual space* $(L\mathfrak{h})^*$ *to* $L\mathfrak{h}$*.*

Proof. First we show that the weight 0 component of $H^1(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ is isomorphic to the space of weight 0 cocycles in $C^1(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$. Indeed, since $\operatorname{wt} a^*_{\alpha,n} = -\alpha$, the weight 0 part of $C^0_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}) = \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ is one-dimensional and consists of the constants. If we apply the differential to a constant we obtain 0, and so there are no coboundaries of weight 0 in $C^1(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$.

Next, we show that any weight 0 one-cocycle ϕ is uniquely determined by its restriction to $L\mathfrak{h} \subset L\mathfrak{g}$, which may be an arbitrary (continuous) linear functional on $L\mathfrak{h}$. Indeed, since the weights occurring in $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ are less than or equal to 0, the restriction of ϕ to $L_+\mathfrak{n}_-$ is equal to 0, and the restriction to $L\mathfrak{h}$ takes values in the constants $\mathbb{C} \subset \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$. Now let us fix $\phi|_{L\mathfrak{h}}$. We identify $(L\mathfrak{h})^*$ with $\mathfrak{h}^*((t))dt$ using the residue pairing, and write $\phi|_{L\mathfrak{h}}$ as $\chi(t)dt$ using this identification. Here

$$\chi(t) = \sum_{n\in\mathbb{Z}} \chi_n t^{-n-1}, \qquad \chi_n \in \mathfrak{h}^*,$$

where $\chi_n(h) = \phi(h_n)$. We denote $\langle \chi(t), h_i \rangle$ by $\chi_i(t)$.

We claim that for any $\chi(t)dt \in \mathfrak{h}^*((t))dt$, there is a unique one-cocycle ϕ of weight 0 in $C^1_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}})$ such that

$$\phi(e_i(z)) = 0, \tag{4.3}$$

$$\phi(h_i(z)) = \chi_i(z), \tag{4.4}$$

$$\phi(f_i(z)) = \chi_i(z) a^*_{\alpha,n}(z). \tag{4.5}$$

Indeed, having fixed $\phi(e_i(z))$ and $\phi(h_i(z))$ as in (4.3) and (4.4), we obtain using the formula

$$\phi\left([e_{i,n}, f_{j,m}]\right) = e_{i,n} \cdot \phi(f_{j,m}) - f_{j,m} \cdot \phi(e_{i,n})$$

that

$$\delta_{i,j}\chi_{i,n+m} = e_{i,n} \cdot \phi(f_{j,m}).$$

This equation on $\phi(f_{j,m})$ has a unique solution in $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$ of weight $-\alpha_i$, namely, the one given by formula (4.5). The cocycle ϕ, if exists, is uniquely determined once we fix its values on $e_{i,n}, h_{i,n}$ and $f_{i,n}$. Let us show that it exists. This is equivalent to showing that the Fourier coefficients of the fields

$$e_i(z) = a_{\alpha_i}(z) + \sum_{\beta\in\Delta_+} {:}P^i_\beta(a^*_\alpha(z))a_\beta(z){:}, \tag{4.6}$$

$$h_i(z) = -\sum_{\beta\in\Delta_+} \beta(h_i){:}a^*_\beta(z)a^*_\beta(z){:} + \chi_i(z), \tag{4.7}$$

$$f_i(z) = \sum_{\beta\in\Delta_+} {:}Q^i_\beta(a^*_\alpha(z))a_\beta(z){:} + c_i\partial_z a^*_{\alpha_i}(z) + \chi_i(z)a^*_{\alpha,n}(z), \tag{4.8}$$

satisfy the relations of $\widehat{\mathfrak{g}}_{\kappa_c}$ with $K = 1$.

Let us remove the normal ordering and set $c_i = 0, i = 1, \dots, \ell$. Then the corresponding Fourier coefficients are no longer well-defined as linear operators on $M_{\mathfrak{g}}$. But they are well-defined linear operators on the space $\operatorname{Fun} LU$. The resulting $L\mathfrak{g}$–module structure is easy to describe. Indeed, the Lie algebra $L\mathfrak{g}$ acts on $\operatorname{Fun} LU$ by vector fields. More generally, for any $L\mathfrak{b}_-$–module R we obtain a natural action of $L\mathfrak{g}$ on the tensor product $\operatorname{Fun} LU \widehat{\otimes} R$ (this is just the topological $L\mathfrak{g}$–module induced from the $L\mathfrak{b}_-$–module R). If we choose as R the one-dimensional representation on which all $f_{i,n}$ act by 0 and $h_{i,n}$ acts by multiplication by $\chi_{i,n}$ for all $i = 1, \dots, \ell$ and $n \in \mathbb{Z}$, then the corresponding $L\mathfrak{g}$–action on $\operatorname{Fun} LU \widehat{\otimes} R \simeq \operatorname{Fun} LU$ is given by the Fourier coefficients of the above formulas, with the normal ordering removed. Hence if we remove the normal ordering and set $c_i = 0$, then these Fourier coefficients do satisfy the commutation relations of $L\mathfrak{g}$.

When we restore the normal ordering, these commutation relations may in general be distorted, due to the double contractions, as explained in Section 2.9. But we know from Theorem 4.3 that when we restore normal ordering and set all $\chi_{i,n} = 0$, then there exist the numbers c_i such that these Fourier coefficients satisfy the commutation relations of $\widehat{\mathfrak{g}}_{\kappa_c}$ with $K = 1$. The terms (4.3)–(4.5) will not generate any new double contractions in the commutators. Therefore computing these commutators with nonzero values of $\chi_{i,n}$, we find that the relations remain the same. This completes the proof. $\square$

Corollary 4.6. *For each $\chi(t) \in \mathfrak{h}^*((t))$ there is a $\widehat{\mathfrak{g}}$–module structure of critical level on $M_{\mathfrak{g}}$, with the action given by formulas* (4.6)–(4.8).

We call these modules the *Wakimoto modules of critical level* and denote them by $W_{\chi(t)}$.

Let π_0 be the commutative algebra $\mathbb{C}[b_{i,n}]_{i=1,\dots,\ell;n<0}$ with the derivation T given by the formula

$$T \cdot b_{i_1,n_1} \dots b_{i_m,n_m} = -\sum_{j=1}^m n_j b_{i_1,n_1} \dots b_{i_j,n_j-1} \cdots b_{i_m,n_m}.$$

Then π_0 is naturally a commutative vertex algebra (see [FB], § 2.3.9). In particular, we have

$$Y(b_{i,-1}, z) = b_i(z) = \sum_{n<0} b_{i,n} z^{-n-1}.$$

Using the same argument as in the proof of Lemma 4.5, we now obtain a stronger version of Corollary 4.2.

Theorem 4.7. *There exists a homomorphism of vertex algebras*

$$w_{\kappa_c} : V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0$$

such that

$$\begin{aligned} e_i(z) &\mapsto a_{\alpha_i}(z) + \sum_{\beta \in \Delta_+} {:}P^i_\beta(a^*_\alpha(z)) a_\beta(z){:}, \\ h_i(z) &\mapsto - \sum_{\beta \in \Delta_+} \beta(h_i) {:}a^*_\beta(z) a_\beta(z){:} + b_i(z), \\ f_i(z) &\mapsto \sum_{\beta \in \Delta_+} {:}Q^i_\beta(a^*_\alpha(z)) a_\beta(z){:} + c_i \partial_z a^*_{\alpha_i}(z) + b_i(z) a^*_{\alpha_i}(z), \end{aligned}$$

where the polynomials P^i_β, Q^i_β *are introduced in formulas* (1.7)–(1.9).

Thus, any module over the vertex algebra $M_{\mathfrak{g}} \otimes \pi_0$ becomes a $V_{\kappa_c}(\mathfrak{g})$–module, and hence a $\widehat{\mathfrak{g}}$–module of critical level. We will not require that the module is necessarily $\mathbb{Z}$–graded. In particular, for any $\chi(t) \in \mathfrak{h}^*((t))$ we have a one-dimensional π_0–module $\mathbb{C}_{\chi(t)}$, on which $b_{i,n}$ acts by multiplication by $\chi_{i,n}$. The corresponding $\widehat{\mathfrak{g}}_{\kappa_c}$–module is the Wakimoto module $W_{\chi(t)}$ introduced above.

4.3. **Application: proof of the Kac–Kazhdan conjecture.** As an application of the above construction of Wakimoto modules, we give a proof of the Kac–Kazhdan conjecture from [KK], following [F1].

Let us recall the notion of a character of a $\widehat{\mathfrak{g}}_\kappa$–module. Suppose that we have a $\widehat{\mathfrak{g}}_\kappa$–module M equipped with an action of the gradation operator $L_0 = -t\partial_t$. Note that if $\kappa \neq \kappa_c$, then any smooth $\widehat{\mathfrak{g}}_\kappa$–module (i.e., such that any vector is annihilated by the Lie subalgebra $\mathfrak{g} \otimes t^N \mathbb{C}[[t]]$ for sufficiently large N) carries an action of the Virasoro algebra obtained via the Segal–Sugawara construction (see Section 5.3 below), and so in particular an L_0–action. However, smooth $\widehat{\mathfrak{g}}_{\kappa_c}$–modules do not necessarily carry an L_0–action.

Suppose in addition that L_0 and $\mathfrak{h} \otimes 1 \subset \widehat{\mathfrak{g}}_\kappa$ act diagonally on M with finite-dimensional common eigenspaces. Then we define the character of M as the formal series

$$\operatorname{ch} M = \sum_{\widehat{\lambda} \in (\mathfrak{h} \oplus \mathbb{C}L_0)^*} \dim M(\widehat{\lambda})\, e^{\widehat{\lambda}}, \tag{4.9}$$

where $M(\widehat{\lambda})$ is the generalized eigenspace of L_0 and $\mathfrak{h} \otimes 1$ corresponding to $\widehat{\lambda} : (\mathfrak{h} \oplus \mathbb{C}L_0)^* \to \mathbb{C}$.

The direct sum $(\mathfrak{h} \otimes 1) \oplus \mathbb{C}L_0 \oplus \mathbb{C}K$ is the Cartan subalgebra of the extended Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa \oplus \mathbb{C}L_0$. Elements of the dual space are called weights. We will consider the weights occurring in modules on which K acts as the identity. Therefore without loss of generality we may view these weights as elements of the dual space to $\widetilde{\mathfrak{h}} = (\mathfrak{h} \otimes 1) \oplus \mathbb{C}L_0$. The set of positive roots of $\widehat{\mathfrak{g}}$ is naturally a subset of $\widetilde{\mathfrak{h}}^*$, and this determines a natural partial order on the set of weights. In what follows we will often denote $e^{-\delta} = e^{-(0,1)}$, where δ is the imaginary root of $\mathfrak{g}$, by q.

For $\widehat{\lambda} = (\lambda, a)$, let $M_{\widehat{\lambda},\kappa}$ be the Verma module over $\widehat{\mathfrak{g}}_\kappa$ with highest weight $\widehat{\lambda}$,

$$M_{\widehat{\lambda},\kappa} = \operatorname{Ind}^{\widehat{\mathfrak{g}}_\kappa}_{\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \oplus K} \mathbb{C}_{\widehat{\lambda}},$$

where the Lie algebra $\widetilde{\mathfrak{n}}_+ = (\mathfrak{g} \otimes t\mathbb{C}[[t]]) \oplus (\mathfrak{n}_+ \otimes 1)$ acts trivially on $\mathbb{C}_{\widehat{\lambda}}$, $\widetilde{\mathfrak{h}}$ acts according to the weight $\widehat{\lambda}$ and K acts as the identity. By the PBW theorem, as a vector space $M_{\widehat{\lambda},\kappa}$ is isomorphic to $U(\widetilde{\mathfrak{n}}_-)$, where $\widetilde{\mathfrak{n}}_- = (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{n}_- \otimes 1)$. Therefore we obtain the following formula for the character of $M_{\widehat{\lambda},\kappa}$:

$$\operatorname{ch} M_{\widehat{\lambda},\kappa} = e^{\widehat{\lambda}} \prod_{\alpha \in \widehat{\Delta}_+} (1 - e^{-\alpha})^{-1},$$

where $\widehat{\Delta}_+$ is the set of positive roots of $\widehat{\mathfrak{g}}_\kappa$.

It is well-known that as a module over the extended Kac-Moody algebra, $\widehat{\mathfrak{g}}_\kappa \oplus \mathbb{C}L_0$, the Verma module $M_{\widehat{\lambda},\kappa}$ has a unique irreducible quotient, which we denote by $L_{\widehat{\lambda},\kappa}$.

In [KK] a certain subset $H^\kappa_{\beta,m} \in \widetilde{\mathfrak{h}}^*$ is defined for any pair (β, m), where β is a positive root of $\widehat{\mathfrak{g}} \oplus \mathbb{C}L_0$ and m is a positive integer. If β is a real root, then $H^\kappa_{\beta,m}$ is a hyperplane in $\widetilde{\mathfrak{h}}^*$ and if β is an imaginary root, then $H^{\kappa_c}_{\beta,m} = \widetilde{\mathfrak{h}}^*$ and $H^\kappa_{\beta,m} = \emptyset$ for $\kappa \neq \kappa_c$. It is shown in [KK] that $L_{\widehat{\lambda},\kappa}$ is a subquotient of $M_{\widehat{\mu},\kappa}$ if and only if the following condition is satisfied: there exists a finite sequence of weights $\widehat{\mu}_0, \ldots, \widehat{\mu}_n$ such that $\widehat{\mu}_0 = \widehat{\lambda}$, $\widehat{\mu}_n = \widehat{\mu}$, $\widehat{\mu}_{i+1} = \widehat{\mu}_i - m_i\beta_i$ for some positive roots β_i and positive integers m_i, and $\widehat{\mu}_i \in H^\kappa_{\beta_i,m_i}$ for all $i = 1, \ldots, n$.

Denote by $\widehat{\Delta}^{\mathrm{re}}_+$ the set of positive real roots of $\widehat{\mathfrak{g}}_\kappa$. Let us call a weight $\widehat{\lambda}$ a *generic weight of critical level* if $\widehat{\lambda}$ does not belong to any of the hyperplanes $H^\kappa_{\beta,m}, \beta \in \widehat{\Delta}^{\mathrm{re}}_+$. It is easy to see from the above condition that $\widehat{\lambda}$ is a generic weight of critical level if and only if the only irreducible subquotients of $M_{\widehat{\lambda},\kappa_c}$ have highest weights $\widehat{\lambda} - n\delta$, where n is a non-negative integer (i.e., their $\mathfrak{h}^*$ components are equal to the $\mathfrak{h}^*$ component of $\widehat{\lambda}$). The following assertion is the Kac–Kazhdan conjecture for the untwisted affine Kac-Moody algebras.

Theorem 4.8. *For generic weight $\widehat{\lambda}$ of critical level*

$$\operatorname{ch} L_{\widehat{\lambda},\kappa_c} = e^{\widehat{\lambda}} \prod_{\alpha \in \widehat{\Delta}^{\mathrm{re}}_+} (1 - e^{-\alpha})^{-1}.$$

Proof. Without loss of generality, we may assume that $\widehat{\lambda} = (\lambda, 0)$.

Introduce the gradation operator L_0 on the Wakimoto module $W_{\chi(t)}$ by using the vertex algebra gradation on $M_{\mathfrak{g}}$. It is clear from the formulas defining the $\widehat{\mathfrak{g}}_{\kappa_c}$–action on $W_{\chi(t)}$ given in Theorem 4.7 that this action is compatible with the gradation if and only if $\chi(t) = \lambda/t$, where $\lambda \in \mathfrak{h}^*$. In that case

$$\operatorname{ch} W_{\lambda/t} = e^{\widehat{\lambda}} \prod_{\alpha \in \widehat{\Delta}_+^{\mathrm{re}}} (1 - e^{-\alpha})^{-1},$$

where $\widehat{\lambda} = (\lambda, 0)$. Thus, in order to prove the theorem we need to show that if $\widehat{\lambda}$ is a generic weight of critical level, then $W_{\lambda/t}$ is irreducible. Suppose that this is not so. Then either $W_{\lambda/t}$ contains a singular vector, i.e., a vector annihilated by the Lie subalgebra $\widetilde{\mathfrak{n}}_+ = (\mathfrak{g} \otimes t\mathbb{C}[[t]]) \oplus (\mathfrak{n}_+ \otimes 1)$, other than the multiples of the highest weight vector, or $W_{\lambda/t}$ is not generated by its highest weight vector.

Suppose that $W_{\lambda/t}$ contains a singular vector other than a multiple of the highest weight vector. Such a vector must then be annihilated by the Lie subalgebra $L_+\mathfrak{n}_+ = \mathfrak{n}_+[[t]]$. We have introduced in Remark 4.4 the right action of $\mathfrak{n}_+((t))$ on $M_{\mathfrak{g}}$, which commutes with the left action. It is clear from formula (4.2) that the monomials

$$\prod_{n_{r(\alpha)} < 0} e^R_{\alpha, n_{r(\alpha)}} \prod_{m_{s(\alpha)} \leq 0} a^*_{\alpha, m_{s(\alpha)}} |0\rangle \tag{4.10}$$

form a basis of $M_{\mathfrak{g}}$. Therefore the space of $L_+\mathfrak{n}_+$–invariants of $W_{\kappa_c,0}$ is equal to the tensor product of the subspace $M_{\mathfrak{g},-}$ of W_{0,κ_c} spanned by all monomials (4.10) not containing $a^*_{\alpha,n}$, and the space of $L_+\mathfrak{n}_+$–invariants in $M_{\mathfrak{g},+} = \mathbb{C}[a^*_{\alpha,n}]_{\alpha \in \Delta_+, n \leq 0}$. According to Section 3.3, $M_{\mathfrak{g},+}$ is an $L_+\mathfrak{g}$–module isomorphic to $\operatorname{Coind}_{L_+\mathfrak{b}_-}^{L_+\mathfrak{g}} \mathbb{C}$. Therefore the action of $L_+\mathfrak{n}_+$ on it is co-free, and the space of $L_+\mathfrak{n}_+$–invariants is one-dimensional, spanned by constants. Thus, we obtain that the space of $L_+\mathfrak{n}_+$–invariants in $W_{\lambda/t}$ is equal to $M_{\mathfrak{g},-}$. In particular, we find that the weight of any singular vector of $W_{\lambda/t}$ which is not equal to the highest weight vector has the form $(\lambda, 0) - \sum_j (n_j \delta - \beta_j)$, where $n_j > 0$ and each β_j is a positive root of $\mathfrak{g}$. But then $W_{\lambda/t}$ contains an irreducible subquotient of such a weight.

Now observe that

$$\operatorname{ch} M_{(\lambda,0),\kappa_c} = \prod_{n>0} (1 - q^n)^{-\ell} \cdot \operatorname{ch} W_{\lambda/t},$$

where $q = e^{-\delta}$. If an irreducible module $L_{\widehat{\mu},\kappa_c}$ appears as a subquotient of $W_{\lambda/t}$, then it appears in the decomposition of $\operatorname{ch} W_{\lambda/t}$ into the sum of characters of irreducible representations and hence in the decomposition of $\operatorname{ch} M_{(\lambda,0),\kappa_c}$. Since the characters of irreducible representations are linearly independent, this implies that $L_{\widehat{\mu},\kappa_c}$ is an irreducible subquotient of $M_{(\lambda,0),\kappa_c}$. But this contradicts our assumption that $(\lambda, 0)$ is a generic weight of critical level. Therefore $W_{\lambda/t}$ does not contain any singular vectors other than the multiples of the highest weight vector.

Now suppose that $W_{\lambda/t}$ is not generated by its highest weight vector. But then there exists a homogeneous linear functional on $W_{\lambda/t}$, whose weight is less than the highest weight and which is invariant under $\widetilde{\mathfrak{n}}_- = (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{n}_- \otimes 1)$, and in particular, under its Lie subalgebra $L_-\mathfrak{n}_+ = \mathfrak{n}_+ \otimes t^{-1}\mathbb{C}[t^{-1}]$. Therefore this functional factors through the space of coinvariants of $W_{\lambda/t}$ by $L_-\mathfrak{n}_+$. But $L_-\mathfrak{n}_+$ acts freely on $W_{\lambda/t}$, and the space of coinvariants is isomorphic to the subspace $\mathbb{C}[a^*_{\alpha,n}]_{\alpha \in \Delta_+, n \leq 0}$ of $W_{\lambda/t}$. Hence we obtain that the weight of this functional has the form $(\lambda, 0) - \sum_j (n_j \delta + \beta_j)$, where

$n_j \geq 0$ and each β_j is a positive root of $\mathfrak{g}$. In the same way as above, it follows that this contradicts our assumption that λ is a generic weight. Therefore $W_{\lambda/t}$ is generated by its highest weight vector. This completes the proof. □

5. Deforming to other levels

5.1. **Homomorphism of vertex algebras.** As before, we denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$. Let $\widehat{\mathfrak{h}}_\kappa$ be the one-dimensional central extension of the loop algebra $L\mathfrak{h} = \mathfrak{h}((t))$ with the two-cocycle obtained by restriction of the two-cocycle on $L\mathfrak{g}$ corresponding to the inner product κ. Then according to formula (2.7), $\widehat{\mathfrak{h}}_\kappa$ is a Heisenberg Lie algebra. We will consider a copy of this Lie algebra with generators $b_{i,n}, i = 1, \dots, \ell, n \in \mathbb{Z}$, and $\mathbf{1}$ with the commutation relations

$$[b_{i,n}, b_{j,m}] = n\kappa(h_i, h_j)\mathbf{1}\delta_{n,-m}.$$

Thus, the $b_{i,n}$'s satisfy the same relations as the $h_{i,n}$'s. Let π_0^κ denote the $\widehat{\mathfrak{h}}_\kappa$–module induced from the one-dimensional representation of the abelian Lie subalgebra of $\widehat{\mathfrak{h}}_\kappa$ spanned by $b_{i,n}, i = 1, \dots, \ell, n \geq 0$, and $\mathbf{1}$, on which $\mathbf{1}$ acts as the indentity and all other generators act by 0. We denote by $|0\rangle$ the generating vector of this module. It satisfies: $b_{i,n}|0\rangle = 0, n \geq 0$. Then π_0^κ has the following structure of a vertex algebra (see Theorem 2.3.7 of [FB]):

- Gradation: $\deg b_{i_1,n_1} \dots b_{i_m,n_m}|0\rangle = -\sum_{i=1}^m n_i$.
- Vacuum vector: $|0\rangle$.
- Translation operator: $T|0\rangle = 0$, $[T, b_{i,n}] = -nb_{i,n-1}$.
- Vertex operators:

$$Y(b_{i,-1}|0\rangle, z) = b_i(z) = \sum_{n\in\mathbb{Z}} b_{i,n} z^{-n-1},$$

$$Y(b_{i_1,n_1} \dots b_{i_m,n_m}|0\rangle, z) = \prod_{j=1}^{n} \frac{1}{(-n_j - 1)!} \,{:}\partial_z^{-n_1-1} b_{i_1}(z) \dots \partial_z^{-n_m-1} b_{i_m}(z){:}\, .$$

The tensor product $M_\mathfrak{g} \otimes \pi_0^{\kappa-\kappa_c}$ also acquires a vertex algebra structure.

Theorem 5.1. *There exists a homomorphism of vertex algebras*

$$w_\kappa : V_\kappa(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0^{\kappa-\kappa_c}$$

such that

$$\begin{aligned} e_i(z) &\mapsto a_{\alpha_i}(z) + \sum_{\beta\in\Delta_+} {:}P^i_\beta(a^*_\alpha(z)) a_\beta(z){:}\,, \\ h_i(z) &\mapsto -\sum_{\beta\in\Delta_+} \beta(h_i){:}a^*_\beta(z) a_\beta(z){:} + b_i(z), \\ f_i(z) &\mapsto \sum_{\beta\in\Delta_+} {:}Q^i_\beta(a^*_\alpha(z)) a_\beta(z){:} + (c_i + (\kappa - \kappa_c)(e_i, f_i))\, \partial_z a^*_{\alpha_i}(z) + b_i(z) a^*_{\alpha_i}(z), \end{aligned} \tag{5.1}$$

where the polynomials P^i_β, Q^i_β are introduced in formulas (1.7)–(1.9).

Proof. Denote by $\widetilde{\mathcal{A}}^{\mathfrak{g}}_{\text{loc}}$ the Lie algebra $U(M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c})$. By Lemma 4.1, in order to prove the theorem, we need to show that formulas (5.1) define a homomorphism of Lie algebras $\widehat{\mathfrak{g}}_\kappa \to \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\text{loc}}$ sending the central element K to the identity.

Formulas (5.1) certainly define a linear map $\overline{w}_\kappa : L\mathfrak{g} \to \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\text{loc}}$. Denote by ω_κ the linear map $\bigwedge^2 L\mathfrak{g} \to \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\text{loc}}$ defined by the formula

$$\omega_\kappa(f,g) = [\overline{w}_\kappa(f), \overline{w}_\kappa(g)] - \overline{w}_\kappa([f,g]).$$

Evaluating it explicitly in the same way as in the proof of Lemma 3.2, we find that ω_κ takes values in $\mathcal{A}^{\mathfrak{g}}_{0,\text{loc}} \subset \widetilde{\mathcal{A}}^{\mathfrak{g}}_{\text{loc}}$. Furthermore, by construction of $\overline{w}_\kappa$, for any $X \in \mathcal{A}^{\mathfrak{g}}_{0,\text{loc}}$ and $f \in L\mathfrak{g}$ we have $[\overline{w}_\kappa(f), X] = f \cdot X$, where in the right hand side we consider the action of f on the $L\mathfrak{g}$–module $\mathcal{A}^{\mathfrak{g}}_{0,\text{loc}}$. This immediately implies that ω_κ is a two-cocycle of $L\mathfrak{g}$ with coefficients in $\mathcal{A}^{\mathfrak{g}}_{0,\text{loc}}$. By construction, ω_κ is local, i.e., belongs to $C^2_{\text{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\text{loc}})$.

Let us compute the restriction of ω_κ to $\bigwedge^2 L\mathfrak{h}$. The calculations made in the proof of Lemma 3.7 imply that

$$\omega_\kappa(h_n, h'_m) = n(\kappa_c(h,h') + (\kappa - \kappa_c)(h,h')) = n\kappa(h,h').$$

Therefore this restriction is equal to the restriction of the Kac-Moody two-cocycle σ_κ on $L\mathfrak{g}$ corresponding to κ. Now Lemma 3.6 implies that the two-cocycle ω_κ is cohomologically equivalent to $i_*(\sigma_\kappa)$. We claim that it is actually equal to $i_*(\sigma_\kappa)$.

Indeed, the difference between these cocycles is the coboundary of some element $\gamma \in C^1_{\text{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g}}_{0,\text{loc}})$. The discussion before Theorem 4.3 implies that $\gamma(e_i(z)) = \gamma(h_i(z)) = 0$ and $\gamma(f_i(z)) = c'_i \partial_z a^*_{\alpha_i}(z)$ for some constants $c'_i \in \mathbb{C}$. In order to find the constants c'_i we compute the value of the corresponding two-cocycle $\omega_\kappa + d\gamma$ on $e_{i,n}$ and $f_{i,-n}$. We find that it is equal to $n\sigma_\kappa(e_{i,n}, f_{i,-n}) + c'_i n(e_i, f_i)$. Therefore $c'_i = 0$ for all $i = 1, \ldots, \ell$, and so $\gamma = 0$ and $\omega_\kappa = i_*(\sigma_\kappa)$. This implies that formulas (5.1) indeed define a homomorphism of Lie algebras $\widehat{\mathfrak{g}}_\kappa \to \mathcal{A}^{\mathfrak{g}}_{\leq 1,\text{loc}}$ sending the central element K to the identity. This completes the proof. □

The following result is useful in applications.

Proposition 5.2. *The homomorphism w_κ of Theorem 5.1 is injective for any κ.*

Proof. We will introduce filtrations on $V_\kappa(\mathfrak{g})$ and $W_{0,\kappa} = M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ which are preserved by w_κ, and then show that the induced map $\operatorname{gr} w_\kappa : \operatorname{gr} V_\kappa(\mathfrak{g}) \to \operatorname{gr} W_{0,\kappa}$ (which turns out to be independent of κ) is injective.

In order to define a filtration on $V_\kappa(\mathfrak{g})$ we observe that

$$V_\kappa(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])v_\kappa$$

and use the Poincaré–Birkhoff–Witt filtration on $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$. Now we define a filtration $\{W^{\leq p}_{0,\kappa}\}$ on $W_{0,\kappa}$ by defining $W^{\leq p}_{0,\kappa}$ to be the span of monomials in the $a_{\alpha,n}$'s, $a^*_{\alpha,n}$'s and $b_{i,n}$'s whose combined degree in the $a_{\alpha,n}$'s and $b_{i,n}$'s is less than or equal to p (this is analogous to the filtration by the order of differential operator). It is clear from the construction of the homomorphism w_κ that it preserves these filtrations.

Now we describe the corresponding operator $\operatorname{gr} w_\kappa : \operatorname{gr} V_\kappa(\mathfrak{g}) \to \operatorname{gr} W_{0,\kappa}$. Let $\widetilde{\mathfrak{g}}$ be the variety of pairs $(\mathfrak{b}, x)$, where $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g}$ and $x \in \mathfrak{b}$. The natural morphism $\widetilde{\mathfrak{g}} \to \mathrm{Fl}$, where Fl is the flag variety of $\mathfrak{g}$, mapping $(\mathfrak{b}, x)$ to $\mathfrak{b} \in \mathrm{Fl}$ identifies $\widetilde{\mathfrak{g}}$ with a vector bundle over the flag variety Fl, whose fiber over $\mathfrak{b} \in \mathrm{Fl}$ is the vector space $\mathfrak{b}$. There is also a morphism $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ sending $(\mathfrak{b}, x)$ to x. Now let U be again the big cell, i.e., the open N_+–orbit of Fl, and $\widetilde{U}$ its preimage in $\widetilde{\mathfrak{g}}$. In other words, $\widetilde{U}$ consists of those pairs $(\mathfrak{b}, x)$ for which $\mathfrak{b}$ is in generic relative position with $\mathfrak{b}_+$. It is isomorphic to an affine space of dimension equal to $\dim \mathfrak{g}$. The induced morphism $p : \widetilde{U} \to \mathfrak{g}$ is dominant and generically one-to-one.

Now let $J\widetilde{U}$ and $J\mathfrak{g}$ be the infinite jet schemes of $\widetilde{U}$ and $\mathfrak{g}$, defined as in [FB], § 9.4.4, and Jp the corresponding morphism $J\widetilde{U} \to J\mathfrak{g}$. Then Jp is clearly a dominant morphism and so the corresponding homomorphism of rings of functions $Jp : \mathbb{C}[J\mathfrak{g}] \to \mathbb{C}[J\widetilde{U}]$ is injective.

Since $J\mathfrak{g} \simeq \mathfrak{g}[[t]]$, it follows that

$$\mathrm{gr}\, V_\kappa(\mathfrak{g}) \simeq \mathrm{Sym}\, \mathfrak{g}((t))/\mathfrak{g}[[t]] \simeq \mathbb{C}[J\mathfrak{g}].$$

Moreover, $J\widetilde{U}$ is isomorphic to $\mathrm{gr}\, W_{0,\kappa}$, and it follows from our construction of w_κ that $\mathrm{gr}\, w_\kappa = Jp$. This implies the statement of the proposition. $\square$

5.2. **Wakimoto modules away from the critical level.** Any module over the vertex algebra $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ now becomes a $V_\kappa(\mathfrak{g})$–module and hence a $\widehat{\mathfrak{g}}_\kappa$–module (with K acting as 1). For $\lambda \in \mathfrak{h}^*$, let $\pi_\lambda^{\kappa-\kappa_c}$ be the Fock representation of $\widehat{\mathfrak{h}}_{\kappa-\kappa_c}$ generated by a vector $|\lambda\rangle$ satisfying

$$b_{i,n}|\lambda\rangle = 0, \quad n > 0, \qquad b_{i,0}|\lambda\rangle = \lambda(h_i)|\lambda\rangle, \qquad \mathbf{1}|\lambda\rangle = |\lambda\rangle.$$

Then

$$W_{\lambda,\kappa} \overset{\mathrm{def}}{=} M_{\mathfrak{g}} \otimes \pi_\lambda^{\kappa-\kappa_c}$$

is an $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$–module, and hence a $\widehat{\mathfrak{g}}_\kappa$–module. We call it the *Wakimoto module of level* κ *and highest weight* λ.

5.3. **Conformal structures at non-critical levels.** In this section we show that the homomorphism w_κ of Theorem 5.1 is a homomorphism of conformal vertex algebras when $\kappa \neq \kappa_c$. This will allow us to obtain a coordinate–independent version of this homomorphism. By taking the limit $\kappa \to \kappa_c$, we will also obtain a coordinate–independent version of the homomorphism w_{κ_c}.

The vertex algebra $V_\kappa(\mathfrak{g}), \kappa \neq \kappa_c$, has the standard conformal structure given by the Segal–Sugawara vector

$$\mathbf{s}_\kappa = \frac{1}{2} \sum_a J^a_{-1} J_{a,-1} v_\kappa, \tag{5.2}$$

where $\{J_a\}$ is the basis of $\mathfrak{g}$ dual to the basis $\{J^a\}$ with respect to the inner product $\kappa - \kappa_c$ (see [FB], § 2.5.10, for more details). We need to calculate the image of $\mathbf{s}_\kappa$ under w_κ.

Lemma 5.3. *The image of* $\mathbf{s}_\kappa$ *under* w_κ *is equal to*

$$\left(\sum_{\alpha \in \Delta_+} a_{\alpha,-1} a^*_{\alpha,-1} + \frac{1}{2} \sum_{i=1}^{\ell} b_{i,-1} b^i_{-1} - \rho_{-2} \right) |0\rangle, \tag{5.3}$$

where $\{b^i\}$ *is a dual basis to* $\{b_i\}$ *and* ρ *is the element of* $\mathfrak{h}$ *corresponding to* $\rho \in \mathfrak{h}^*$ *under the isomorphism induced by the inner product* $(\kappa - \kappa_c)|_{\mathfrak{h}}$.

Proof. The vector $w_\kappa(\mathbf{s}_\kappa)$ is of degree 2 with respect to the vertex algebra gradation on $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ and of weight 0 with respect to the gradation by the root system such that $\operatorname{wt} a_{\alpha,n} = -\operatorname{wt} a^*_{\alpha,n} = \alpha, \operatorname{wt} b_{i,n} = 0$. The basis in the corresponding subspace of $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ consists of the monomials of the form

$$b_{i,-1}b_{j,-1}, \qquad b_{i,-2}, \tag{5.4}$$

$$a_{\alpha,-1}a^*_{\alpha,-1}, \qquad a_{\alpha,-2}a^*_{\alpha,0}, \tag{5.5}$$

$$a_{\alpha,-1}a_{\beta,-1}a^*_{\alpha,0}a^*_{\beta,0}, \qquad a_{\alpha,-1}a^*_{\alpha,0}b_{i,-1}, \tag{5.6}$$

applied to the vacuum vector $|0\rangle$.

The Fourier coefficients $L_n, n \in \mathbb{Z}$, of the vertex operator $w_\kappa(\mathbf{s}_\kappa)$ preserve the weight gradation on $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ and $\deg L_n = -n$ with respect to the vertex algebra gradation on $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$. This implies that the vectors $a_{\alpha,-1}|0\rangle$ and $a^*_{\alpha,0}|0\rangle, \alpha \in \Delta_+$, are primary vectors, i.e., they are annihilated by $L_n, n > 0$, and are eigenvectors of L_0. We claim that L_0 acts by 1 on $a_{\alpha,-1}|0\rangle$ and by 0 on $a^*_{\alpha,0}|0\rangle$.

Indeed, the vectors $J^a_{-1}v_\kappa \in V_\kappa(\mathfrak{g})$ are primary of degree 1. Therefore the same is true for their images in $M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$. Let $\{e_\alpha\}_{\alpha\in\Delta_+}$ be a root basis of $\mathfrak{n}_+ \subset \mathfrak{g}$ such that $e_{\alpha_i} = e_i$. Then we have

$$w_\kappa(e_{\alpha,-1}v_\kappa) = \left(a_{\alpha,-1} + \sum_{\beta\in\Delta_+} P^\alpha_\beta(a^*_{\alpha,0})a_{\beta,-1} \right) |0\rangle, \tag{5.7}$$

where the polynomials P^α_β are found from the formula of the action of e_α on U:

$$e_\alpha = \frac{\partial}{\partial y_\alpha} + \sum_{\beta\in\Delta_+} P^\alpha_\beta(y_\alpha)\frac{\partial}{\partial y_\beta}.$$

In particular, $P^{\alpha_i}_\beta = P^i_\beta$ considered above. In addition we have the following formula for $w_\kappa(h_{i,-1}v_\kappa)$ which follows from Theorem 5.1:

$$w_\kappa(h_{i,-1}v_\kappa) = \left(- \sum_{\beta\in\Delta_+} \beta(h_i)a^*_{0,\beta}a_{\beta,-1} + b_{i,-1} \right) |0\rangle. \tag{5.8}$$

Using these formulas, we obtain that the vectors $a_{\alpha,-1}|0\rangle$ (resp., $a^*_{\alpha,0}|0\rangle$) are primary vectors of degree 1 (resp., 0) with respect to $w_\kappa(\mathbf{s}_\kappa)$. This readily implies that the basis vectors (5.6) do not appear in the decomposition of $w_\kappa(\mathbf{s}_\kappa)$ and that the vectors (5.5) enter in the combination

$$\sum_{\alpha\in\Delta_+} a_{\alpha,-1}a^*_{\alpha,-1}|0\rangle. \tag{5.9}$$

It remains to determine the coefficients with which the monomials (5.4) enter the formula for $w_\kappa(\mathbf{s}_\kappa)$.

Using formula (5.8) and our knowledge of the aa^* component of $w_\kappa(\mathbf{s}_\kappa)$ we find the following action of the Fourier coefficients L_n of $Y(w_\kappa(\mathbf{s}_\kappa), z)$ on the first summand of $w_\kappa(h_{i,-1}v_\kappa)$:

$$\begin{aligned} L_0 \cdot - \sum_{\beta \in \Delta_+} \beta(h_i) a^*_{0,\beta} a_{\beta,-1}|0\rangle &= - \sum_{\beta \in \Delta_+} \beta(h_i) a^*_{0,\beta} a_{\beta,-1}|0\rangle, \\ L_1 \cdot - \sum_{\beta \in \Delta_+} \beta(h_i) a^*_{0,\beta} a_{\beta,-1}|0\rangle &= - \sum_{\beta \in \Delta_+} \beta(h_i)|0\rangle = -2\rho(h_i)|0\rangle. \end{aligned}$$

and $L_n, n > 1$, act by 0. There is a unique combination of monomials (5.4) which, when added to (5.9), makes a conformal vector with respect to which $w_\kappa(h_{i,-1}v_\kappa)$ is a vector of degree 1 annihilated by $L_n, n > 0$, namely,

$$\frac{1}{2} \sum_{i=1}^{\ell} b_{i,-1} b^i_{-1} - \rho_{-2}. \tag{5.10}$$

This completes the proof. □

5.4. **Quasi-conformal structures at the critical level.** Now we use this lemma to obtain additional information about the homomorphism w_{κ_c}. Denote by O the complete topological ring $\mathbb{C}[[t]]$ and by $\operatorname{Der} O$ the Lie algebra of its continuous derivations. Note that $\operatorname{Der} O \simeq \mathbb{C}[[t]]\partial_t$.

Recall that a vertex algebra is called *quasi-conformal* if it carries an action of the Lie algebra $\operatorname{Der} O$ satisfying the conditions of Definition 6.3.4 of [FB]. In particular, a conformal vertex algebra is automatically quasi-conformal (with the $\operatorname{Der} O$–action coming from the Virasoro action).

The Lie algebra $\operatorname{Der} O$ acts naturally on $\widehat{\mathfrak{g}}_{\kappa_c}$ preserving $\mathfrak{g}[[t]]$, and hence it acts on $V_{\kappa_c}(\mathfrak{g})$. The $\operatorname{Der} O$–action on $V_{\kappa_c}(\mathfrak{g})$ coincides with the limit $\kappa \to \kappa_c$ of the $\operatorname{Der} O$–action on $V_\kappa(\mathfrak{g}), \kappa \neq \kappa_c$, obtained from the Sugawara conformal structure. Therefore this action defines the structure of a quasi-conformal vertex algebra on $V_{\kappa_c}(\mathfrak{g})$.

Next, we define the structure of a quasi-conformal algebra on $M_{\mathfrak{g}} \otimes \pi_0$ as follows. The vertex algebra $M_{\mathfrak{g}}$ is conformal with the conformal vector (5.9), and hence it is also quasi-conformal. The commutative vertex algebra π_0 is the $\kappa \to \kappa_c$ limit of the family of conformal vertex algebras $\pi_0^{\kappa-\kappa_c}$ with the conformal vector (5.10). The induced action of the Lie algebra $\operatorname{Der} O$ on $\pi_0^{\kappa-\kappa_c}$ is well-defined in the limit $\kappa \to \kappa_c$ and so it induces a $\operatorname{Der} O$–action on π_0. Therefore it gives rise to the structure of a quasi-conformal vertex algebra on π_0. The $\operatorname{Der} O$–action is in fact given by derivations of the algebra structure on $\pi_0 \simeq \mathbb{C}[b_{i,n}]$, and hence by Lemma 6.3.5 of [FB] it defines the structure of a quasi-conformal vertex algebra on π_0. Explicitly, the action of the basis elements $L_n = -t^{n+1}\partial_t, n \geq -1$, of $\operatorname{Der} O$ on π_0 is determined by the following formulas:

$$\begin{aligned} L_n \cdot b_{i,m} &= -m b_{i,n+m}, \qquad -1 \leq n < -m, \\ L_n \cdot b_{i,-n} &= n, \qquad n > 0, \\ L_n \cdot b_{i,m} &= 0, \qquad n > -m \end{aligned} \tag{5.11}$$

(note that $\rho(h_i) = 1$ for all i). Now we obtain a quasi-conformal vertex algebra structure on $M_{\mathfrak{g}} \otimes \pi_0$ by taking the sum of the above $\operatorname{Der} O$–actions.

Since the quasi-conformal structures on $V_{\kappa_c}(\mathfrak{g})$ and $M_{\mathfrak{g}} \otimes \pi_0$ both came as the limits of conformal structures as $\kappa \to \kappa_c$, we obtain the following corollary of Lemma 5.3:

Corollary 5.4. *The homomorphism* $w_{\kappa_c} : V_{\kappa_c} \to M_{\mathfrak{g}} \otimes \pi_0$ *preserves quasi-conformal structures. In other words, it commutes with the* $\operatorname{Der} O$*–action on both sides.*

5.5. **Transformation formulas for the fields.** We can now obtain the transformation formulas for the fields $a_\alpha(z), a^*_\alpha(z)$ and $b_i(z)$. According to the computations we made in the proof of Lemma 5.3, we have the following action of the basis elements $L_n, n \geq 0$, of $\operatorname{Der}_+ O$ on the vectors $a_{\alpha,-1}|0\rangle$ and $a^*_{\alpha,0}|0\rangle$:

$$L_0 \cdot a_{\alpha,-1}|0\rangle = a_{\alpha,-1}|0\rangle, \qquad L_n \cdot a_{\alpha,-1}|0\rangle = 0, \quad n > 0,$$

$$L_n \cdot a^*_{\alpha,0}|0\rangle = 0, \quad n \geq 0.$$

According to Proposition 6.4.7 of [FB], this implies that the field $a_\alpha(z) = Y(a_{\alpha,-1}|0\rangle, z)$ transforms as a one-form on the punctured disc $D^\times = \operatorname{Spec} \mathbb{C}((z))$, while the field $a^*_\alpha(z) = Y(a^*_{\alpha,0}|0\rangle, z)$ transforms as a function on $D^\times$. In particular, we obtain the following description of the module $M_{\mathfrak{g}}$. Consider the Heisenberg Lie algebra Γ which is a central extension of the commutative Lie algebra $U((t)) \oplus U^*((t))dt$ with the cocycle given by the formula

$$f(t), g(t)dt \mapsto \int \langle f(t), g(t) \rangle dt.$$

This cocycle is coordinate–independent, and therefore Γ carries natural actions of $\operatorname{Der} O$, which preserve the Lie subalgebra $\Gamma_+ = U[[t]] \oplus U[[t]]dt$. We identify the completed Weyl algebra $\widetilde{\mathcal{A}}^{\mathfrak{g}}$ with a completion of $U(\Gamma)/(\mathbf{1}-1)$, where $\mathbf{1}$ is the central element. The module $M_{\mathfrak{g}}$ is then identified with the Γ–module induced from the one-dimensional representation of $\Gamma_+ \oplus \mathbb{C}\mathbf{1}$, on which Γ_+ acts by 0, and $\mathbf{1}$ acts as the identity. The $\operatorname{Der} O$–action on $M_{\mathfrak{g}}$ considered above is nothing but the natural action on the induced module.

Now we consider the fields $b_i(z)$. We have

$$L_0 \cdot b_{i,-1}|0\rangle = b_{i,-1}|0\rangle, \qquad L_1 \cdot b_{i,-1}|0\rangle = 2|0\rangle, \qquad L_n \cdot b_{i,-1}|0\rangle = 0, \quad n > 1. \tag{5.12}$$

Recall that $b_i(z) = Y(b_{i,-1}|0\rangle, z)$. According to [FB], § 8.1.11, these formulas imply that $\partial_z + b_i(z)$ transforms as a connection on the line bundle $\Omega^{-\rho(h_i)}$ over $D^\times$.

More precisely, let ${}^L H$ be the dual group to H, i.e., it is the torus that is determined by the property that its lattice of characters ${}^L H \to \mathbb{C}^\times$ is the lattice of cocharacters $\mathbb{C}^\times \to H$, and the lattice of cocharacters of ${}^L H$ is the lattice of characters of H. The Lie algebra ${}^L\mathfrak{h}$ of ${}^L H$ is then naturally identified with $\mathfrak{h}^*$. Denote by $\Omega^{-\rho}$ the unique principal ${}^L H$–bundle on a smooth curve or a (punctured) disc such that the line bundle associated any character $\check{\lambda} : {}^L H \to \mathbb{C}^\times$ (equivalently, a cocharacter of H) is $\Omega^{-\langle \rho, \check{\lambda} \rangle}$. The space of connections on this bundle is a torsor over $\mathfrak{h}^* \otimes \Omega$.

The above statement about $b_i(z)$ may be reformulated as follows: consider the $\mathfrak{h}^*$–valued field $\mathbf{b}(z) = \sum_{i=1}^{\ell} b_i(z)\omega_i$ such that $\langle \mathbf{b}(z), h_i \rangle = b_i(z)$. Then the operator $\partial_z + \mathbf{b}(z)$ transforms as a connection on the ${}^L H$–bundle $\Omega^{-\rho}$. Equivalently, $\partial_z + b_i(z)$ transforms as a connection on the line bundle $\Omega^{-\rho(h_i)}$ over $D^\times$.

If w is a new coordinate such that $z = \varphi(w)$, then the same connection will appear as $\partial_w + \widetilde{\mathbf{b}}(w)$, where

$$\widetilde{\mathbf{b}}(w) = \varphi' \cdot \mathbf{b}(\varphi(w)) + \rho \frac{\varphi''}{\varphi'}. \tag{5.13}$$

5.6. **Coordinate–independent version.** Up to now, we have considered representations of the Lie algebra $\widehat{\mathfrak{g}}_{\kappa_c}$, which is the central extension of $L\mathfrak{g} = \mathfrak{g}((t))$. In applications, it is important to develop a theory which applies to the central extension of the Lie algebra $\mathfrak{g}(\mathcal{K}) = \mathfrak{g} \otimes \mathcal{K}$, where $\mathcal{K}$ is a topological algebra which is isomorphic to $\mathbb{C}((t))$, but non-canonically. For instance, we can take as $\mathcal{K}$ the completion of the field of rational functions on an algebraic curve X defined by a point $x \in X$. If we choose a formal coordinate t at x, we may identify $\mathcal{K}$ with $\mathbb{C}((t))$, but this identification is non-canonical as there is usually no preferred choice of coordinate t. Formula (2.7) defining the central extension of $\mathfrak{g}((t))$ is independent of the choice of t, and so the central extension is well-defined for any algebra $\mathcal{K}$ as above. We will denote it by $\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K})$ to emphasize this fact.

What is needed in order to recast the above construction of Wakimoto modules in a coordinate–independent way is precisely the knowledge of the action of the Lie algebra $\operatorname{Der} O$ and its Lie group, the group of changes of coordinates.

Let $\operatorname{Aut} O$ be the group of continuous automorphisms of $\mathbb{C}[[t]]$. Any such automorphism is determined by its action on the topological generator t of $\mathbb{C}[[t]]$, $t \mapsto \rho(t)$. This allows us to identify $\operatorname{Aut} O$ with the group of formal power series $\rho(t) = \rho_1 t + \rho_2 t^2 + \ldots$, where $\rho_1 \neq 0$ (see [FB], § 6.2, for more details). The Lie algebra of $\operatorname{Aut} O$ is $\operatorname{Der}_+ O = t\mathbb{C}[[t]]\partial_t$. Recall that $\operatorname{Der} O$ denotes the Lie algebra $\mathbb{C}[[t]]\partial_t$. Then $(\operatorname{Der} O, \operatorname{Aut} O)$ is an example of a Harish-Chandra pair (see [FB], Ch. 6 and § 17.2).

Now let $\mathcal{O} \subset \mathcal{K}$ be a pair consisting of a complete local ring $\mathcal{O}$ isomorphic to $\mathbb{C}[[t]]$ and its field of fractions $\mathcal{K}$. Let $\mathfrak{m}$ be the unique maximal ideal of $\mathcal{O}$, and $\mathcal{A}ut$ the set of topological generators of $\mathfrak{m}$ (these are the coordinates on the disc $D = \operatorname{Spec} \mathcal{O}$). Then $\mathcal{A}ut$ is naturally an $\operatorname{Aut} O$–torsor, with the (right) action given by the formula $z \mapsto \rho(z)$ for each $\rho(t) \in \operatorname{Aut} O$. Given any $\operatorname{Aut} O$–module V, we may form its twist by $\mathcal{A}ut$,

$$\mathcal{V} \overset{\text{def}}{=} \mathcal{A}ut \underset{\operatorname{Aut} O}{\times} V.$$

Let V be the module $V_\kappa(\mathfrak{g})$ defined in Section 4.1 (where we identify $L\mathfrak{g}$ with $\mathfrak{g}((t))$). Since $\mathfrak{g}((t))$ and $\mathfrak{g}[[t]]$ carry natural $\operatorname{Aut} O$–actions, so does $V_\kappa(\mathfrak{g})$. The corresponding $\mathcal{A}ut$–twist $\mathcal{V}_\kappa(\mathfrak{g})$ may alternatively be described as follows:

$$\mathcal{V}_\kappa(\mathfrak{g}) = \mathcal{A}ut \underset{\operatorname{Aut} O}{\times} V_\kappa(\mathfrak{g}) = \operatorname{Ind}_{\mathfrak{g}\otimes\mathcal{O}\oplus\mathbb{C}K}^{\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K})} \mathbb{C}_1, \tag{5.14}$$

where $\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K})$ is now the central extension of $\mathfrak{g} \otimes \mathcal{K}$ rather than $\mathfrak{g}((t))$.

We have described above an action of $\operatorname{Der} O$ on $M_{\mathfrak{g}}$ and π_0 such that the homomorphism $V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0$ of Theorem 4.7 commutes with the action of $\operatorname{Der} O$. The action of its Lie subalgebra $\operatorname{Der}_+ O$ may be exponentiated to an action of the Lie group $\operatorname{Aut} O$. Twisting the homomorphism $w_\kappa : \mathcal{V}_\kappa(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0^{\kappa-\kappa_c}$ by an $\operatorname{Aut} O$–torsor $\mathcal{A}ut$, we then obtain a description of the Wakimoto modules in a coordinate–independent fashion.

Let $\Gamma(\mathcal{K})$ the Heisenberg Lie algebra obtained in the same way as above as the central extension of $(U \otimes \mathcal{K}) \oplus (U^* \otimes \Omega_{\mathcal{K}})$. Let $\Gamma_+(\mathcal{K})$ be its commutative Lie subalgebra $(U \otimes \mathcal{O}) \oplus (U^* \otimes \Omega_{\mathcal{O}})$. Let $\mathcal{V}_\kappa(\mathfrak{g})$ be the $\widehat{\mathfrak{g}}_\kappa(\mathcal{K})$–module defined by formula (5.14) and $\mathcal{M}_{\mathfrak{g}}(\mathcal{K})$ be the $\Gamma(\mathcal{K})$–module

$$\mathcal{M}_{\mathfrak{g}}(\mathcal{K}) = \mathcal{A}ut \underset{\operatorname{Aut} O}{\times} M_{\mathfrak{g}}.$$

According to the above description of $M_{\mathfrak{g}}$ as the induced module, we have

$$\begin{aligned} \mathcal{M}_{\mathfrak{g}}(\mathcal{K}) &= \operatorname{Ind}_{\Gamma_+(\mathcal{K}) \oplus \mathbb{C}\mathbf{1}}^{\Gamma(\mathcal{K})} \mathbb{C} \\ &\simeq \operatorname{Sym}(U \otimes \mathcal{K} \oplus U^* \otimes \Omega_{\mathcal{K}})/(U \otimes \mathcal{O} \oplus U^* \otimes \Omega_{\mathcal{O}}) \\ &\simeq \operatorname{Fun}(U \otimes \Omega_{\mathcal{O}} \oplus U^* \otimes \mathcal{O}). \end{aligned}$$

The last isomorphism is obtained using the residue pairing.

Now consider the twist

$$\Pi_0^\nu = \mathcal{A}ut \underset{\operatorname{Aut} O}{\times} \pi_0^\nu,$$

This is a module over the Lie algebra

$$\widehat{\mathfrak{h}}_\nu(\mathcal{K}) = \mathcal{A}ut \underset{\operatorname{Aut} O}{\times} \widehat{\mathfrak{h}}_\nu,$$

which is a central extension of $\mathfrak{h} \otimes \mathcal{K}$. According to the transformation formula for $\mathbf{b}(z)$ obtained above, $\widehat{\mathfrak{h}}_\nu(\mathcal{K})$ may be described in the following way.

Consider the vector space $\operatorname{Conn}_{\{\lambda\}}(\Omega^{-\rho})_{D^\times}$ of λ–connections on the ${}^L H$–bundle $\Omega^{-\rho}$ on $D^\times = \operatorname{Spec} \mathcal{K}$, for all possible complex values of λ. Recall that if we choose an isomorphism $\mathcal{K} \simeq \mathbb{C}((z))$, then a λ–connection is an operator $\nabla = \lambda \partial_z + \chi(z)$, where $\chi(z) \in \mathfrak{h}^*((z))$. We have an exact sequence

$$0 \to \mathfrak{h}^* \otimes \Omega_{D^\times} \to \operatorname{Conn}_{\{\lambda\}}(\Omega^{-\rho})_{D^\times} \to \mathbb{C}\partial_z \to 0,$$

where the penultimate map sends ∇ as above to $\lambda \partial_z$. Then $\widehat{\mathfrak{h}}_\nu(\mathcal{K})$ is by definition the topological dual vector space to $\operatorname{Conn}_{\{\lambda\}}(\Omega^{-\rho})_{D^\times}$, together with the corresponding action of $\operatorname{Aut} O$. It fits into an exact sequence

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{h}}_\nu(\mathcal{K}) \to \mathfrak{h} \otimes \mathcal{K} \to 0,$$

where $\mathbf{1}$ be the element dual to ∂_z. In contrast to the exact sequence defining $\widehat{\mathfrak{h}}_\nu$, this sequence (which does not depend on ν) does not split as a sequence of $\operatorname{Aut} O$–modules. The Lie bracket is given by the old formula (see Section 5.1); it is easy to see that this formula (which depends on ν) is coordinate-independent.

In particular, for $\nu = 0$ the Lie algebra $\widehat{\mathfrak{h}}_0(\mathcal{K})$ is commutative. Any connection ∇ on $\Omega^{-\rho}$ over the punctured disc $\operatorname{Spec} \mathcal{K}$ defines a linear functional on $\widehat{\mathfrak{h}}_0(\mathcal{K})$, and hence a one-dimensional representation $\mathbb{C}_\nabla$ of $\widehat{\mathfrak{h}}_0(\mathcal{K})$. If we choose an isomorphism $\mathcal{K} \simeq \mathbb{C}((z))$, then the connection is given by the formula $\nabla = \partial_z + \chi(z)$, where $\chi(z) \in \mathfrak{h}^*((z))$. The action of the generators $b_{i,n}$ on $\mathbb{C}_\nabla$ is then given by the formula

$$b_{i,n} \mapsto \int \langle \chi(z), h_i \rangle z^n dz.$$

The space Π_0, considered as an $\widehat{\mathfrak{h}}_0(\mathcal{K})$–module, is identified with the space of functions on the subspace $\operatorname{Conn}(\Omega^{-\rho})_D$ of connections on the disc $D = \operatorname{Spec}\mathcal{O}$. Thus, the annihilator of Π_0 in $\widehat{\mathfrak{h}}_0(\mathcal{K})$ is the subspace $\mathfrak{h} \otimes \mathcal{O}$, which is the orthogonal complement of $\operatorname{Conn}(\Omega^{-\rho})_D$ in $\widehat{\mathfrak{h}}_0(\mathcal{K})$.

Proposition 5.5. *For any connection on the ${}^L H$–bundle $\Omega^{-\rho}$ over the punctured disc* $\operatorname{Spec}\mathcal{K}$*, there is a canonical $\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K})$–module structure on $\mathcal{M}_{\mathfrak{g}}(\mathcal{K})$.*

Proof. By Theorem 4.7, there exists a homomorphism of vertex algebras $w_{\kappa_c} : V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0$. According to Corollary 5.4, it commutes with the action of $\operatorname{Der} O$ and $\operatorname{Aut} O$ on both sides. Therefore the corresponding homomorphism of Lie algebras $\widehat{\mathfrak{g}} \to U(M_{\mathfrak{g}} \otimes \pi_0)$ also commutes with the action of $\operatorname{Der} O$ and $\operatorname{Aut} O$. Hence we may twist this homomorphism with the $\operatorname{Aut} O$–torsor $\mathcal{A}ut$. Then we obtain a homomorphism of Lie algebras

$$\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K}) \to U(M_{\mathfrak{g}} \otimes \pi_0)(\mathcal{K}) = \mathcal{A}ut \underset{\operatorname{Aut} O}{\times} U(M_{\mathfrak{g}} \otimes \pi_0).$$

Let us call a $\Gamma(\mathcal{K}) \oplus \widehat{\mathfrak{h}}(\mathcal{K})$–module smooth if any vector in this module is annihilated by the Lie subalgebra

$$(U \otimes \mathfrak{m}^N) \oplus (U^* \otimes \mathfrak{m}^N \Omega_{\mathcal{O}}) \oplus (\mathfrak{h} \otimes \mathfrak{m}^N),$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$, for sufficiently large N. Then clearly any smooth $\Gamma(\mathcal{K}) \oplus \widehat{\mathfrak{h}}(\mathcal{K})$–module is automatically a $U(M_{\mathfrak{g}} \otimes \pi_0)(\mathcal{K})$–module and hence a $\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K})$–module. Note that $\mathcal{M}_{\mathfrak{g}}(\mathcal{K})$ is a smooth $\Gamma(\mathcal{K})$–module, and $\mathbb{C}_{\nabla}$ is a smooth $\widehat{\mathfrak{h}}(\mathcal{K})$–module for any connection ∇ on the ${}^L H$–bundle $\Omega^{-\rho}$ over the punctured disc $\operatorname{Spec}\mathcal{K}$. Taking the tensor product of these two modules we obtain a $\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K})$–module, which is isomorphic to $\mathcal{M}_{\mathfrak{g}}(\mathcal{K})$ as a vector space. If we choose an isomorphism $\mathcal{K} \simeq \mathbb{C}((z))$, this is nothing but the Wakimoto module $W_{\chi(z)}$ introduced in Corollary 4.6. □

Thus, we obtain a family of $\widehat{\mathfrak{g}}_{\kappa_c}(\mathcal{K})$–modules parameterized by flat connections on the ${}^L H$–bundle $\Omega^{-\rho}$ over the punctured disc $\operatorname{Spec}\mathcal{K}$.

6. Semi-infinite parabolic induction

6.1. **Wakimoto modules as induced representations.**

The construction of the Wakimoto modules presented above may be summarized as follows: for each representation N of the Heisenberg Lie algebra $\widehat{\mathfrak{h}}_\kappa$, we have constructed a $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$–module structure on $M_{\mathfrak{g}} \otimes N$. The procedure consists of the extension the $\widehat{\mathfrak{h}}_\kappa$–module by 0 to $\widehat{\mathfrak{b}}_{-,\kappa}$ followed by what may be viewed as a semi-infinite analogue of induction from $\widehat{\mathfrak{b}}_{-,\kappa}$ to $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$. An important feature of this construction, as opposed to the ordinary induction, is that the level gets shifted by κ_c. In particular, if we start with an $\widehat{\mathfrak{h}}_0$–module, or equivalently, a representation of the commutative Lie algebra $L\mathfrak{h}$, then we obtain a $\widehat{\mathfrak{g}}_{\kappa_c}$–module of critical level, rather than of level 0. Irreducible smooth representations of $L\mathfrak{h}$ are one-dimensional and are in one-to-one correspondence with the elements $\chi(t)$ of the (topological) dual space $(L\mathfrak{h})^* \simeq \mathfrak{h}^*((t))dt$. Thus, we obtain the Wakimoto modules $W_{\chi(t)}$ of critical level. But as we have seen above, if we look at the transformation properties of $\chi(t)$ under the action of the group $\operatorname{Aut} O$ of changes of the coordinate

t, we find that $\chi(t)$ actually transforms not as a one-form, but as a connection on a specific ${}^L H$–bundle.

In contrast, if $\kappa \neq 0$, the irreducible smooth $\widehat{\mathfrak{h}}_\kappa$–modules are just the Fock representations π^κ_λ. To each of them we attach a $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$–module $W_{\lambda,\kappa+\kappa_c}$.

Now we want to generalize this construction replacing the Borel subalgebra $\mathfrak{b}_-$ and its Levi quotient $\mathfrak{h}$ by an arbitrary parabolic subalgebra $\mathfrak{p}$ and its Levi quotient $\mathfrak{m}$. Then we wish to attach to a module over a central extension of the loop algebra $L\mathfrak{m}$ a $\widehat{\mathfrak{g}}$–module. It turns out that this is indeed possible provided that we pick a suitable central extension of $L\mathfrak{m}$. We call the resulting $\widehat{\mathfrak{g}}$–modules the generalized Wakimoto modules corresponding to $\mathfrak{p}$. Thus, we obtain a functor from the category of smooth $\widehat{\mathfrak{m}}$–modules to the category of smooth $\widehat{\mathfrak{g}}$–modules. It is natural to call it the functor of *semi-infinite parabolic induction* (by analogy with a similar construction for representations of reductive groups).

6.2. **The main result.** Let $\mathfrak{p}$ be a parabolic Lie subalgebra of $\mathfrak{g}$. We will assume that $\mathfrak{p}$ contains the lower Borel subalgebra $\mathfrak{b}_-$ (and so in particular, $\mathfrak{p}$ contains $\widehat{\mathfrak{h}}$). Let

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}$$

be the direct sum decomposition of $\mathfrak{p}$, where $\mathfrak{m}$ is the Levi subgroup containing $\widehat{\mathfrak{h}}$ and $\mathfrak{r}$ is the nilpotent radical of $\mathfrak{p}$. Further, let

$$\mathfrak{m} = \bigoplus_{i=1}^{s} \mathfrak{m}_i \oplus \mathfrak{m}_0$$

be the decomposition of $\mathfrak{m}$ into the direct sum of simple Lie subalgebras $\mathfrak{m}_i, i = 1, \ldots, s$, and an abelian subalgebra $\mathfrak{m}_0$ such that these direct summands are mutually orthogonal with respect to the inner product on $\mathfrak{g}$. We denote by $\kappa_{i,c}$ the critical inner product on $\mathfrak{m}_i, i = 1, \ldots, s$, defined as in Section 2.3. We also set $\kappa_{0,c} = 0$.

Given a set of inner products κ_i on $\mathfrak{m}_i, 0 = 1, \ldots, s$, we obtain an inner product on $\mathfrak{m}$. Let $\widehat{\mathfrak{m}}_{(\kappa_i)}$ be the corresponding affine Kac-Moody algebra, i.e., the one-dimensional central extension of $L\mathfrak{m}$ with the commutation relations given by formula (2.7). We denote by $V_{\kappa_i}(\mathfrak{m}_i), i = 1, \ldots, s$, the vacuum module over $\widehat{\mathfrak{m}}_i$ with the vertex algebra defined as in Section 4.1. We also denote by $V_{\kappa_0}(\mathfrak{m}_0)$ the Fock representation $\pi^{\kappa_0}_0$ of the Heisenberg Lie algebra $\mathfrak{m}_{\kappa_0}$ with its vertex algebra structure defined as in Section 5.1. Let

$$V_{(\kappa_i)}(\mathfrak{m}) \overset{\text{def}}{=} \bigotimes_{i=0}^{s} V_{\kappa_i}(\mathfrak{m}_i)$$

be the vacuum module over $\widehat{\mathfrak{m}}_{(\kappa_i)}$ with the tensor product vertex algebra structure.

Denote by Δ'_+ the set of positive roots of $\mathfrak{g}$ which do not belong to $\mathfrak{p}$. Let $\mathcal{A}^{\mathfrak{g},\mathfrak{p}}$ be the Weyl algebra with generators $a_{\alpha,n}, a^*_{\alpha,n}, \alpha \in \Delta'_+, n \in \mathbb{Z}$, and relations (2.2). Let $M_{\mathfrak{g},\mathfrak{p}}$ be the Fock representation of $\mathcal{A}^{\mathfrak{g},\mathfrak{p}}$ generated by a vector $|0\rangle$ such that

$$a_{\alpha,n}|0\rangle = 0, \quad n \geq 0; \qquad a^*_{\alpha,n}|0\rangle = 0, \quad n > 0.$$

Then $M_{\mathfrak{g},\mathfrak{p}}$ carries a vertex algebra structure defined as in Section 2.4.

We have the following analogue of Theorem 5.1.

Theorem 6.1. *Suppose that* $\kappa_i, i = 0, \ldots, s$, *is a set of inner products such that there exists an inner product* κ *on* $\mathfrak{g}$ *whose restriction to* $\mathfrak{m}_i$ *equals* $\kappa_i - \kappa_{i,c}$ *for all* $i = 0, \ldots, s$. *Then there exists a homomorphism of vertex algebras*

$$w^{\mathfrak{p}}_{\kappa} : V_{\kappa+\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g},\mathfrak{p}} \otimes V_{(\kappa_i)}(\mathfrak{m}).$$

Proof. The proof is a generalization of the proof of Theorem 5.1 (in fact, Theorem 5.1 is a special case of Theorem 6.1 when $\mathfrak{p} = \mathfrak{b}_-$). Let P be the Lie subgroup of G corresponding to $\mathfrak{p}$, and consider the homogeneous space G/P. It has an open dense subset $U_{\mathfrak{p}} = N_{\mathfrak{p}} \cdot [1]$, where $N_{\mathfrak{p}}$ is the subgroup of N_+ corresponding to the subset $\Delta'_+ \subset \Delta_+$. We identify $U_{\mathfrak{p}}$ with $N_{\mathfrak{p}}$ and with its Lie algebra $\mathfrak{n}_{\mathfrak{p}}$ using the exponential map.

Set $LU_{\mathfrak{p}} = U_{\mathfrak{p}}((t))$. We define functions and vector fields on $LU_{\mathfrak{p}}$, denoted by $\operatorname{Fun} LU_{\mathfrak{p}}$ and $\operatorname{Vect} LU_{\mathfrak{p}}$, respectively, in the same way as in Section 2.1. The action of $L\mathfrak{g}$ on $U_{\mathfrak{p}}((t))$ gives rise to a Lie algebra homomorphism

$$\widehat{\rho}_{\mathfrak{p}} : L\mathfrak{g} \to \operatorname{Vect} LU_{\mathfrak{p}} \oplus \operatorname{Fun} U_{\mathfrak{p}} \widehat{\otimes} L\mathfrak{m}.$$

Moreover, the image of this homomorphism is contained in the "local part", i.e., the direct sum of the local part $\mathcal{T}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}}$ of $\operatorname{Vect} LU_{\mathfrak{p}}$ defined as in Section 2.6 and the local part $\mathcal{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}}$ of $\operatorname{Fun} U_{\mathfrak{p}} \widehat{\otimes} L\mathfrak{m}$. By definition, $\mathcal{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}}$ is the span of the Fourier coefficients of the formal power series $P(\partial_z^n a^*_{\alpha}(z)) J^b(z)$, where P is a differential polynomial in $a^*_{\alpha}(z), \alpha \in \Delta'_+$, and $J^a \in \mathfrak{m}$.

Let $\mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\mathrm{loc}}$ and $\mathcal{A}^{\mathfrak{g}}_{\leq 1,\mathrm{loc}}$ be the zeroth and the first terms of the natural filtration on the local completion of the Weyl algebra $\mathcal{A}^{\mathfrak{g},\mathfrak{p}}$, defined as in Section 2.3. We have a non-split exact sequence

$$0 \to \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\mathrm{loc}} \to \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{\leq 1,\mathrm{loc}} \to \mathcal{T}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}} \to 0. \tag{6.1}$$

Set

$$\mathfrak{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}} \overset{\mathrm{def}}{=} \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{\leq 1,\mathrm{loc}} \oplus \mathcal{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}},$$

and note that $\mathfrak{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}}$ is naturally a Lie subalgebra of the local Lie algebra $U(M_{\mathfrak{g},\mathfrak{p}} \otimes V_{(\kappa_i)}(\mathfrak{m}))$. Using the splitting of the sequence (6.1) as a vector space via the normal ordering, we obtain a linear map $\overline{w}_{(\kappa_i)} : L\mathfrak{g} \to \mathfrak{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}}$.

We need to compute the failure of $\overline{w}_{(\kappa_i)}$ to be a Lie algebra homomorphism. Thus, we consider the corresponding linear map $\omega_{(\kappa_i)} : \bigwedge^2 L\mathfrak{g} \to \mathfrak{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}}$ defined by the formula

$$\omega_{(\kappa_i)}(f, g) = [\overline{w}_{\kappa}(f), \overline{w}_{\kappa}(g)] - \overline{w}_{\kappa}([f, g]).$$

Evaluating it explicitly in the same way as in the proof of Lemma 3.2, we find that $\omega_{(\kappa_i)}$ takes values in $\mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\mathrm{loc}} \subset \mathfrak{J}^{\mathfrak{g},\mathfrak{p}}_{\mathrm{loc}}$. Furthermore, $\mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\mathrm{loc}}$ is naturally an $L\mathfrak{g}$–module, and by construction of $\overline{w}_{(\kappa_i)}$, for any $X \in \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\mathrm{loc}}$ and $f \in L\mathfrak{g}$ we have $[\overline{w}_{(\kappa_i)}(f), X] = f \cdot X$. This implies that $\omega_{(\kappa_i)}$ is a two-cocycle of $L\mathfrak{g}$ with coefficients in $\mathcal{A}^{\mathfrak{g}}_{0,\mathrm{loc}}$. By construction, it is local, i.e., belongs to $C^2_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\mathrm{loc}})$.

Following the argument used in the proof of Lemma 3.6, we show that any two cocycles in $C^2_{\mathrm{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\mathrm{loc}})$, whose restrictions to $\bigwedge^2(L\mathfrak{m})$ coincide, represent the same cohomology class.

Let us compute the restriction of $\omega_{(\kappa_i)}$ to $\bigwedge^2 L\mathfrak{m}$. For that we evaluate $\overline{w}_{(\kappa_i)}$ on elements of $L\mathfrak{m}$. Let $\{J^i\}_{i=1,\dots,\dim\mathfrak{m}}$ be a basis of $\mathfrak{m}$. The adjoint action of the Lie algebra $\mathfrak{m}$ on $\mathfrak{g}$ preserves $\mathfrak{n}_{\mathfrak{p}}$. Under this action, which we denote by $\rho_{\mathfrak{n}_{\mathfrak{p}}}$, an element $A \in \mathfrak{m}$ acts by the formula

$$\rho_{\mathfrak{n}_{\mathfrak{p}}}(A) \cdot J^\alpha = \sum_{\beta \in \Delta'_+} c^\alpha_\beta(A) J^\beta$$

for some $c^\alpha_\beta(A) \in \mathbb{C}$. Therefore

$$\overline{w}_{(\kappa_i)}(A(z)) = - \sum_{\beta \in \Delta'_+} c^\alpha_\beta(A) {:}a^*_\beta(z) a_\beta(z){:} + \widetilde{A}(z),$$

where $\widetilde{A}(z)$ is the field $\sum_{n\in\mathbb{Z}}(A \otimes t^n) z^{-n-1}$, considered as a generating series of elements of $\mathcal{I}^{\mathfrak{g},\mathfrak{p}}_{\text{loc}}$. Let $\kappa_{\mathfrak{n}_{\mathfrak{p}}}$ be the inner product on $\mathfrak{m}$ defined by the formula

$$\kappa_{\mathfrak{n}_{\mathfrak{p}}}(A, B) = \operatorname{Tr}_{\mathfrak{n}_{\mathfrak{p}}} \rho_{\mathfrak{n}_{\mathfrak{p}}}(A) \rho_{\mathfrak{n}_{\mathfrak{p}}}(B).$$

Moreover, we find that for $A \in \mathfrak{m}_i, B \in \mathfrak{m}_j$,

$$\omega_{(\kappa_i)}(A_n, B_m) = n(-\kappa_{\mathfrak{n}_{\mathfrak{p}}}(A, B) + \kappa_i(A, B))\delta_{n,-m}, \tag{6.2}$$

if $i = j$, and

$$\omega_{(\kappa_i)}(A_n, B_m) = -n\kappa_{\mathfrak{n}_{\mathfrak{p}}}(A, B)\delta_{n,-m}, \tag{6.3}$$

if $i \neq j$. Thus, the restriction of $\omega_{(\kappa_i)}$ to $\bigwedge^2 L\mathfrak{m}$ takes values in the constants $\mathbb{C} \subset \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\text{loc}}$.

Let κ_K be the Killing form on $\mathfrak{g}$ and $\kappa_{i,K}$ be the Killing form on $\mathfrak{m}_i$ (in particular, $\kappa_{0,K} = 0$). Then

$$\begin{aligned} &\kappa_K(A, B) = \kappa_{i,K}(A, B) + 2\kappa_{\mathfrak{n}_{\mathfrak{p}}}(A, B), \qquad \text{if} \quad i = j, \\ &\kappa_K(A, B) = 2\kappa_{\mathfrak{n}_{\mathfrak{p}}}(A, B), \qquad \text{if} \quad i \neq j. \end{aligned}$$

Since $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are orthogonal with respect to κ_K for all $i \neq j$ by construction, we obtain that $\kappa_{\mathfrak{n}_{\mathfrak{p}}}(A, B) = 0$, if $i \neq j$. Recall that by definition $\kappa_c = -\frac{1}{2}\kappa_K$, and $\kappa_{i,c} = -\frac{1}{2}\kappa_{i,K}$. Hence $\kappa_{\mathfrak{n}_{\mathfrak{p}}}|_{\mathfrak{m}_i} = -\kappa_c + \kappa_{i,c}$, and so if κ_i equals $\kappa|_{\mathfrak{m}_i} + \kappa_{i,c}$ for some invariant inner product κ on $\mathfrak{g}$, then $-\kappa_{\mathfrak{n}_{\mathfrak{p}}}|_{\mathfrak{m}_i} + \kappa_i = (\kappa + \kappa_c)|_{\mathfrak{m}_i}$.

Thus, we obtain that if κ is an invariant inner product on $\mathfrak{g}$ whose restriction to $\mathfrak{m}_i$ equals $\kappa_i - \kappa_{i,c}$ for all $i = 0, \dots, s$, then the restriction of the two-cocycle $\omega_{(\kappa_i)}$ to $\bigwedge^2(L\mathfrak{m})$ is equal to the restriction to $\bigwedge^2(L\mathfrak{m})$ of the two-cocycle $\sigma_{\kappa+\kappa_c}$ on $L\mathfrak{g}$ (corresponding to the one-dimensional central extension with respect to the inner product $\kappa + \kappa_c$ on $\mathfrak{g}$). Therefore the $\omega_{(\kappa_i)}$ is equivalent to $\sigma_{\kappa+\kappa_c}$ in this case and we obtain that the linear map $\overline{w}_{(\kappa_i)} : L\mathfrak{g} \to \mathcal{J}^{\mathfrak{g},\mathfrak{p}}_{\text{loc}}$ may be modified by the addition of an element of $C^1_{\text{loc}}(L\mathfrak{g}, \mathcal{A}^{\mathfrak{g},\mathfrak{p}}_{0,\text{loc}})$ to give us a Lie algebra homomorphism

$$\widehat{\mathfrak{g}}_{\kappa+\kappa_c} \to \mathcal{J}^{\mathfrak{g},\mathfrak{p}}_{\text{loc}} \subset U(M_{\mathfrak{g},\mathfrak{p}} \otimes V_{(\kappa_i)}(\mathfrak{m})).$$

Now Lemma 4.1 implies that there exists a homomorphism of vertex algebras

$$w^{\mathfrak{p}}_\kappa : V_{\kappa+\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g},\mathfrak{p}} \otimes V_{(\kappa_i)}(\mathfrak{m}).$$

This completes the proof. □

We will call an $\widehat{\mathfrak{m}}_{(\kappa_i)}$–module *smooth* if any vector in it is annihilated by the Lie subalgebra $\mathfrak{m} \otimes t^N \mathbb{C}[[t]]$ for sufficiently large N.

Corollary 6.2. *For any smooth* $\widehat{\mathfrak{m}}_{(\kappa_i)}$*–module* R *with the* κ_i*'s satisfying the conditions of Theorem 6.1, the tensor product* $M_{\mathfrak{g},\mathfrak{p}} \otimes R$ *is naturally a smooth* $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$*–module. There is a functor from the category of smooth* $\widehat{\mathfrak{m}}_{(\kappa_i)}$*–modules to the category of smooth* $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$*–module sending a module* R *to* $M_{\mathfrak{g},\mathfrak{p}} \otimes R$ *and* $\widehat{\mathfrak{m}}_{(\kappa_i)}$*–homomorphism* $R_1 \to R_2$ *to the* $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$*–homomorphism* $M_{\mathfrak{g},\mathfrak{p}} \otimes R_1 \to M_{\mathfrak{g},\mathfrak{p}} \otimes R_2$*.*

We call the $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$–module $M_{\mathfrak{g},\mathfrak{p}} \otimes R$ the *generalized Wakimoto module corresponding to* R*.*

Consider the special case when R is the tensor product of the Wakimoto modules W_{λ_i,κ_i} over $\widehat{\mathfrak{m}}_i, i = 1, \ldots, s$, and the Fock representation $\pi^{\kappa_0}_{\lambda_0}$ over the Heisenberg Lie algebra $\widehat{\mathfrak{m}}_0$. In this case it follows from the construction that the corresponding $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$–module $M_{\mathfrak{g},\mathfrak{p}} \otimes R$ is isomorphic to the Wakimoto module $W_{\lambda,\kappa+\kappa_c}$ over $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$, where $\lambda = (\lambda_i)$.

6.3. **General parabolic subalgebras.** So far we have worked under the assumption that the parabolic subalgebra $\mathfrak{p}$ contains $\mathfrak{b}_-$. It is also possible to construct Wakimoto modules associated to other parabolic subalgebras. Let us explain how to do this in the case when $\mathfrak{p} = \mathfrak{b}_+$. We have the involution of $\mathfrak{g}$ sending e_i to f_i and h_i to $-h_i$. Under this involution $\mathfrak{b}_-$ goes to $\mathfrak{b}_+$.

Let N be any module over the vertex algebra $M_{\mathfrak{g}} \otimes \pi^{\kappa-\kappa_c}_0$. Then Theorem 5.1 implies that the following formulas define a $\widehat{\mathfrak{g}}_\kappa$–structure on N (with K acting as the identity):

$$f_i(z) \mapsto a_{\alpha_i}(z) + \sum_{\beta\in\Delta_+} {:}P^i_\beta(a^*_\alpha(z)) a_\beta(z){:}\ ,$$

$$h_i(z) \mapsto \sum_{\beta\in\Delta_+} \beta(h_i){:}a^*_\beta(z) a_\beta(z){:} - b_i(z),$$

$$e_i(z) \mapsto \sum_{\beta\in\Delta_+} {:}Q^i_\beta(a^*_\alpha(z)) a_\beta(z){:} + (c_i + (\kappa-\kappa_c)(e_i,f_i))\, \partial_z a^*_{\alpha_i}(z) + b_i(z) a^*_{\alpha_i}(z),$$

where the polynomials P^i_β, Q^i_β are introduced in formulas (1.7)–(1.9).

In order to make N into a module with highest weight, we choose N as follows. Let $M'_{\mathfrak{g}}$ be the Fock representation of the Weyl algebra $\mathcal{A}^{\mathfrak{g}}$ generated by a vector $|0\rangle'$ such that

$$a_{\alpha,n}|0\rangle' = 0, \quad n > 0; \qquad a^*_{\alpha,n}|0\rangle' = 0, \quad n \geq 0.$$

We take as N the module $M'_{\mathfrak{g}} \otimes \pi^{\kappa-\kappa_c}_{-2\rho-\lambda}$, where $\pi^{\kappa-\kappa_c}_{-2\rho-\lambda}$ is the $\pi^{\kappa-\kappa_c}_0$–module defined in Section 5.2. It is easy to see that the corresponding $\widehat{\mathfrak{g}}_\kappa$–module has a highest weight vector on which $\mathfrak{h} \otimes 1$ acts through the weight λ. We denote this module by $W^+_{\lambda,\kappa}$. This is the Wakimoto module corresponding to $\mathfrak{b}_+$.

The following result will be used in Section 9.3.

Proposition 6.3. *The Wakimoto module* W^+_{0,κ_c} *is isomorphic to the Verma module* M_{0,κ_c}*.*

Proof. The proof is analogous to the proof of Theorem 4.8. Since W^+_{0,κ_c} has a highest weight vector of the same weight as that of the highest weight vector of M_{0,κ_c}, there is a non-zero homomorphism $M_{0,\kappa_c} \to W^+_{0,\kappa_c}$. But the characters of M_{0,κ_c} and W^+_{0,κ_c} are equal. Therefore the theorem will follow if we show that W^+_{0,κ_c} is generated by its highest weight vector.

Suppose that W^+_{0,κ_c} is not generated by its highest weight vector. Then there exists a homogeneous linear functional on W^+_{0,κ_c}, whose weight is less than the highest weight $(\lambda, 0)$ and which is invariant under the Lie subalgebra

$$\widetilde{\mathfrak{n}}_- = (\mathfrak{n}_- \otimes 1) \oplus (\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]),$$

and in particular, under its Lie subalgebra $L_-\mathfrak{n}_- = \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}]$. Therefore this functional factors through the space of coinvariants of W_{0,κ_c} by $L_-\mathfrak{n}_-$. But it follows from the construction that $L_-\mathfrak{n}_-$ acts freely on W^+_{0,κ_c}, and the space of coinvariants is isomorphic to the subspace $\mathbb{C}[a^*_{\alpha,n}]_{\alpha\in\Delta_+,n<0} \otimes \mathbb{C}[b_{i,n}]_{i=1,\dots,\ell;n<0}$ of W^+_{0,κ_c}. Hence we obtain that the weight of this functional has the form

$$-\sum_j (n_j\delta - \beta_j), \qquad n_j > 0, \quad \beta_j \in \Delta_+. \tag{6.4}$$

But then M_{0,κ_c} must have an irreducible subquotient of highest weight of this form.

Now recall the Kac–Kazhdan theorem [KK] describing the set of highest weights of irreducible subquotients of Verma modules (see Section 4.3). In the case at hand the statement is as follows. A weight $\widehat{\mu} = (\mu, n)$ appears in the decomposition of M_{0,κ_c} if and only $n \leq 0$ and either $\mu = 0$ or there exists a finite sequence of weights $\mu_0, \dots, \mu_m \in \mathfrak{h}^*$ such that $\mu_0 = \mu, \mu_m = 0$, $\mu_{i+1} = \mu_i \pm m_i\beta_i$ for some positive roots β_i and positive integers m_i which satisfy

$$2(\mu_i + \rho, \beta_i) = m_i(\beta_i, \beta_i) \tag{6.5}$$

(here $(\cdot,\cdot)$ is the inner product on $\mathfrak{h}^*$ induced by an arbitrary non-degenerate invariant inner product on $\mathfrak{g}$).

Now observe that the equations (6.5) coincide with the equations appearing in the analysis of irreducible subquotients of the Verma module over $\mathfrak{g}$ of highest weight 0. This implies that the above statement is equivalent to the following: a weight $\widehat{\mu} = (\mu, n)$ appears in the decomposition of M_{0,κ_c} if and only $n \leq 0$ and $\mu = w(\rho) - \rho$ for some element w of the Weyl group of $\mathfrak{g}$. But for any w, the weight $w(\rho) - \rho$ equals the sum of negative simple roots of $\mathfrak{g}$. Hence the weight of any irreducible subquotient of M_{0,κ_c} has the form $-n\delta - \sum_i m_i\alpha_i, m \geq 0$. Such a weight cannot be of the form (6.4). Therefore W^+_{0,κ_c} is generated by the highest weight vector and hence is isomorphic to M_{0,κ_c}. ☐

Remark 6.4. The same argument as in the proof of Proposition 6.3 shows that the Wakimoto module W^+_{λ,κ_c} is isomorphic to M_{λ,κ_c} if λ is such that all weights (μ, n) of irreducible subquotients of M_{λ,κ_c} satisfy $\mu = \lambda - \sum_i m_i\alpha_i, m_i \geq 0$. Likewise, the Wakimoto module W_{λ,κ_c} is isomorphic to M_{λ,κ_c} if we have $\mu = \lambda + \sum_i m_i\alpha_i, m_i \geq 0$ for all such μ. ☐

In Section 9.3 we will need one more result on the structure of W^+_{0,κ_c}. Consider the Lie algebra $\widetilde{\mathfrak{b}}_+ = (\mathfrak{b}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]])$.

Lemma 6.5. *The space of $\widetilde{\mathfrak{b}}_+$–invariants of W^+_{0,κ_c} is equal to $\pi_0 \subset W^+_{0,\kappa_c}$.*

Proof. It follows from the above formulas for the action of $\widehat{\mathfrak{g}}_{\kappa_c}$ on W^+_{0,κ_c} that all vectors in π_0 are annihilated by $\widetilde{\mathfrak{b}}_+$. Let us show that there are no other $\widetilde{\mathfrak{b}}_+$–invariant vector in W^+_{0,κ_c}.

A $\widetilde{\mathfrak{b}}_+$–invariant vector is in particular annihilated by the Lie subalgebra $L_+\mathfrak{n}_- = \mathfrak{n}_+ \otimes t\mathbb{C}[[t]]$. In the same way as in the proof of Theorem 4.8 we show that the space of $L_+\mathfrak{n}_+$–invariants of $W_{\kappa_c,0}$ is equal to the tensor product of π_0 and the subspace of W^+_{0,κ_c} spanned by all monomials of the form $\prod\limits_{m_{s(\alpha)} \leq 0} a^*_{\alpha,m_{s(\alpha)}}|0\rangle'$. But a $\widetilde{\mathfrak{b}}_+$–invariant vector is also annihilated by $\mathfrak{h} \otimes 1$. A monomials of the above form is annihilated by $\mathfrak{h} \otimes 1$ only if it belongs to π_0. Hence the space of $\widetilde{\mathfrak{b}}_+$–invariants of W^+_{0,κ_c} is equal to π_0. ☐

7. Wakimoto modules over $\widehat{\mathfrak{sl}}_2$

In this section we describe explicitly the Wakimoto modules over $\widehat{\mathfrak{sl}}_2$ and some intertwining operators between them.

Let $\{e, h, f\}$ be the standard basis of the Lie algebra $\mathfrak{sl}_2$. Let κ_0 be the invariant inner product on $\mathfrak{sl}_2$ normalized in such a way that $\kappa_0(h, h) = 2$. We will write an arbitrary invariant inner product κ on $\mathfrak{sl}_2$ as $k\kappa_0$, where $k \in \mathbb{C}$, and will use k in place of κ in our notation. In particular, κ_c corresponds to $k = -2$. The set Δ_+ consists of one element in the case of $\mathfrak{sl}_2$, so we will drop the index α in $a_\alpha(z)$ and $a^*_\alpha(z)$. Likewise, we will drop the index i in $b_i(z)$, etc., and will write M for $M_{\mathfrak{sl}_2}$ throughout this section. We will also identify the dual space to the Cartan subalgebra $\mathfrak{h}^*$ with $\mathbb{C}$ by sending $\chi \in \mathfrak{h}^*$ to $\chi(h)$.

7.1. The homomorphism w_k.

The Weyl algebra $\mathcal{A}_{\mathfrak{sl}_2}$ has generators $a_n, a^*_n, n \in \mathbb{Z}$, with the commutation relations

$$[a_n, a^*_m] = \delta_{n,-m}.$$

Its Fock representation is denoted by M. The Heisenberg Lie algebra $\widehat{\mathfrak{h}}_k$ has generators $b_n, n \in \mathbb{Z}$, and $\mathbf{1}$, with the commutation relations

$$[b_n, b_m] = 2kn\delta_{n,-m}\mathbf{1},$$

and π^k_λ is its Fock representation generated by a vector $|\lambda\rangle$ such that

$$b_n|\lambda\rangle = 0, \quad n > 0; \qquad b_0|\lambda\rangle = \lambda|\lambda\rangle; \qquad \mathbf{1}|\lambda\rangle = |\lambda\rangle.$$

The module π^k_0 and the tensor product $M \otimes \pi^k_0$ are vertex algebras.

The homomorphism $w_k : V_k(\mathfrak{sl}_2) \to M \otimes \pi_0^{k+2}$ of vertex algebras is given by the following formulas:

$$\begin{aligned} e(z) &\mapsto a(z) \\ h(z) &\mapsto -2{:}a^*(z)a(z){:} + b(z) \\ f(z) &\mapsto -{:}a^*(z)^2 a(z){:} + k\partial_z a^*(z) + a^*(z)b(z). \end{aligned} \tag{7.1}$$

The Wakimoto module $M \otimes \pi_\lambda^{k+2}$ will be denoted by $W_{\lambda,k}$ and its highest weight vector will be denoted by $v_{\lambda,k}$.

Recall that under the homomorphism w_k the generators e_n of $\widehat{\mathfrak{sl}}_2$ are mapped to a_n (from now on, by abuse of notation, we will identify the elements of $\widehat{\mathfrak{sl}}_2$ with their images under w_k). The commutation relations of $\widehat{\mathfrak{sl}}_2$ imply the following formulas

$$[e_n, a_{-1}] = 0, \qquad [h_n, a_{-1}] = 2a_{n-1}, \quad [f_n, a_{-1}] = -h_{n-1} + k\delta_{n,1}.$$

In addition, it follows from formulas (7.1) that

$$e_n v_{-2,k} = a_n v_{-2,k} = 0, \quad n \geq 0; \qquad h_n v_{-2,k} = f_n v_{-2,k} = 0, \quad n > 0,$$
$$h_0 v_{-2,k} = -2v_{-2,k}.$$

Therefore

$$e_n \cdot a_{-1} v_{-2,k} = h_n \cdot a_{-1} v_{-2,k} = 0, \qquad n \geq 0,$$

and

$$f_n \cdot a_{-1} v_{-2,k} = 0, \qquad n > 1.$$

We also find that

$$f_1 \cdot a_{-1} v_{-2,k} = (k+2) v_{-2,k},$$

and

$$\begin{aligned} f_0 \cdot a_{-1} v_{-2,k} &= a_{-1} f_0 v_{-2,k} - h_0 v_{-2,k} \\ &= -b_{-1} v_{-2,k} = (k+2) T v_{-2,k}, \end{aligned}$$

where T is the translation operator.

7.2. **Vertex operators associated to a module over a vertex algebra.** We need to recall some general results on the vertex operators associated to a module over a vertex algebra, following [FHL], § 5.1. Let V be a conformal vertex algebra V and M a V–module, i.e., a vector space together with a linear map

$$Y_M : V \to \operatorname{End} M[[z^{\pm 1}]]$$

satisfying the axioms of Definition 5.1.1 of [FB]. Define a linear map

$$Y_{V,M} : M \to \operatorname{Hom}(V, M)[[z^{\pm 1}]]$$

by the formula

$$Y_{V,M}(A, z)B = e^{zT} Y_M(B, -z)A, \qquad A \in M, B \in V.$$

By Proposition 5.1.2 of [FHL], this map satisfies the following property: for any $A \in V, B \in M, C \in V$, there exists an element

$$f \in M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

such that the formal power series

$$Y_M(A,z)Y_{V,M}(B,w)C, \qquad Y_{V,M}(B,w)Y(A,z)C,$$

$$Y_{V,M}(Y_{V,M}(B,w-z)A,z)C, \qquad Y_{V,M}(Y(A,z-w)B,w)C$$

are expansions of f in

$$M((z))((w)), \qquad M((w))((z)), \qquad M((z))((z-w)), \qquad M((w))((z-w)),$$

respectively (compare with Corollary 3.2.3 of [FB]). Abusing notation, we will write

$$Y_M(A,z)Y_{V,M}(B,w) = Y_{V,M}(Y(A,z-w)B,w),$$

and call this formula the operator product expansion (OPE), as in the case $M = V$ when $Y_{V,M} = Y$ (see [FB], § 3.3).

This property implies the following commutation relations between the Fourier coefficients of $Y(A,z)$ and $Y_{V,M}(B,w)$, which is proved in exactly the same way as in the case $M = V$ (see [FB], § 3.3.6). If we write

$$Y_M(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)}z^{-n-1}, \qquad Y_{V,M}(B,w) = \sum_{n\in\mathbb{Z}} B_{(n)}w^{-n-1},$$

then we have

$$[B_{(m)}, A_{(k)}] = \sum_{n\geq 0} \begin{pmatrix} m \\ n \end{pmatrix} (B_{(n)} \cdot A)_{(m+k-n)}. \tag{7.2}$$

In particular, we obtain that

$$\left[\int Y_{V,M}(B,z)dz, Y(A,w)\right] = Y_{V,M}\left(\int Y_{V,M}(B,z)dz \cdot A, w\right). \tag{7.3}$$

7.3. **The screening operator.** Let us apply the results of Section 7.2 in the case of the vertex algebra $W_{0,k}$ and its module $W_{-2,k}$ for $k \neq -2$. Then we find, in the same way as in [FB], § 5.2.6, that

$$Y_{W_{0,k},W_{-2,k}}(a_{-1}v_{-2,k}) = S_k(z) \overset{\mathrm{def}}{=} a(z)V_{-2}(z),$$

where $V_{-2}(z)$ is the bosonic vertex operator acting from π_0^{k+2} to π_{-2}^{k+2} given by the formula

$$V_{-2}(z) = T_{-2}\exp\left(\frac{1}{k+2}\sum_{n<0}\frac{b_n}{n}z^{-n}\right)\exp\left(\frac{1}{k+2}\sum_{n>0}\frac{b_n}{n}z^{-n}\right). \tag{7.4}$$

Here we denote by T_{-2} the translation operator $\pi_0^{k+2} \to \pi_{-2}^{k+2}$ sending the highest weight vector to the highest weight vector and commuting with all $b_n, n \neq 0$.

Direct computation using formula (7.2) implies that the operator $S_k(z)$ has the following OPEs:

$$\begin{aligned} e(z)S_k(w) &= \text{reg.}, \qquad h(z)S_k(w) = \text{reg.}, \\ f(z)S_k(w) &= \frac{(k+2)V_{-2}(w)}{(z-w)^2} + \frac{(k+2)\partial_w V_{-2}(w)}{z-w} + \text{reg.} \\ &= (k+2)\partial_w \frac{V_{-2}(w)}{z-w} + \text{reg.} \end{aligned}$$

Since the residue of a total derivative is equal to 0, we obtain the following:

Proposition 7.1. *The residue* $S_k = \displaystyle\int S_k(w)dw$ *is an intertwining operator between the* $\widehat{\mathfrak{sl}}_2$*–modules* $W_{0,k}$ *and* $W_{-2,k}$*.*

We call S_k the *screening operator of the first kind* for $\widehat{\mathfrak{sl}}_2$.

Proposition 7.2. *For* $k \notin -2 + \mathbb{Q}_{\geq 0}$ *the sequence*

$$0 \to V_k(\mathfrak{sl}_2) \to W_{0,k} \overset{S_k}{\to} W_{-2,k} \to 0$$

is exact.

Proof. By Proposition 5.2, $V_k(\mathfrak{sl}_2)$ is naturally an $\widehat{\mathfrak{sl}}_2$–submodule of $W_{0,k}$ for any value of k. The module $V_k(\mathfrak{sl}_2)$ is generated from the vector v_k, whose image in $W_{0,k}$ is the highest weight vector. We have

$$S_k = \sum_{n\in\mathbb{Z}} a_n V_{-2,-n},$$

where $V_{-2,-n}$ is the coefficients in front of z^n in $V_{-2}(z)$. It is clear from formula (7.4) that $V_{-2,m}v_k = 0$ for all $m > 0$. We also have $a_n v_k = 0$ for all $n \geq 0$. Therefore v_k belongs to the kernel of S_k. But since S_k commutes with the action of $\widehat{\mathfrak{sl}}_2$, this implies that the entire submodule $V_k(\mathfrak{sl}_2)$ lies in the kernel of S_k.

In order to prove that $V_k(\mathfrak{sl}_2)$ coincides with the kernel of S_k, we compare their characters. See Section 4.3 for the definition of the character. We will use the notation $q = e^{-\delta}, u = e^{\alpha}$.

Since $V_k(\mathfrak{g})$ is isomorphic to the universal enveloping algebra of the Lie algebra spanned by e_n, h_n and f_n with $n < 0$, we find that

$$\operatorname{ch} V_k(\mathfrak{g}) = \prod_{n>0}(1-q^n)^{-1}(1-uq^n)^{-1}(1-u^{-1}q^n)^{-1}. \tag{7.5}$$

Similarly, we obtain that

$$\operatorname{ch} W_{\lambda,k} = u^{\lambda/2}\prod_{n>0}(1-q^n)^{-1}(1-uq^n)^{-1}(1-u^{-1}q^{n-1})^{-1}.$$

Thus, $\operatorname{ch} W_{\lambda,k} = \operatorname{ch} M_{\lambda,k}$, where $M_{\lambda,k}$ is the Verma module over $\widehat{\mathfrak{sl}}_2$ with highest weight (λ, k). Therefore if $M_{\lambda,k}$ is irreducible, then so is $W_{\lambda,k}$. The set of values (λ, k) for which $M_{\lambda,k}$ is irreducible is described in [KK]. It follows from this description that if $k \notin -2 + \mathbb{Q}_{\geq 0}$, then $M_{-2,k}$, and hence $W_{-2,k}$, is irreducible. It is easy to check that

$S_k(a_0^* v_{0,k}) = v_{-2,k}$, so that S_k is a non-zero homomorphism. Hence it is surjective for such values of k. Therefore the character of its kernel for such k is equal to $\operatorname{ch} W_{0,k} - \operatorname{ch} W_{-2,k} = \operatorname{ch} V_k(\mathfrak{sl}_2)$. This completes the proof. $\square$

Next, we will describe the second screening operator for $\widehat{\mathfrak{sl}}_2$. For that we need to recall the Friedan–Martinec–Shenker bosonization.

7.4. **Friedan–Martinec–Shenker bosonization.** Consider the Heisenberg Lie algebra with the generators $p_n, q_n, n \in \mathbb{Z}$, and the central element $\mathbf{1}$ with the commutation relations

$$[p_n, p_m] = n\delta_{n,-m}\mathbf{1}, \qquad [q_n, q_m] = -n\delta_{n,-m}\mathbf{1}, \qquad [p_n, q_m] = 0.$$

We set

$$p(z) = \sum_{n\in\mathbb{Z}} p_n z^{-n-1}, \qquad q(z) = \sum_{n\in\mathbb{Z}} q_n z^{-n-1}.$$

For $\lambda \in \mathbb{C}, \mu \in \mathbb{C}$, let $\Pi_{\lambda,\mu}$ be the Fock representation of this Lie algebra generated by a highest weight vector $|\lambda,\mu\rangle$ such that

$$p_n|\lambda,\mu\rangle = \lambda\delta_{n,0}|\lambda,\mu\rangle, \quad q_n|\lambda,\mu\rangle = \mu\delta_{n,0}|\lambda,\mu\rangle, \quad n \geq 0; \quad \mathbf{1}|\lambda,\mu\rangle = |\lambda,\mu\rangle.$$

Consider the vertex operators $V_{\lambda,\mu}(z) : \Pi_{\lambda',\mu'} \to \Pi_{\lambda+\lambda',\mu+\mu'}$ given by the formula

$$V_{\lambda,\mu}(z) = T_{\lambda,\mu} z^{\lambda\lambda' - \mu\mu'} \exp\left(-\sum_{n<0} \frac{\lambda p_n + \mu q_n}{n} z^{-n}\right) \exp\left(-\sum_{n>0} \frac{\lambda p_n + \mu q_n}{n} z^{-n}\right),$$

where $T_{\lambda,\mu}$ is the translation operator $\Pi_{0,0} \to \Pi_{\lambda,\mu}$ sending the highest weight vector to the highest weight vector and commuting with all $p_n, q_n, n \neq 0$.

Abusing notation, we will write these operators as $e^{\lambda u + \mu v}$, where $u(z)$ and $v(z)$ stand for the anti-derivatives of $p(z)$ and $q(z)$, respectively, i.e., $p(z) = \partial_z u(z), q(z) = \partial_z v(z)$.

For $\alpha \in \mathbb{C}$, set

$$\Pi_\alpha = \bigoplus_{n\in\mathbb{Z}} \Pi_{n+\alpha, n+\alpha}.$$

Using the vertex operators, one defines a vertex algebra structure on the direct sum Π_0 (with the vacuum vector $|0,0\rangle$) as in [FB], § 5.2.6.

Moreover, Π_α is a module over the vertex algebra Π_0 for any $\alpha \in \mathbb{C}$.

The following theorem is due to Friedan, Martinec and Shenker [FMS] (see also [FF5]).

Theorem 7.3. *There is a (unique) embedding of vertex algebras $M \hookrightarrow \Pi_0$ under which the fields $a(z)$ and $a^*(z)$ are mapped to the fields*

$$\widetilde{a}(z) = e^{u+v}, \qquad \widetilde{a}^*(z) = (\partial_z e^{-u})e^{-v} = -{:}p(z)e^{-u-v}{:}\,.$$

Further, the image of M in Π_0 is equal to the kernel of the operator $\int e^u dz$.

Equivalently, Π_0 may be described as the localization of M with respect to a_{-1}, i.e.,

$$\Pi_0 \simeq M[(a_{-1})^{-1}] = \mathbb{C}[a_n]_{n<-1} \otimes \mathbb{C}[a_n^*]_{n\leq 0} \otimes \mathbb{C}[(a_{-1})^{\pm 1}]. \tag{7.6}$$

The vertex algebra structure on Π_0 is obtained by a natural extension of the vertex algebra structure on M.

Remark 7.4. Under the embedding $M \hookrightarrow \Pi_0$ the Virasoro vertex operator of M described in Section 5.3 maps to the following conformal vector in Π_0:

$$\frac{1}{2}{:}p(z)^2{:} - \frac{1}{2}\partial_z p(z) - \frac{1}{2}{:}q(z)^2{:} + \frac{1}{2}\partial_z q(z).$$

Thus, the map $M \hookrightarrow \Pi_0$ becomes a homomorphism of conformal vertex algebras if we choose these conformal structures. $\square$

The above "bosonization" allows us to make sense of the field $a(z)^\alpha$, where α is an arbitrary complex number. Namely, we replace $a(z)^\alpha$ with the field

$$\widetilde{a}(z)^\alpha = e^{\alpha(u+v)} : \Pi_0 \to \Pi_\alpha.$$

Now we take the tensor product $\Pi_0 \otimes \pi_0^{k+2}$, where we again assume that $k \neq -2$. This is a vertex algebra which contains $M \otimes \pi_0^{k+2}$, and hence $V_k(\mathfrak{sl}_2)$, as vertex subalgebras. In particular, for any $\alpha, \lambda \in \mathbb{C}$, the tensor product $\Pi_\alpha \otimes \pi_\lambda^{k+2}$ is a module over $V_k(\mathfrak{sl}_2)$ and hence over $\widehat{\mathfrak{sl}}_2$. We denote it by $\widetilde{W}_{\alpha,\lambda,k}$. In addition, we introduce the bosonic vertex operator

$$V_{2(k+2)}(z) = T_{2(k+2)} \exp\left(-\sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(-\sum_{n>0} \frac{b_n}{n} z^{-n}\right), \tag{7.7}$$

A straightforward computation similar to the one performed in Section 7.3 yields:

Proposition 7.5. *The residue*

$$\widetilde{S}_k = \int \widetilde{a}(z)^{-(k+2)} V_{2(k+2)}(z) : \widetilde{W}_{0,0,k} \to \widetilde{W}_{-(k+2),2(k+2),k}$$

is an intertwining operator between the $\widehat{\mathfrak{sl}}_2$*–modules* $\widetilde{W}_{0,0,k}$ *and* $\widetilde{W}_{-(k+2),2(k+2),k}$.

We call $\widetilde{S}_k$, or its restriction to $W_{0,k} \subset \widetilde{W}_{0,0,k}$, the *screening operator of the second kind* for $\widehat{\mathfrak{sl}}_2$. This operator was first introduced by V. Dotsenko [D].

Proposition 7.6. *For generic* k *the* $\widehat{\mathfrak{sl}}_2$*–submodule* $V_\kappa(\mathfrak{sl}_2) \subset W_{0,k}$ *is equal to the kernel of* $\widetilde{S}_k : W_{0,k} \to \widetilde{W}_{-(k+2),2(k+2),k}$.

Proof. We will show that for generic k the kernel of $\widetilde{S}_k : W_{0,k} \to \widetilde{W}_{-(k+2),2(k+2),k}$ coincides with the kernel of the screening operator of the first kind, $S_k : W_{0,k} \to W_{-2,k}$. This, together with Proposition 7.2, will imply the statement of the proposition. The above two kernels are equal to the intersection of $W_{0,k} \subset \widetilde{W}_{0,0,k}$ and the kernels of the operators $\widetilde{S}_k$ and S_k acting from $\widetilde{W}_{0,0,k}$ to $\widetilde{W}_{-(k+2),2(k+2),k}$ and $\widetilde{W}_{0,-2,k}$, respectively, (note that the operator S_k has an obvious extension to an operator $\widetilde{W}_{0,0,k} \to \widetilde{W}_{0,-2,k}$). Therefore it is sufficient to show that the kernels of $\widetilde{S}_k$ and S_k in $\widetilde{W}_{0,0,k}$ are equal for generic k.

Now let $\phi(z)$ be the anti-derivative of $b(z)$, i.e., $b(z) = \partial_z \phi(z)$. By abusing notation we will write $V_{-2}(z) = e^{-(k+2)^{-1}\phi}$ and $V_{2(k+2)}(z) = e^{\phi}$. Then the screening currents $S_k(z)$ and $\widetilde{S}_k(z)$ become:

$$S_k(z) = e^{u+v-(k+2)^{-1}\phi(z)}, \qquad \widetilde{S}_k(z) = e^{-(k+2)u-(k+2)v+\phi}.$$

Consider a more general situation: let $\mathfrak{h}$ be an abelian Lie algebra with a non-degenerate inner product κ. Using this inner product, we identify $\mathfrak{h}$ with $\mathfrak{h}^*$. Let $\widehat{\mathfrak{h}}_\kappa$ be the Heisenberg Lie algebra and $\pi^\kappa_\lambda, \lambda \in \mathfrak{h}^* \simeq \mathfrak{h}$ be its Fock representations, defined as in Section 5.1. Then for any $\chi \in \mathfrak{h}^*$ there is a vertex operator $V^\kappa_\chi(z) : \pi^\kappa_0 \to \pi^\kappa_\chi$ given by the formula

$$V^\kappa_\chi(z) = T_\chi \exp\left(-\sum_{n<0} \frac{\chi_n}{n} z^{-n} \right) \exp\left(-\sum_{n>0} \frac{\chi_n}{n} z^{-n} \right), \tag{7.8}$$

where the $\chi_n = \chi \otimes t^n$ are the elements of the Heisenberg Lie algebra $\widehat{\mathfrak{h}}_\kappa$ corresponding to $\chi \in \mathfrak{h}^*$, which we identify with $\mathfrak{h}$ using the inner product κ.

Suppose that $\kappa(\chi,\chi) \neq 0$, and denote by $\check{\chi}$ the element of $\mathfrak{h}$ equal to $-2\chi/\kappa(\chi,\chi)$. We claim that if χ is generic (i.e., away from countably many hypersurfaces in $\mathfrak{h}$) then the kernels of $\displaystyle\int V^\kappa_\chi(z)dz$ and $\displaystyle\int V^\kappa_{\check{\chi}}(z)dz$ in π^κ_0 coincide. Indeed, each of these kernels is equal to the tensor product of the subspace in π^κ_0 generated from the vacuum vector by elements of $\widehat{\mathfrak{h}}_\kappa$ which commute with χ_n (they span a Heisenberg Lie subalgebra of $\widehat{\mathfrak{h}}_\kappa$ corresponding to the orthogonal complement of χ in $\mathfrak{h}$; note that by our assumption on χ, this orthogonal complement does not contain χ) and the kernel of our operator on the Fock representation of the Heisenberg Lie algebra generated by $\chi_n, n \in \mathbb{Z}$. But the latter two kernels coincide for generic values of $\kappa(\chi,\chi)$, as shown in [FB], § 15.4.15. Hence the kernels of $\displaystyle\int V^\kappa_\chi(z)dz$ and $\displaystyle\int V^\kappa_{\check{\chi}}(z)dz$ also coincide generically.

Our situation corresponds to the 3–dimensional Lie algebra $\mathfrak{h}$ with a basis $\overline{u}, \overline{v}, \overline{\phi}$ with the following non-zero inner products of the basis elements:

$$\kappa(\overline{u},\overline{u}) = -\kappa(\overline{v},\overline{v}) = 1, \qquad \kappa(\overline{\phi},\overline{\phi}) = 2(k+2).$$

Our screening currents $S_k(z)$ and $\widetilde{S}_k(z)$ are equal to $V^\kappa_\chi(z)$ and $V^\kappa_{\check{\chi}}(z)$, where

$$\chi = \overline{u} + \overline{v} - (k+2)^{-1}\overline{\phi}, \qquad \check{\chi} = -(k+2)\chi = -(k+2)\overline{u} - (k+2)\overline{v} + \overline{\phi}.$$

Therefore for generic k the kernels of the screening operators S_k and $\widetilde{S}_k$ coincide. □

8. Intertwining operators for an arbitrary $\mathfrak{g}$

In this section we construct screening operators between Wakimoto modules over $\widehat{\mathfrak{g}}_\kappa$ for an arbitrary simple Lie algebra $\mathfrak{g}$ and use them to characterize $V_\kappa(\mathfrak{g})$ inside $W_{0,\kappa}$.

8.1. **Parabolic induction.** Denote by $\mathfrak{sl}_2^{(i)}$ the Lie subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}_2$, which is generated by e_i, h_i, and f_i. Let $\mathfrak{p}^{(i)}$ be the parabolic subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{b}_-$ and e_i, and $\mathfrak{m}^{(i)}$ its Levi subalgebra. Thus, $\mathfrak{m}^{(i)}$ is equal to the direct sum of $\mathfrak{sl}_2^{(i)}$ and the orthogonal complement $\mathfrak{h}_i^\perp$ of h_i in $\mathfrak{h}$.

We apply to $\mathfrak{p}^{(i)}$ the results on semi-infinite parabolic induction of Section 6. According to Corollary 6.2, we obtain a functor from the category of smooth representations of $\widehat{\mathfrak{sl}}_2 \oplus \widehat{\mathfrak{h}}^\perp_{i,\kappa_0}$, with k and κ_0 satisfying the conditions of Theorem 6.1, to the category of smooth $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$–modules. The condition on k and κ_0 is that the inner products on $\mathfrak{sl}_2^{(i)}$ corresponding to $(k+2)$ and κ_0 are both restrictions of an invariant inner product

$(\kappa - \kappa_c)$ on $\mathfrak{g}$. In other words, $(\kappa - \kappa_c)(h_i, h_i) = 2(k+2)$ and $\kappa_0 = \kappa|_{\mathfrak{h}_i^\perp}$. By abuse of notation we will write κ for κ_0. If k and κ satisfy this condition, then for any smooth $\widehat{\mathfrak{sl}}_2$–module R of level k and any smooth $\widehat{\mathfrak{h}}_\kappa^\perp$–module L the tensor product $M_{\mathfrak{g},\mathfrak{p}^{(i)}} \otimes R \otimes L$ is a smooth $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$–module.

In particular, if we choose R to be the Wakimoto module $W_{\lambda,k}$ over $\mathfrak{sl}_2$, and L to be the Fock representation $\pi^\kappa_{\lambda_0}$, the corresponding $\widehat{\mathfrak{g}}_\kappa$–module will be isomorphic to the Wakimoto module $W_{(\lambda,\lambda_0),\kappa+\kappa_c}$, where (λ, λ_0) is the weight of $\mathfrak{g}$ built from λ and λ_0. Under this isomorphism the generators $a_{\alpha_i,n}, n \in \mathbb{Z}$, will have a special meaning: they correspond to the *right* action of the elements $e_{i,n}$ of $\widehat{\mathfrak{g}}$, which was defined in Remark 4.4. In other words, making the above identification of modules forces us to choose a system of coordinates $\{y_\alpha\}_{\alpha \in \Delta_+}$ on N_+ such that $\rho^R(e_i) = \partial/\partial y_{\alpha_i}$ (in the notation of Section 1.5), and so $w^R(e_i(z)) = a_{\alpha_i}(z)$ (in the notation of Remark 4.4). From now on we will denote $w^R(e_i(z))$ by $e_i^R(z)$. For a general coordinate system on N_+ we have

$$e_i^R(z) = a_{\alpha_i}(z) + \sum_{\beta \in \Delta_+} P_\beta^{R,i}(a^*_\alpha(z)) a_\beta(z) \tag{8.1}$$

(see formula (4.1)).

Now any intertwining operator between Wakimoto modules $W_{\lambda_1,k}$ and $W_{\lambda_2,k}$ over $\widehat{\mathfrak{sl}}_2$ gives rise to an intertwining operator between the Wakimoto modules $W_{(\lambda_1,\lambda_0),\kappa+\kappa_c}$ and $W_{(\lambda_2,\lambda_0),\kappa+\kappa_c}$ over $\widehat{\mathfrak{g}}_{\kappa+\kappa_c}$ for any weight λ_0 of $\mathfrak{h}_i^\perp$. We will use this fact and the $\widehat{\mathfrak{sl}}_2$ screening operators introduced in the previous section to construct intertwining operators between Wakimoto modules over $\mathfrak{g}$.

8.2. **Screening operators of the first kind.** Let κ be a non-zero invariant inner product on $\mathfrak{g}$. We will use the same notation for the restriction of κ to $\mathfrak{h}$. Now to any $\chi \in \mathfrak{h}$ we associate a vertex operator $V^\kappa_\chi(z) : \pi^\kappa_0 \to \pi^\kappa_\chi$ given by formula (7.8). For $\kappa \neq \kappa_c$, we set

$$S_{i,\kappa}(z) \overset{\text{def}}{=} e_i^R(z) V^{\kappa-\kappa_c}_{-\alpha_i}(z) : W_{0,\kappa} \to W_{-\alpha_i,\kappa},$$

where $e_i^R(z)$ is given by formula (4.1). Note that

$$S_{i,\kappa}(z) = Y_{W_{0,\kappa},W_{-\alpha_i,\kappa}}(e^R_{i,-1} v_{-\alpha_i,\kappa}, z) \tag{8.2}$$

in the notation of Section 7.2.

According to Proposition 7.1 and the above discussion, the operator

$$S_{i,\kappa} = \int S_{i,\kappa}(z) dz : W_{0,\kappa} \to W_{-\alpha_i,\kappa}$$

is induced by the screening operator of the first kind S_k for the ith $\widehat{\mathfrak{sl}}_2$ subalgebra, where k is determined from the formula $(\kappa - \kappa_c)(h_i, h_i) = 2(k+2)$. Hence Proposition 7.1 implies:

Proposition 8.1. *The operator $S_{i,\kappa}$ is an intertwining operator between the $\widehat{\mathfrak{g}}_\kappa$–modules $W_{0,\kappa}$ and $W_{-\alpha_i,\kappa}$ for each $i = 1, \ldots, \ell$.*

We call $S_{i,\kappa}$ the ith screening operator of the first kind for $\widehat{\mathfrak{g}}_\kappa$.

Recall that by Proposition 5.2 $V_\kappa(\mathfrak{g})$ is naturally a $\widehat{\mathfrak{g}}_\kappa$–submodule and a vertex subalgebra of $W_{0,\kappa}$. On the other hand, the intersection of the kernels of $S_{i,\kappa}, i = 1, \ldots, \ell$, is a $\widehat{\mathfrak{g}}_\kappa$–submodule of $W_{0,\kappa}$, by Proposition 8.1, and a vertex subalgebra of $W_{0,\kappa}$, due to formula (8.2) and the commutation relations (7.3). The following proposition is proved in [FF8].

Proposition 8.2. *For generic κ, $V_\kappa(\mathfrak{g})$ is equal to the intersection of the kernels of the screening operators $S_{i,\kappa}, i = 1, \ldots, \ell$.*

Furthermore, in [FF8], § 3, a complex $C^\bullet(\mathfrak{g})$ of $\widehat{\mathfrak{g}}_\kappa$–modules is constructed for generic κ. Its ith degree term is the direct sum of the Wakimoto modules $W_{w(\rho)-\rho,\kappa}$, where w runs over all elements of the Weyl group of $\mathfrak{g}$ of length i. Its 0th cohomology is isomorphic to $V_\kappa(\mathfrak{g})$, and all other cohomologies vanish. For $\mathfrak{g} = \mathfrak{sl}_2$ this complex has length one and coincides with the one appearing in Proposition 7.2. In general, the degree zero term of the complex is $W_{0,\kappa}$, the degree one term is $\bigoplus_{i=1}^{\ell} W_{-\alpha_i,\kappa}$, and the zeroth differential is the sum of the screening operators $S_{i,\kappa}$.

In [FF8] it is also explained how to construct other intertwining operators as compositions of the screening operators $S_{i,\kappa}$ using the Bernstein-Gelfand-Gelfand resolution of the trivial representation of the quantum group $U_q(\mathfrak{g})$. Roughly speaking, the screening operators $S_{i,\kappa}$ satisfy the q–Serre relations, i.e., the defining relations of the quantized enveloping algebra $U_q(\mathfrak{n}_+)$ with appropriate parameter q. Then for generic κ we attach to a singular vector in the Verma module M_λ over $U_q(\mathfrak{g})$ of weight μ an intertwining operator $W_{\lambda,\kappa} \to W_{\mu,\kappa}$. This operator is equal to the integral of a product of the screening currents $S_{i,\kappa}(z)$ over a certain cycle on the configuration space with coefficients in a local system that is naturally attached to the above singular vector.

8.3. **Screening operators of the second kind.** In order to define the screening operators of the second kind, we need to make sense of the series $(e_i^R(z))^\alpha$. So we choose a system of coordinates on N_+ in such a way that $e_i^R(z) = a_{\alpha_i}(z)$ (note that this cannot be achieved for all $i = 1 \ldots, \ell$ simultaneously). This is automatically so if we define the Wakimoto modules over $\widehat{\mathfrak{g}}$ via the semi-infinite parabolic induction from Wakimoto modules over the ith subalgebra $\widehat{\mathfrak{sl}}_2$ (see Section 8.1).

Having chosen such a coordinate system, we define the series $a_{\alpha_i}(z)^\alpha$ using the Friedan–Martinec–Shenker bosonization of the Weyl algebra generated by $a_{\alpha_i,n}, a^*_{\alpha_i,n}$, $n \in \mathbb{Z}$, as explained in Section 7.4. Namely, we have a vertex algebra

$$\Pi_0^{(i)} = \mathbb{C}[a_{\alpha_i,n}]_{n<-1} \otimes \mathbb{C}[a^*_{\alpha_i,n}]_{n\leq 0} \otimes \mathbb{C}[a^{\pm 1}_{\alpha_i,-1}]$$

containing

$$M_{\mathfrak{g}}^{(i)} = \mathbb{C}[a_{\alpha_i,n}]_{n\leq -1} \otimes \mathbb{C}[a^*_{\alpha_i,n}]_{n\leq 0}$$

and a $\Pi_0^{(i)}$–module $\Pi_\gamma^{(i)}$ defined as in Section 7.4. We then set

$$\widetilde{W}^{(i)}_{\gamma,\lambda,\kappa} = W_{\lambda,\kappa} \underset{M_{\mathfrak{g}}^{(i)}}{\otimes} \Pi_\gamma^{(i)}.$$

This is a $\widehat{\mathfrak{g}}_\kappa$–module, which contains $W_{\lambda,\kappa}$ if $\alpha = 0$. Note that $\widetilde{W}^{(i)}_{0,0,\kappa_c}$ is the $\widehat{\mathfrak{g}}$–module obtained by the semi-infinite parabolic induction from the $\widehat{\mathfrak{sl}}_2$–module $\widetilde{W}_{0,0,-2}$.

Now let $\beta = \frac{1}{2}(\kappa - \kappa_c)(h_i, h_i)$ and define the field

$$\widetilde{S}_{\kappa,i}(z) \overset{\mathrm{def}}{=} (e_i^R(z))^{-\beta} V_{\check{\alpha}_i}(z) : \widetilde{W}^{(i)}_{0,0,\kappa} \to \widetilde{W}^{(i)}_{-\beta,\beta\check{\alpha}_i,\kappa}.$$

Here $\check{\alpha}_i = h_i \in \mathfrak{h}$ denotes the ith coroot of $\mathfrak{g}$. Then the operator $\widetilde{S}_{\kappa,i} = \displaystyle\int \widetilde{S}_{\kappa,i}(z)dz$ is induced by the screening operator of the second kind $\widetilde{S}_k$ for the ith $\widehat{\mathfrak{sl}}_2$ subalgebra, where k is determined from the formula $(\kappa - \kappa_c)(h_i, h_i) = 2(k+2)$. Hence Proposition 7.1 implies (see also [PRY]):

Proposition 8.3. *The operator $\widetilde{S}_{\kappa,i}$ is an intertwining operator between the $\widehat{\mathfrak{g}}_\kappa$–modules $\widetilde{W}^{(i)}_{0,0,\kappa}$ and $\widetilde{W}^{(i)}_{-\beta,\beta\check{\alpha}_i,\kappa}$.*

Note that $\widetilde{S}_{i,\kappa}$ is the residue of $Y_{V,M}((e^R_{i,-1})^{-\beta} v_{\check{\alpha}_i,\kappa}, z)$, where $V = \widetilde{W}^{(i)}_{0,0,\kappa}$ and $M = \widetilde{W}^{(i)}_{-\beta,\beta\check{\alpha}_i,\kappa}$ (see Section 7.2). Therefore, according to the commutation relations (7.3), the intersection of kernels of $\widetilde{S}_{i,\kappa}, i = 1, \ldots, \ell$, is naturally a vertex subalgebra of $\widetilde{W}^{(i)}_{0,0,\kappa}$ or $W_{0,\kappa}$. Combining Proposition 8.2 and Proposition 7.6, we obtain:

Proposition 8.4. *For generic κ, $V_\kappa(\mathfrak{g})$ is equal, as a $\widehat{\mathfrak{g}}_\kappa$–module and as a vertex algebra, to the intersection of the kernels of the screening operators*

$$\widetilde{S}_{i,\kappa} : W_{0,\kappa} \to \widetilde{W}^{(i)}_{-\beta,\beta\check{\alpha}_i,\kappa}, \qquad i = 1, \ldots, \ell.$$

One can use the screening operators $\widetilde{S}_{i,\kappa}$ to construct more general intertwining operators following the procedure of [FF8].

8.4. **Screening operators of second kind at the critical level.** Now we define the limits of the intertwining operators $\widetilde{S}_{i,\kappa}$ as $\kappa \to \kappa_c$.

First we consider the case when $\mathfrak{g} = \mathfrak{sl}_2$. In order to define the limit of $\widetilde{S}_k$ as $k \to -2$ we make $W_{0,k}$ and $\widetilde{W}_{-(k+2),2(k+2),k}$ into free modules over $\mathbb{C}[\beta]$, where β is a formal variable representing $k+2$, and then consider the quotient of these modules by the ideal generated by β.

More precisely, let $\pi_0[\beta]$ (resp., $\pi_{2\beta}[\beta]$) be the free $\mathbb{C}[\beta]$–module spanned by the lexicographically ordered monomials in $b_n, n < 0$, applied to a vector $|0\rangle$ (resp., $|2\beta\rangle$). We define the structure of vertex algebra over $\mathbb{C}[\beta]$ on $\pi_0[\beta]$ as in Section 5.1. Then $\pi_{2\beta}[\beta]$ is a module over $\pi_0[\beta]$. The dependence on β comes from the commutation relations

$$[b_n, b_m] = 2\beta n \delta_{n,-m}$$

and the fact that b_0 acts on $\pi_{2\beta}[\beta]$ by multiplication by 2β. Taking the quotient of $\pi_0[\beta]$ (resp., $\pi_{2\beta}[\beta]$) by the ideal generated by $(\beta - k), k \in \mathbb{C}$, we obtain the vertex algebra π_0^k (resp., the module π_{2k}^k over π_0^k) introduced in Section 7.1.

We define free $\mathbb{C}[\beta]$–modules $W_0[\beta]$ and $\widetilde{W}_{0,0}[\beta]$ as the tensor products $M \otimes_{\mathbb{C}} \pi_0[\beta]$ and $\Pi_0 \otimes_{\mathbb{C}} \pi_0[\beta]$, respectively. These are vertex algebras over $\mathbb{C}[\beta]$, and their quotients by the ideals generated by $(\beta - k), k \in \mathbb{C}$, are the vertex algebras $W_{0,k}$ and $\widetilde{W}_{0,0,k}$, respectively.

Next, let $\Pi_{-\beta+n,-\beta+n}$ be the free $\mathbb{C}[\beta]$–module spanned by the lexicographically ordered monomials in $p_n, q_n, n<0$, applied to a vector $|-\beta+n,-\beta+n\rangle$. Set

$$\Pi_{-\beta}=\bigoplus_{n\in\mathbb{Z}}\Pi_{-\beta+n,-\beta+n},$$

and

$$\widetilde{\widetilde{W}}_{-\beta,2\beta}=\Pi_{-\beta}\otimes_{\mathbb{C}[\beta]}\pi_{2\beta}[\beta].$$

Each Fourier coefficient of the formal power series $V_{2\beta}(z):\pi_0[\beta]\to\pi_{2\beta}[\beta]$ and $\widetilde{a}(z)^{-\beta}:\Pi_0\otimes_{\mathbb{C}}\mathbb{C}[\beta]\to\Pi_{-\beta}$, given by the formulas above, is a well-defined linear operator commuting with the action of $\mathbb{C}[\beta]$. Hence the Fourier coefficients of their product are also well-defined linear operators from $\widetilde{W}_0[\beta]$ to $\widetilde{\widetilde{W}}_{-\beta,2\beta}$. The corresponding operators on the quotients by the ideals generated by $(\beta-k), k\in\mathbb{C}$, coincide with the operators introduced above. We need to compute explicitly the leading term in the β–expansion of the residue $\int\widetilde{a}(z)^{-\beta}V_{2\beta}(z)dz$ and its restriction to $W_0[\beta]$.

Consider first the expansion of $V_{2\beta}(z)$ in powers of β. Let us write $V_{2\beta}(z)=\sum_{n\in\mathbb{Z}}V_{2\beta}[n]z^{-n}$. Introduce the operators $\overline{V}[n], n\leq 0$, via the formal power series

$$\sum_{n\leq 0}\overline{V}[n]z^{-n}=\exp\left(\sum_{m>0}\frac{b_{-m}}{m}z^m\right).$$

Using formula (7.7), we obtain the following expansion of $V_{2\beta}[n]$:

$$V_{2\beta}[n]=\begin{cases}\overline{V}[n]+\beta(\ldots), & n\leq 0,\\ -2\beta\sum_{m\leq 0}\overline{V}[m]\dfrac{\partial}{\partial b_{m-n}}+\beta^2(\ldots), & n>0.\end{cases}$$

We denote the β–linear term of $V_{2\beta}[n], n>0$, by $\overline{V}[n]$.

Next, we consider the expansion of $\widetilde{a}(z)^{-\beta}=e^{-\beta(u+v)}$ in powers of β. Let us write $\widetilde{a}(z)^{-\beta}=\sum_{n\in\mathbb{Z}}\widetilde{a}(z)^{-\beta}_{[n]}z^{-n}$. We will identify $\Pi_{-\beta}$ with $\Pi_0\otimes_{\mathbb{C}}\mathbb{C}[\beta]$, as $\mathbb{C}[\beta]$–modules. Then we find that

$$\widetilde{a}(z)^{-\beta}_{[n]}=\begin{cases}1+\beta(\ldots), & n=0,\\ \beta\dfrac{p_n+q_n}{n}+\beta^2(\ldots), & n\neq 0.\end{cases}$$

The above formulas imply the following expansion of the screening operator:

$$\int\widetilde{a}(z)^{-\beta}V_{2\beta}(z)dz=\beta\left(\overline{V}[1]+\sum_{n>0}\frac{1}{n}\overline{V}[-n+1](p_n+q_n)\right)+\beta^2(\ldots).$$

Therefore we define the limit of the screening operator $\widetilde{S}_k$ at $k=-2$ as the operator

$$\overline{S}\overset{\text{def}}{=}\overline{V}[1]+\sum_{n>0}\frac{1}{n}\overline{V}[-n+1](p_n+q_n), \tag{8.3}$$

acting on $\widetilde{\widetilde{W}}_{0,0,-2}=\Pi_0\otimes\pi^0$. It follows from the construction that $\overline{S}$ commutes with the $\widehat{\mathfrak{sl}}_2$–action on $\widetilde{\widetilde{W}}_{0,0,-2}$.

It is possible to express the operators

$$\frac{p_n+q_n}{n}=-(u_n+v_n), \qquad n>0,$$

in terms of the Heisenberg algebra generated by $a_m, a^*_m, m \in \mathbb{Z}$. Namely, from the definition $\widetilde{a}(z)=e^{u+v}$ it follows that $u(z)+v(z)=\log \widetilde{a}(z)$, so that the field $u(z)+v(z)$ commutes with $a(w)$, and we have the following OPE with $a^*(w)$:

$$(u(z)+v(z))a^*(w)=\frac{1}{z-w}\widetilde{a}(w)^{-1}+\text{reg.}$$

Therefore, writing $\widetilde{a}(z)^{-1}=\sum_{n\in\mathbb{Z}}\widetilde{a}(z)^{-1}_{[n]}z^{-n}$, we obtain the following commutation relations:

$$\left[\frac{p_n+q_n}{n}, a^*_m\right]=-\widetilde{a}(z)^{-1}_{[n+m-1]}, \qquad n>0. \tag{8.4}$$

Using the realization (7.6) of Π_0, the series $\widetilde{a}(z)^{-1}$ is expressed as follows:

$$\widetilde{a}(z)^{-1}=(a_{-1})^{-1}\left(1+(a_{-1})^{-1}\sum_{n\neq -1}a_n z^{-n-1}\right)^{-1}, \tag{8.5}$$

where the right hand side is expanded as a formal power series in positive powers of $(a_{-1})^{-1}$. It is easy to see that each Fourier coefficient of this power series is well-defined as a linear operator on Π_0.

The above formulas completely determine the action of $(p_n+q_n)/n, n>0$, and hence of $\overline{S}$, on any vector in $W_{0,-2}=M\otimes\pi^0\subset\widetilde{W}_{0,0,-2}$. Namely, we use the commutation relations (8.4) to move $(p_n+q_n)/n$ through the a^*_m's. As the result, we obtain Fourier coefficients of $\widetilde{a}(z)^{-1}$, which are given by formula (8.5). Applying each of them to any vector in $W_{0,-2}$, we always obtain a finite sum.

This completes the construction of the limit $\overline{S}$ of the screening operator $\widetilde{S}_k$ as $k\to -2$ in the case of $\mathfrak{sl}_2$. Now we consider the case of an arbitrary $\mathfrak{g}$.

The limit of $\widetilde{S}_{i,\kappa}$ as $\kappa\to\kappa_c$ is by definition the operator $\overline{S}_i$ on $\widetilde{W}^{(i)}_{0,0,\kappa_c}$, obtained from $\overline{S}$ via the functor of semi-infinite parabolic induction. Therefore it is given by the formula

$$\overline{S}_i=\overline{V}_i[1]+\sum_{n>0}\frac{1}{n}\overline{V}_i[-n+1](p_{i,n}+q_{i,n}). \tag{8.6}$$

Here $\overline{V}_i[n]:\pi^0\to\pi^0$ are the linear operators given by the formulas

$$\sum_{n\leq 0}\overline{V}_i[n]z^{-n}=\exp\left(\sum_{m>0}\frac{b_{i,-m}}{m}z^m\right), \tag{8.7}$$

$$\overline{V}_i[1]=-\sum_{m\leq 0}\overline{V}_i[m]\,\mathbf{D}_{b_{i,m-1}},$$

where $\mathbf{D}_{b_{i,m}}$ denotes the derivative in the direction of $b_{i,m}$ given by the formula

$$\mathbf{D}_{b_{i,m}}\cdot b_{j,n}=a_{ji}\delta_{n,m}, \tag{8.8}$$

and (a_{kl}) is the Cartan matrix of $\mathfrak{g}$ (it is normalized so that we have $\mathbf{D}_{b_{i,m}} \cdot b_{i,n} = 2\delta_{n,m}$). The operators $(p_{i,n} + q_{i,n})/n$ acting on $\Pi_0^{(i)}$ are defined in the same way as above.

Thus, we obtain well-defined linear operators $\overline{S}_i : W_{0,\kappa_c} \to \widetilde{W}^{(i)}_{0,0,\kappa_c}$. By construction, they commute with the action of $\widehat{\mathfrak{g}}_{\kappa_c}$ on both modules. It is clear that the operators $\overline{S}_i, i = 1, \ldots, \ell$, annihilate the highest weight vector of W_{0,κ_c}. Therefore they annihilate all vectors obtained from the highest weight vector under the action of $\widehat{\mathfrak{g}}_{\kappa_c}$, i.e., all vectors in $V_{\kappa_c}(\mathfrak{g}) \subset W_{0,\kappa_c}$. Thus we obtain

Proposition 8.5. *The vacuum module $V_{\kappa_c}(\mathfrak{g})$ is contained in the intersection of the kernels of the operators $\overline{S}_i : W_{0,\kappa_c} \to \widetilde{W}^{(i)}_{0,0,\kappa_c}, i = 1, \ldots, \ell$.*

9. DESCRIPTION OF THE CENTER OF $V_{\kappa_c}(\mathfrak{g})$

In this section we use Proposition 8.5 to describe the center of $V_{\kappa_c}(\mathfrak{g})$. Recall from [FB], § 5.7.2, that the center of a vertex algebra V is by definition its commutative vertex subalgebra spanned by all vectors $v \in V$ such that $Y(A,z)v \in V[[z]]$ for all $A \in V$. In the case when $V = V_\kappa(\mathfrak{g})$, the center is equal to the space of invariants of the Lie subalgebra $L_+\mathfrak{g} = \mathfrak{g}[[t]]$ of $\widehat{\mathfrak{g}}_\kappa$. According to Lemma 18.4.1 of [FB], it is trivial, i.e., equal to the span of the vacuum vector v_κ if $\kappa \neq \kappa_c$. So let us consider the center of $V_{\kappa_c}(\mathfrak{g})$, which we denote by $\mathfrak{z}(\widehat{\mathfrak{g}})$.

9.1. **The center of $V_{\kappa_c}(\mathfrak{g})$ and Wakimoto modules.** Recall that by Proposition 5.2 the homomorphism $w_{\kappa_c} : V_{\kappa_c}(\mathfrak{g}) \hookrightarrow W_{0,\kappa_c}$ is injective. The vertex algebra $V_{\kappa_c}(\mathfrak{g})$ contains the commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$, its center. On the other hand, $W_{0,\kappa_c} = M_\mathfrak{g} \otimes \pi_0$ contains the commutative subalgebra π_0, which is its center.

Lemma 9.1. *The image of $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset V_{\kappa_c}(\mathfrak{g})$ in $W_{\kappa_c,0}$ under w_{κ_c} is contained in $\pi_0 \subset W_{\kappa_c,0}$.*

Proof. Observe that the lexicographically ordered monomials of the form

$$\prod_{n_{r(\alpha)}<0} e^R_{\alpha,n_{r(\alpha)}} \prod_{m_{s(\alpha)}\leq 0} a^*_{\alpha,m_{s(\alpha)}} \prod_{n_{t(i)}<0} b_{i,n_{t(i)}}|0\rangle, \tag{9.1}$$

where the $e^R_{i,n}$'s are given by formula (4.1), form a basis of W_{0,κ_c}. The image of any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ in $W_{\kappa_c,0}$ is an $L_+\mathfrak{g}$–invariant vector. In particular, it is annihilated by $L_+\mathfrak{n}_+ = \mathfrak{n}_+[[t]]$ and by $\mathfrak{h}$. Since $L_+\mathfrak{n}_+$ commutes with $e^R_{\alpha,n}$ and $b_{i,n}$, the space of $L_+\mathfrak{n}_+$–invariants of $W_{\kappa_c,0}$ is equal to the tensor product of the subspace $M_{\mathfrak{g},-}$ of W_{0,κ_c} spanned by all monomials (9.1) not containing $a^*_{\alpha,n}$, and the space of $L_+\mathfrak{n}_+$–invariants in $M_{\mathfrak{g},+} = \mathbb{C}[a^*_{\alpha,n}]_{\alpha\in\Delta_+, n\leq 0}$. According to Section 3.3, $M_{\mathfrak{g},+}$ is an $L_+\mathfrak{g}$–module isomorphic to $\mathrm{Coind}^{L_+\mathfrak{g}}_{L_+\mathfrak{b}_-} \mathbb{C}$. Therefore the action of $L_+\mathfrak{n}_+$ on it is co-free, and the space of $L_+\mathfrak{n}_+$–invariants is one-dimensional, spanned by constants.

Thus, we obtain that the space of $L_+\mathfrak{n}_+$–invariants in W_{0,κ_c} is equal to $M_{\mathfrak{g},-}$. However, its subspace of $\mathfrak{h}$–invariants is the span of the monomials (9.1), which only involve the $b_{i,n}$'s, i.e., $\pi_0 \subset W_{0,\kappa_c}$. This completes the proof. □

Using Proposition 8.5, we now obtain that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is contained in the intersection of the kernels of the operators $\overline{S}_i, i = 1, \ldots, \ell$, restricted to $\pi_0 \subset W_{0,\kappa_c}$. But according to formula (8.6), the restriction of $\overline{S}_i$ to π_0 is nothing but the operator $\overline{V}_i[1] : \pi_0 \to \pi_0$ given by the formula

$$\overline{V}_i[1] = -\sum_{m\leq 0} \overline{V}_i[m]\, \mathbf{D}_{b_{i,m-1}} \tag{9.2}$$

where $\overline{V}_i[m], \mathbf{D}_{b_{i,m-1}}, m \leq 0$, are given by formulas (8.7) and (8.8), respectively. Therefore we obtain the following

Proposition 9.2. *The center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $V_{\kappa_c}(\mathfrak{g})$ is contained in the intersection of the kernels of the operators $\overline{V}_i[1], i = 1, \dots, \ell$, in π_0.*

We now show that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is actually equal to the intersection of the kernels of the operators $\overline{V}_i[1], i = 1, \dots, \ell$.

9.2. **The associated graded of $\mathfrak{z}(\widehat{\mathfrak{g}})$.**

The $\widehat{\mathfrak{g}}_{\kappa_c}$–module $V_{\kappa_c}(\mathfrak{g})$ has a natural filtration induced by the Poincaré–Birkhoff–Witt filtration on the universal enveloping algebra $U(\widehat{\mathfrak{g}}_{\kappa_c})$, and the action of $\widehat{\mathfrak{g}}_{\kappa_c}$ preserves this filtration. Furthermore, it is clear that the associated graded space $\operatorname{gr} V_{\kappa_c}(\mathfrak{g})$ is isomorphic to

$$\operatorname{Sym} \mathfrak{g}((t))/\mathfrak{g}[[t]] \simeq \operatorname{Fun} \mathfrak{g}[[t]],$$

where we use the following (coordinate-dependent) pairing

$$\langle A \otimes f(t), B \otimes g(t)\rangle = \kappa(A, B) \operatorname{Res} f(t)g(t)\frac{dt}{t}, \tag{9.3}$$

and κ is an arbitrary fixed non-degenerate inner product on $\mathfrak{g}$.

Recall that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is equal to the space of $\mathfrak{g}[[t]]$–invariants in $V_{\kappa_c}(\mathfrak{g})$. The symbol of a $\mathfrak{g}[[t]]$–invariant vector in $V_{\kappa_c}(\mathfrak{g})$ is a $\mathfrak{g}[[t]]$–invariant vector in $\operatorname{gr} V_{\kappa_c}(\mathfrak{g})$, i.e., an element of the space of $\mathfrak{g}[[t]]$–invariants in $\operatorname{Fun} \mathfrak{g}[[t]]$. Hence we obtain an injective map

$$\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) \hookrightarrow (\operatorname{Fun} \mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}. \tag{9.4}$$

In order to describe the space $(\operatorname{Fun} \mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}$, recall that $(\operatorname{Fun} \mathfrak{g})^{\mathfrak{g}}$ is a polynomial algebra in generators $P_i, i = 1, \dots, \ell$, of degrees $d_i + 1$, where d_i is the ith exponent of $\mathfrak{g}$. Each P_i is a polynomial in the basis elements J^a of $\mathfrak{g}$ (considered as linear functionals of $\mathfrak{g}$ via the inner product κ). Substituting the generating functions

$$J^a(t) = \sum_{n<0} J^a_n t^{-n-1}$$

of the basis elements J^a_n of $t^{-1}\mathfrak{g}[t^{-1}]$ into P_i instead of the J^a's, we obtain a generating series

$$P_i(J^a(t)) = \sum_{n\geq 0} P_{i,n} t^n$$

of elements of $\operatorname{Fun} \mathfrak{g}[[t]]$. Clearly, its coefficients $P_{i,n}$ is $\mathfrak{g}[[t]]$–invariant. Furthermore we have the following result due to Beilinson and Drinfeld, [BD1], § 2.4.1. For reader's convenience, we reproduce the proof (see also [M], Appendix, Prop. A.1):

Proposition 9.3. *The algebra $(\operatorname{Fun} \mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}$ is the free polynomial algebra in the generators $P_{i,n}, i = 1, \dots, \ell; n \geq 0$.*

Proof. Let G be the connected simply-connected algebraic group with the Lie algebra $\mathfrak{g}$. Then the space of $\mathfrak{g}[[t]]$–invariants in $\operatorname{Fun}\mathfrak{g}[[t]]$ coincides with the space of $G[[t]]$–invariants in $\operatorname{Fun}\mathfrak{g}[[t]]$.

Recall that given an algebraic variety X, we denote by JX its infinite jet scheme, as defined in [FB], § 9.4.4 (see also [M], where the notation X_∞ is used). In particular, $J\mathfrak{g} = \mathfrak{g}[[t]]$ and $JG = G[[t]]$.

Let $\mathfrak{g}_{\mathrm{reg}}$ be the smooth open subset of $\mathfrak{g}$ consisting of regular elements. It is known that the morphism

$$\chi : \mathfrak{g}_{\mathrm{reg}} \to \mathcal{P} := \operatorname{Spec}(\operatorname{Fun}\mathfrak{g})^{\mathfrak{g}} = \operatorname{Spec}\mathbb{C}[P_1, \ldots, P_\ell]$$

is smooth and surjective (see [Ko]). Therefore the morphism

$$J\chi : J\mathfrak{g}_{\mathrm{reg}} \to J\mathcal{P} := \operatorname{Spec}\mathbb{C}[P_{1,m}, ..., P_{\ell,m}]_{m \geq 0}$$

is also smooth and surjective.

Consider the map $a : G \times \mathfrak{g}_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{reg}} \underset{\mathcal{P}}{\times} \mathfrak{g}_{\mathrm{reg}}$ defined by the formula $a(g, x) = (x, g \cdot x)$. The map a is smooth, and since G acts transitively along the fibers of χ, it is also surjective. Hence the corresponding map of jet schemes $Ja : JG \times J\mathfrak{g}_{\mathrm{reg}} \to J\mathfrak{g}_{\mathrm{reg}} \underset{J\mathcal{P}}{\times} J\mathfrak{g}_{\mathrm{reg}}$ is surjective. Given two points y_1, y_2 in the same fiber of $J\chi$, let (h, y_1) be a point in the (non-empty) fiber $(Ja)^{-1}(y_1, y_2)$. Then $y_2 = h \cdot y_1$. Hence JG acts transitively along the fibers of the map $J\chi$. This implies that the ring of JG–invariant functions on $J\mathfrak{g}_{\mathrm{reg}}$ is equal to $\mathbb{C}[P_{1,m}, ..., P_{\ell,m}]_{m \geq 0}$. Because $\mathfrak{g}_{\mathrm{reg}}$ is an open dense subset of $\mathfrak{g}$, we obtain that $J\mathfrak{g}_{\mathrm{reg}}$ is dense in $J\mathfrak{g}$, and so any JG–invariant function on $J\mathfrak{g}$ is determined by its restriction to $J\mathfrak{g}_{\mathrm{reg}}$. This proves the proposition. $\square$

We have an action of the operator $L_0 = -t\partial_t$ on $\operatorname{Fun}\mathfrak{g}[[t]]$. It defines a $\mathbb{Z}$–gradation on $\operatorname{Fun}\mathfrak{g}[[t]]$ such that $\deg J^a_n = -n$. Then $\deg P_{i,n} = d_i + n + 1$, and we obtain the following formula for the character of $(\operatorname{Fun}\mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}$ (i.e., the generating function of its graded dimensions):

$$\operatorname{ch}(\operatorname{Fun}\mathfrak{g}[[t]])^{\mathfrak{g}[[t]]} = \prod_{i=1}^{\ell} \prod_{n_i \geq d_i+1} (1 - q^{n_i})^{-1}. \tag{9.5}$$

9.3. **Computation of the character of $\mathfrak{z}(\widehat{\mathfrak{g}})$.** We now show that the map (9.4) is an isomorphism using Proposition 6.3.

Consider the Lie subalgebra $\widetilde{\mathfrak{b}}_+ = (\mathfrak{b}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]])$ of $\mathfrak{g}[[t]]$. The natural surjective homomorphism $M_{0,\kappa_c} \to V_{\kappa_c}(\mathfrak{g})$ gives rise to a map of $\widetilde{\mathfrak{b}}_+$–invariants

$$\phi : M_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+} \to V_{\kappa_c}(\mathfrak{g})^{\widetilde{\mathfrak{b}}_+}.$$

Both M_{0,κ_c} and $V_{\kappa_c}(\mathfrak{g})$ have natural filtrations which are preserved by the homomorphism between them. Therefore we have the corresponding map of associated graded

$$\phi_{\mathrm{cl}} : (\operatorname{gr} M_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} \to (\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\widetilde{\mathfrak{b}}_+}.$$

Since $V_{\kappa_c}(\mathfrak{g})$ is a direct sum of finite-dimensional representations of the constant subalgebra $\mathfrak{g} \otimes 1$ of $\mathfrak{g}[[t]]$, we find that any $\widetilde{\mathfrak{b}}_+$–invariant in $V_{\kappa_c}(\mathfrak{g})$ or $\operatorname{gr} V_{\kappa_c}(\mathfrak{g})$ is automatically a $\mathfrak{g}[[t]]$–invariant. Thus, we have

$$\begin{aligned} V_{\kappa_c}(\mathfrak{g})^{\widetilde{\mathfrak{b}}_+} &= V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}, \\ (\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\widetilde{\mathfrak{b}}_+} &= (\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]} = \mathbb{C}[P_{1,m}, ..., P_{\ell,m}]_{m \geq 0}. \end{aligned}$$

We need to describe $(\operatorname{gr} M_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+}$. First, observe that

$$\operatorname{gr} M_{0,\kappa_c} = \operatorname{Sym} \mathfrak{g}((t))/\widetilde{\mathfrak{b}}_+ \simeq \operatorname{Fun} \widetilde{\mathfrak{n}}_+^{(-1)},$$

where

$$\widetilde{\mathfrak{n}}_+^{(-1)} = (\mathfrak{n}_+ \otimes t^{-1}) \oplus \mathfrak{g}[[t]],$$

and we use the pairing (9.3) on $\mathfrak{g}((t))$. In terms of this identification, the map ϕ_{cl} becomes a ring homomorphism

$$\operatorname{Fun} \widetilde{\mathfrak{n}}_+^{(-1)} \to \operatorname{Fun} \mathfrak{g}[[t]]$$

induced by the natural embedding $\mathfrak{g}[[t]] \to \widetilde{\mathfrak{n}}_+^{(-1)}$.

We construct $\widetilde{\mathfrak{b}}_+$–invariant functions on $\widetilde{\mathfrak{n}}_+^{(-1)}$ in the same way as above, by substituting the generating functions $J^a(t)$ into the P_i's. However, now $J^a(t)$ has non-zero t^{-1} coefficients if the inner product of J^a with $\mathfrak{n}_+$ is non-zero. Therefore the resulting series $P_i(J^a(t))$ will have non-zero coefficients $P_{i,m}$ in front of t^m if and only if $m \geq -d_i$. Thus, we obtain a natural inclusion

$$\mathbb{C}[P_{i,m_i}]_{i=1,\ldots,\ell; m_i \geq -d_i} \subset (\operatorname{Fun} \widetilde{\mathfrak{n}}_+^{(-1)})^{\widetilde{\mathfrak{b}}_+}.$$

Lemma 9.4. *This inclusion is an equality.*

Proof. Let $\widetilde{\mathfrak{n}}_+ = (\mathfrak{n}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]]) = t\widetilde{\mathfrak{n}}_+^{(-1)}$. Clearly, the spaces of $\widetilde{\mathfrak{b}}_+$–invariant functions on $\widetilde{\mathfrak{n}}_+^{(-1)}$ and $\widetilde{\mathfrak{n}}_+$ are isomorphic, so we will consider the latter space.

Denote by $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$ the intersection of $\widetilde{\mathfrak{n}}_+$ and $J\mathfrak{g}_{\mathrm{reg}} = \mathfrak{g}_{\mathrm{reg}} \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]])$. Thus, $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}} = (\mathfrak{n}_+^{\mathrm{reg}} \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]])$, where $\mathfrak{n}_+^{\mathrm{reg}} = \mathfrak{n}_+ \cap \mathfrak{g}_{\mathrm{reg}}$ is an open dense subset of $\mathfrak{n}_+$, so that $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$ is open and dense in $\widetilde{\mathfrak{n}}_+$.

Recall the morphism $J\chi : J\mathfrak{g}_{\mathrm{reg}} \to J\mathcal{P}$ introduced in the proof of Proposition 9.3. It was shown there that the group $JG = G[[t]]$ acts transitively along the fibers of $J\chi$. Let $\widetilde{B}_+$ be the subgroup of $G[[t]]$ corresponding to the Lie algebra $\widetilde{\mathfrak{b}}_+ \subset \mathfrak{g}[[t]]$. Note that for any $x \in \widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$, the group $\widetilde{B}_+$ is equal to the subgroup of all elements g of $G[[t]]$ such that $g \cdot x \in \widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$. Therefore $\widetilde{B}_+$ acts transitively along the fibers of the restriction of the morphism $J\chi$ to $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$.

This implies that the ring of $\widetilde{B}_+$–invariant (equivalently, $\widetilde{\mathfrak{b}}_+$–invariant) polynomials on $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$ is the ring of functions on the image of $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$ in $J\mathcal{P}$ under the map $J\chi$. But it follows from the construction that the image of $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$ in $J\mathcal{P}$ is the subspace determined by the equations $P_{i,m} = 0, i = 1, \ldots, \ell; m > 0$. Hence the ring of $\widetilde{\mathfrak{b}}_+$–invariant polynomials on $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$ is equal to $\mathbb{C}[P_{i,m}]_{i=1,\ldots,\ell;m>0}$. Since $\widetilde{\mathfrak{n}}_+^{\mathrm{reg}}$ is dense in $\widetilde{\mathfrak{n}}_+$, we obtain that this is also the ring of invariant polynomials on $\widetilde{\mathfrak{n}}_+$. When we pass from $\widetilde{\mathfrak{n}}_+$ to $\widetilde{\mathfrak{n}}_+^{(-1)}$, we obtain the statement of the lemma. ☐

Corollary 9.5. *The map ϕ_{cl} is surjective.*

Since $\deg P_{i,m} = d_i + m + 1$, we obtain the character of $(\operatorname{gr} M_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} = (\operatorname{Fun} \widetilde{\mathfrak{n}}_+^{(-1)})^{\widetilde{\mathfrak{b}}_+}$:

$$\operatorname{ch} (\operatorname{gr} M_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} = \prod_{m>0} (1 - q^m)^{-\ell}.$$

Now recall that by Theorem 6.3, the Verma module M_{0,κ_c} is isomorphic to the Wakimoto module W^+_{0,κ_c}. Hence $(M_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} = (W^+_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+}$. In addition, according to Lemma 6.5 we have $(W^+_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} = \pi_0$, and so its character is also equal to $\prod\limits_{m>0} (1-q^m)^{-\ell}$. Therefore we find that the natural embedding

$$\operatorname{gr}(M_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}) \hookrightarrow (\operatorname{gr} M_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+}$$

is an isomorphism. Now consider the commutative diagram

$$\begin{tikzcd}
\operatorname{gr}(M_{0,\kappa_c}^{\widetilde{\mathfrak{b}}_+}) \arrow[r] \arrow[d] & \operatorname{gr}(V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}) \arrow[d] \\
(\operatorname{gr} M_{0,\kappa_c})^{\widetilde{\mathfrak{b}}_+} \arrow[r] & (\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}.
\end{tikzcd}$$

It follows from the above discussion that the left vertical arrow is an isomorphism. Moreover, by Corollary 9.5 the lower horizontal arrow is surjective. Therefore we obtain an isomorphism

$$\operatorname{gr}(V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}) \simeq (\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}.$$

In particular, this implies that the character of $\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) = \operatorname{gr}(V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}$ is equal to that of $(\operatorname{gr} V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]})$ given by formula (9.5). Thus we find that

$$\operatorname{ch} \mathfrak{z}(\widehat{\mathfrak{g}}) = \prod_{i=1}^{\ell} \prod_{n_i \geq d_i+1} (1 - q^{n_i})^{-1}. \tag{9.6}$$

9.4. **The center and the classical $\mathcal{W}$–algebra.** According to Proposition 9.2, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is contained in the intersection of the kernels of the operators $\overline{V}_i[1], i = 1, \ldots, \ell$, on π_0. Now we compute the character of this intersection and compare it with the character formula (9.6) for $\mathfrak{z}(\widehat{\mathfrak{g}})$ to show that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is equal to the intersection of the kernels of the operators $\overline{V}_i[1]$.

First, we identify this intersection with a classical limit of a one-parameter family of vertex algebras, called the $\mathcal{W}$–algebras. The $\mathcal{W}$–algebra $\mathcal{W}_\nu(\mathfrak{g})$ associated to a simple Lie algebra $\mathfrak{g}$ and an invariant inner product ν on $\mathfrak{g}$ is defined via the quantum Drinfeld-Sokolov reduction in [FB], Ch. 15. It is shown there that $\mathcal{W}_\nu(\mathfrak{g})$ is a vertex subalgebra of π_0^ν, which for generic values of ν is equal to the intersection of the kernels of screening operators.

More precisely, consider another copy of the Heisenberg Lie algebra $\widehat{\mathfrak{h}}_\nu$ introduced in Section 5.1. To avoid confusion, we will denote the generators of this Heisenberg Lie algebra by $\mathbf{b}_{i,n}$. We have the vertex operator $V^\nu_{-\alpha_i}(z) : \pi_0^\nu \to \pi^\nu_{-\alpha_i}$ defined by formula (7.8), and let $V^\nu_{-\alpha_i}[1] = \int V^\nu_{-\alpha_i}(z) dz$ be its residue. We call it the $\mathcal{W}$–algebra screening operator (to distinguish it from the Kac-Moody screening operators defined above). Since $V^\nu_{-\alpha_i}(z) = Y_{\pi^\nu_0,\pi^\nu_{-\alpha_i}}(z)$ (in the notation of Section 7.2), we obtain that the intersection of the kernels of $V^\nu_{-\alpha_i}[1], i = 1,\dots,\ell$, in π^ν_0 is a vertex subalgebra of π^ν_0. By Theorem 15.4.12 of [FB], for generic values of ν the $\mathcal{W}$–algebra $\mathcal{W}_\nu(\mathfrak{g})$ is isomorphic to the intersection of the kernels of the operators $V^\nu_{-\alpha_i}[1], i = 1,\dots,\ell$, in π^ν_0.

Now we consider the limit when $\nu \to \infty$. We fix an invariant inner product ν_0 on $\mathfrak{g}$ and denote by ϵ the ratio between ν_0 and ν. We have the following formula for the ith simple root $\alpha_i \in \mathfrak{h}^*$ as an element of $\mathfrak{h}$ using the identification between $\mathfrak{h}^*$ and $\mathfrak{h}$ induced by $\nu = \nu_0/\epsilon$:

$$\alpha_i = \epsilon \frac{2}{\nu_0(h_i,h_i)} h_i.$$

Let

$$\mathbf{b}'_{i,n} = \epsilon \frac{2}{\nu_0(h_i,h_i)} \mathbf{b}_{i,n}. \tag{9.7}$$

Consider the $\mathbb{C}[\epsilon]$–lattice in $\pi^\nu_0 \otimes \mathbb{C}[\epsilon]$ spanned by all monomials in $\mathbf{b}'_{i,n}, i = 1,\dots,\ell; n < 0$. We denote by π_{0,ν_0} the specialization of this lattice at $\epsilon = 0$; it is a commutative vertex algebra. In the limit $\epsilon \to 0$, we obtain the following expansion of the operator $V^\nu_{-\alpha_i}[1]$:

$$V^\nu_{-\alpha_i}[1] = \epsilon \frac{2}{\nu_0(h_i,h_i)} \mathbf{V}_i[1] + \dots,$$

where the dots denote terms of higher order in ϵ, and the operator $\mathbf{V}_i[1]$ acting on π_{0,ν_0} is given by the formula

$$\mathbf{V}_i[1] = \sum_{m \le 0} \mathbf{V}_i[m]\, \mathbf{D}_{\mathbf{b}'_{i,m-1}}, \tag{9.8}$$

where

$$\mathbf{D}_{\mathbf{b}'_{i,m}} \cdot b_{j,n} = a_{ij} \frac{\partial}{\partial \mathbf{b}'_{i,n}} \delta_{n,m}, \tag{9.9}$$

(a_{ij}) is the Cartan matrix of $\mathfrak{g}$, and

$$\sum_{n \le 0} \mathbf{V}_i[n] z^{-n} = \exp\left(- \sum_{m>0} \frac{\mathbf{b}'_{i,-m}}{m} z^m \right).$$

The intersection of the kernels of the operators $\mathbf{V}_i[1], i = 1,\dots,\ell$, is a commutative vertex subalgebra of π_{0,ν_0}, which we denote by $\mathbf{W}_{\nu_0}(\mathfrak{g})$ and call the *classical $\mathcal{W}$–algebra* associated to $\mathfrak{g}$ and ν_0. Note that the structures of commutative vertex algebra on π_{0,ν_0} and $\mathbf{W}_{\nu_0}(\mathfrak{g})$ and the operators $\mathbf{V}_i[1]$ are independent of the choice of ν_0. However, as we will see below, both π_{0,ν_0} and $\mathbf{W}_{\nu_0}(\mathfrak{g})$ also carry vertex Poisson algebra structures, and those structures do depend on ν_0.

9.5. **The appearance of the Langlands dual Lie algebra.** Comparing formulas (9.2) and (9.8), we find that, up to the substitution $b_{i,n} \mapsto -\mathbf{b}'_{i,n}$, the operators $\overline{V}_i[1]$and $\mathbf{V}_i[1]$ are almost identical. The difference is that the matrix coefficient a_{ji} in formula (8.8) gets replaced by a_{ij} in formula (9.9). In other words, the Cartan matrix gets transposed. This transposed Cartan matrix is the Cartan matrix of another Lie algebra, called the *Langlands dual Lie algebra* of $\mathfrak{g}$ and denoted by ${}^L\mathfrak{g}$. Because of this transposition, we may identify the Cartan subalgebra ${}^L\mathfrak{h}$ of ${}^L\mathfrak{g}$ with the dual space $\mathfrak{h}^*$ to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, so that the simple roots of $\mathfrak{g}$ (which are vectors in $\mathfrak{h}^*$) become the simple coroots of ${}^L\mathfrak{g}$ (which are cevtors in ${}^L\mathfrak{h}$).

According to the above discussion, if we make the substitution $b_{i,n} \mapsto -\mathbf{b}'_{i,n}$, then the operator $\overline{V}_i[1]$ attached to $\mathfrak{g}$ becomes the operator $\mathbf{V}_i[1]$ attached to ${}^L\mathfrak{g}$. Therefore the intersection of the kernels of the operators $\overline{V}_i[1], i = 1, \ldots, \ell$, attached to a simple Lie algebra $\mathfrak{g}$ is isomorphic to the intersection of the kernels of the operators $\mathbf{V}_i[1], i = 1, \ldots, \ell$, attached to ${}^L\mathfrak{g}$, i.e., to the $\mathcal{W}$–algebra $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$ attached to ${}^L\mathfrak{g}$.

Using Proposition 9.2, we now find that $\mathfrak{z}(\widehat{\mathfrak{g}})$ may be embedded into $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$. In order to prove that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is isomorphic to $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$, it suffices to show that the character of $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$ is given by the right hand side of formula (9.6).

This may be done in several ways. One of the possible ways is presented in Section 11.2 below. Another possibility is to use the results of [FF7]. According to [FF7], Prop. 2.2.8, the operators $\mathbf{V}_i[1], i = 1, \ldots, \ell$, satisfy the Serre relations of $\mathfrak{g}$, and so they generate an action of the Lie algebra $\mathfrak{n}_+$ on π_{0,ν_0}. Moreover, by [FF7], Prop. 2.4.6, the action of $\mathfrak{n}_+$ on π_{0,ν_0} is co-free, and this enables us to compute the character of $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$. The result is that the character of $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$ is given by the right hand side of formula (9.6) (see [FF7], Prop. 2.4.7). Hence we obtain that the character of the intersection of the kernels of the operators $\overline{V}_i[1], i = 1, \ldots, \ell$, is equal to the character of $\mathfrak{z}(\widehat{\mathfrak{g}})$. Therefore $\mathfrak{z}(\widehat{\mathfrak{g}})$ is isomorphic to the intersection of the kernels of the operators $\overline{V}_i[1], i = 1, \ldots, \ell$. Thus, we obtain the following result.

Theorem 9.6. *The center* $\mathfrak{z}(\widehat{\mathfrak{g}})$ *is isomorphic to the intersection of the kernels of the operators* $\overline{V}_i[1], i = 1, \ldots, \ell$, *on* π_{0,ν_0}, *and to the classical* $\mathcal{W}$*–algebra* $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$.

The natural map $\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) \to (\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}$ *is an isomorphism.*

The theorem implies in particular that for each generator $P_{i,0}$ of $(\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}$ there exists an element S_i of $V_{\kappa_c}(\mathfrak{g})^{\mathfrak{g}[[t]]}$ whose symbol is equal to $P_{i,0}$. Moreover, we have

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[S_i^{(n)}]_{i=1,\ldots,\ell;n\geq 0},$$

where $S_i^{(n)} = T^n S_i$, and we use the commutative algebra structure on $\mathfrak{z}(\widehat{\mathfrak{g}})$ which comes from its commutative vertex algebra structure (see [FB], § 1.4). Note that the symbol of $S_i^{(n)}$ is then equal to $n! P_{i,n}$.

For example, the vector S_1, which is unique up to a scalar, is the Segal–Sugawara vector

$$S_1 = \frac{1}{2} \sum_a J^a_{-1} J_{a,-1} v_{\kappa_c}, \tag{9.10}$$

where $\{J_a\}$ is the basis of $\mathfrak{g}$ dual to the basis $\{J^a\}$ with respect to a non-degenerate invariant inner product. Explicit formulas for other elements $S_i, i > 1$, are unknown in general (see however [H, GW]).

We note that the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ may be identified with the algebra of $\widehat{\mathfrak{g}}_{\kappa_c}$–endomorphisms of $V_{\kappa_c}(\mathfrak{g})$. Indeed, a $\mathfrak{g}[[t]]$–invariant vector $x \in V_{\kappa_c}(\mathfrak{g})$ gives rise to a non-trivial endomorphism of V, which maps the highest weight vector v_{κ_c} to x. On the other hand, given an endomorphism e of $V_{\kappa_c}(\mathfrak{g})$, we obtain a $\mathfrak{g}[[t]]$–invariant vector $e(v)$. It is clear that the two maps are inverse to each other. Now let x_1 and x_2 be two $\mathfrak{g}[[t]]$–invariant vectors in $V_{\kappa_c}(\mathfrak{g})$. Let e_1 and e_2 be the corresponding endomorphisms of $V_{\kappa_c}(\mathfrak{g})$. Then the image of v under the composition $e_1 e_2$ equals $e_1(x_2) = x_2 x_1$, where in the right hand side we use the product structure on $\mathfrak{z}(\widehat{\mathfrak{g}})$. Therefore we find that as an algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is isomorphic to the algebra $\operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} V_{\kappa_c}(\mathfrak{g})$ with the opposite multiplication. But since $\mathfrak{z}(\widehat{\mathfrak{g}})$ is commutative, we obtain an isomorphism of algebras

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{End}_{\widehat{\mathfrak{g}}_{\kappa_c}} V_{\kappa_c}(\mathfrak{g}).$$

9.6. **The vertex Poisson algebra structure.** In addition to the structures of commutative vertex algebras, both $\mathfrak{z}(\widehat{\mathfrak{g}})$ and $\mathbf{W}_{\nu_0}(\mathfrak{g})$ also carry the structures of vertex Poisson algebra. For the definition of vertex Poisson algebra, see, e.g., [FB], § 16.2. In particular, we recall that to a vertex Poisson algebra P we attach a Lie algebra

$$\operatorname{Lie}(P) = P \otimes \mathbb{C}((t)) / \operatorname{Im}(T \otimes 1 + 1 \otimes \partial_t)$$

(see [FB], § 16.1.7).

According to Proposition 16.2.7 of [FB], if V_ϵ is a one-parameter family of vertex algebras, then the center $\mathcal{Z}(V_0)$ of V_0 acquires a natural vertex Poisson algebra structure. Namely, at $\epsilon = 0$ the polar part of the operation Y, restricted to $\mathcal{Z}(V_0)$, vanishes, so we define the operation Y_- on $\mathcal{Z}(V_0)$ as the ϵ–linear term of the polar part of Y.

Let us fix a non-zero inner product κ_0 on $\mathfrak{g}$, and let ϵ be the ratio between the inner products $\kappa - \kappa_c$ and κ_0. Consider the vertex algebras $V_\kappa(\mathfrak{g})$ as a one-parameter family using ϵ as a parameter. Then we obtain a vertex Poisson structure on $\mathfrak{z}(\widehat{\mathfrak{g}})$, the center of $V_{\kappa_c}(\mathfrak{g})$ (corresponding to $\epsilon = 0$). We will denote $\mathfrak{z}(\widehat{\mathfrak{g}})$, equipped with this vertex Poisson structure, by $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}$.

Likewise, we obtain a vertex Poisson structure on π_{0,κ_0} by considering the vertex algebras $\pi_0^{\kappa-\kappa_c}$ as a one-parameter family with the same parameter ϵ, in the same way as in Section 9.4. The corresponding vertex Poisson structure on π_{0,κ_0} is uniquely characterized by the Poisson brackets

$$\{b_{i,n}, b_{j,m}\} = n\kappa_0(h_i, h_j)\delta_{n,-m}.$$

Recall that the homomorphism $w_{\kappa_c} : V_{\kappa_c}(\mathfrak{g}) \to W_{0,\kappa_c} = M \otimes \pi_0$ may be deformed to a homomorphism $w_\kappa : V_\kappa(\mathfrak{g}) \to W_{0,\kappa} = M \otimes \pi_0^{\kappa-\kappa_c}$. Therefore the ϵ–linear term of the polar part of the operation Y of $V_\kappa(\mathfrak{g})$, restricted to $\mathfrak{z}(\widehat{\mathfrak{g}})$, which is used in the definition of the vertex Poisson structure on $\mathfrak{z}(\widehat{\mathfrak{g}})$, may be computed by restricting to $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset \pi_0$ the ϵ–linear term of the polar part of the operation Y of $\pi_0^{\kappa-\kappa_c} \subset W_{0,\kappa}$. But the latter defines the vertex Poisson structure on π_{0,κ_0}. Therefore we obtain the following

Lemma 9.7. *The embedding* $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \hookrightarrow \pi_{0,\kappa_0}$ *is a homomorphism of vertex Poisson algebras and the corresponding map of local Lie algebras* $\operatorname{Lie}(\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}) \hookrightarrow \operatorname{Lie}(\pi_{0,\kappa_0})$ *is a Lie algebra homomorphism.*

On the other hand, it follows from the definition of the classical $\mathcal{W}$–algebra $\mathbf{W}_{\nu_0}({}^L\mathfrak{g})$ that it also carries a vertex Poisson algebra structure. The isomorphism of Theorem 9.6 preserves the vertex Poisson algebra structures in the following sense.

Note that the restriction of a non-zero invariant inner product κ_0 on $\mathfrak{g}$ to $\mathfrak{h}$ defines a non-zero inner product on $\mathfrak{h}^*$, which is the restriction of an invariant inner product $\kappa_0^\vee$

on ${}^L\mathfrak{g}$. To avoid confusion, let us denote by $\pi_0(\mathfrak{g})_{\kappa_0}$ and $\pi_0({}^L\mathfrak{g})_{\kappa_0^\vee}$ the classical limits of the Heisenberg vertex algebras for $\mathfrak{g}$ and ${}^L\mathfrak{g}$, respectively, with their vertex Poisson structures corresponding to κ_0 and $\kappa_0^\vee$, respectively. We have an isomorphism of vertex Poisson algebras

$$\begin{aligned} \imath : \pi_0(\mathfrak{g})_{\kappa_0} &\xrightarrow{\sim} \pi_0({}^L\mathfrak{g})_{\kappa_0^\vee}, \\ b_{i,n} \in \widehat{\mathfrak{h}} &\mapsto -\mathbf{b}'_{i,n} \in \widehat{{}^L\mathfrak{h}}, \end{aligned}$$

where $\mathbf{b}'_{i,n}$ is given by formula (9.7). The restriction of the isomorphism $\imath$ to the subspace $\bigcap_{1 \leq i \leq \ell} \operatorname{Ker} \overline{V}_i[1]$ of $\pi_0(\mathfrak{g})_{\kappa_0}$ gives us an isomorphism of vertex Poisson algebras

$$\bigcap_{1 \leq i \leq \ell} \operatorname{Ker} \overline{V}_i[1] \simeq \bigcap_{1 \leq i \leq \ell} \operatorname{Ker} \mathbf{V}_i^\vee[1],$$

where by $\mathbf{V}_i^\vee[1]$ we denote the operator (9.8) attached to ${}^L\mathfrak{g}$. Recall that we have the following isomorphisms of vertex Poisson algebras:

$$\begin{aligned} \mathfrak{z}(\widehat{\mathfrak{g}}) &\simeq \bigcap_{1 \leq i \leq \ell} \operatorname{Ker} \overline{V}_i[1], \\ \mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g}) &\simeq \bigcap_{1 \leq i \leq \ell} \operatorname{Ker} \mathbf{V}_i^\vee[1]. \end{aligned}$$

Therefore we obtain the following strengthening of Theorem 9.6:

Theorem 9.8. *The center $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}$ is isomorphic, as a vertex Poisson algebra, to the classical $\mathcal{W}$–algebra $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$. The Lie algebras $\operatorname{Lie}(\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0})$ and $\operatorname{Lie}(\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g}))$ are also isomorphic.*

9.7. Aut O–**module structures.** Both $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}$ and $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$ carry actions of the group Aut O and the isomorphism of Theorem 9.8 intertwines these actions. To see that, we describe the two actions as coming from the vertex Poisson algebra structures.

In both cases the action of the group Aut O is obtained by exponentiation of the action of the Lie algebra Der O. In the case of the center $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}$, the action of Der O is the restriction of the natural action on $V_{\kappa_c}(\mathfrak{g})$ which comes from its action on $\widehat{\mathfrak{g}}_{\kappa_c}$ (and preserving its Lie subalgebra $\mathfrak{g}[[t]]$) by infinitesimal changes of variables. But away from the critical level, i.e., when $\kappa \neq \kappa_c$, the action of Der O is obtained thtough the action of the Virasoro algebra which comes from the conformal vector $\mathbf{s}_\kappa$ given by formula (5.3) which we rewrite as follows:

$$\mathbf{s}_\kappa = \frac{\kappa_0}{\kappa - \kappa_c} S_1,$$

where S_1 is given by formula (9.10) and κ_0 is the inner product used in that formula. Thus, the Fourier coefficients $\mathbf{s}_{\kappa,(n)}, n \geq -1$, of the vertex operator

$$Y(\mathbf{s}_\kappa, z) = \sum_{n \in \mathbb{Z}} \mathbf{s}_{\kappa,(n)} z^{-n-1}$$

generate the Der O–action on $V_\kappa(\mathfrak{g})$ when $\kappa \neq \kappa_c$.

In the limit $\kappa \to \kappa_c$, we have $\mathbf{s}_\kappa = \epsilon^{-1} S_1$, where as before $\epsilon = \dfrac{\kappa - \kappa_c}{\kappa_0}$. Therefore the action of $\operatorname{Der} O$ is obtained through the vertex Poisson operation Y_- (defined as the limit of ϵ^{-1} times the polar part of Y when $\epsilon \to 0$, see [FB], § 16.2) on $\mathfrak{z}(\widehat{\mathfrak{g}})$ applied to $S_1 \in \mathfrak{z}(\widehat{\mathfrak{g}})$. In other words, the $\operatorname{Der} O$–action is generated by the Fourier coefficients $S_{1,(n)}, n \geq 0$, of the series

$$Y_-(S_1, z) = \sum_{n \geq 0} S_{1,(n)} z^{-n-1},$$

namely, $L_n \mapsto S_{(n)}$. Thus, we see that the natural $\operatorname{Der} O$–action (and hence $\operatorname{Aut} O$–action) on $\mathfrak{z}(\widehat{\mathfrak{g}})$ is encoded in the vector $S_1 \in \mathfrak{z}(\widehat{\mathfrak{g}})$ through the vertex Poisson algebra structure on $\mathfrak{z}(\widehat{\mathfrak{g}})$. Note that this action endows $\mathfrak{z}(\widehat{\mathfrak{g}})$ with a quasi-conformal structure (see Section 5.4).

Likewise, there is a $\operatorname{Der} O$–action on $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$ coming from its vertex Poisson algebra structure. The vector generating this action is also equal to the limit of a conformal vector in the quantum $\mathcal{W}$–algebra $\mathcal{W}_\nu({}^L\mathfrak{g})$ as $\nu \to \infty$. This conformal vector is unique because as we see from the character formula for $\mathcal{W}_\nu({}^L\mathfrak{g})$ given in the right hand side of (9.6) the homogeneous component of $\mathcal{W}_\nu({}^L\mathfrak{g})$ of degree two (where all conformal vectors live) is one-dimensional. The limit of this vector as $\nu \to \infty$ gives rise to a vector $\mathbf{t}$ in $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$ such that the Fourier coefficients of $Y_-(t, z)$ generate a $\operatorname{Der} O$–action on $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$ (in the next section we will explain the geometric meaning of this action). Since such a vector is unique, it must be the image of $S_1 \in \mathfrak{z}(\widehat{\mathfrak{g}})$ under the isomorphism $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$. This implies the following result.

Proposition 9.9. *The isomorphism Theorem 9.8 intertwines the actions of* $\operatorname{Der} O$ *and* $\operatorname{Aut} O$ *on* $\mathfrak{z}(\widehat{\mathfrak{g}})$ *and* $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$.

We have already calculated in formula (5.3) the image of $\mathbf{s}_\kappa$ in $W_{0,\kappa}$ when $\kappa \neq \kappa_c$. By passing to the limit $\kappa \to \kappa_c$ we find that the image of $S_1 = \epsilon \mathbf{s}_\kappa$ belongs to $\pi_0(\mathfrak{g})_{\kappa_0} \subset W_{0,\kappa}$ and is equal to

$$\frac{1}{2} \sum_{i=1}^{\ell} b_{i,-1} b^i_{-1} - \rho_{-2},$$

where $\{b_i\}$ and $\{b^i\}$ are dual bases with respect to the inner product κ_0 used in the definition of S_1, restricted to $\mathfrak{h}$, and ρ is the element of $\mathfrak{h}$ corresponding to $\rho \in \mathfrak{h}^*$ under the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$ induced by κ_0. Under the isomorphism of Theorem 9.8, this vector becomes the vector $\mathbf{t} \in \pi_0({}^L\mathfrak{g})_{\kappa_0^\vee} \simeq \pi_0(\mathfrak{g})_{\kappa_0}$, which automatically lies in $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g}) \subset \pi_0({}^L\mathfrak{g})_{\kappa_0^\vee}$ and is responsible for the $\operatorname{Der} O$ action on it.

The action of the corresponding operators $L_n \in \operatorname{Der} O, n \geq -1$, on $\pi_0(\mathfrak{g})_{\kappa_0}$ is given by derivations of the algebra structure which are uniquely determined by formulas (5.11). Therefore the action of the L_n's on $\mathbf{W}_{\kappa_0^\vee}({}^L\mathfrak{g})$ is as follows (recall that $b_{i,n} \mapsto -\mathbf{b}'_{i,n}$):

$$\begin{aligned} L_n \cdot \mathbf{b}'_{i,m} &= -m \mathbf{b}'_{i,n+m}, \qquad -1 \leq n < -m, \\ L_n \cdot \mathbf{b}'_{i,-n} &= -n, \qquad n > 0, \\ L_n \cdot \mathbf{b}'_{i,m} &= 0, \qquad n > -m. \end{aligned} \tag{9.11}$$

10. Opers and Miura opers

In this section we introduce opers, (generic) Miura opers and a natural map between them. We will then show in the next section that the corresponding rings of functions are isomorphic to $\mathbf{W}_{\nu_0}(\mathfrak{g})$ and $\pi_0(\mathfrak{g})_{\nu_0}$, respectively, and the corresponding homomorphism $\mathbf{W}_{\nu_0}(\mathfrak{g}) \to \pi_0(\mathfrak{g})_{\nu_0}$ is the embedding constructed in Section 9.4. In other words, the ring of functions on opers is isomorphic to the subring of the ring of functions on generic Miura opers equal to the intersection of the kernels of the screening operators $\mathbf{V}_i[1], i = 1, \ldots, \ell$.

10.1. **Opers.** Let G be a simple algebraic group of adjoint type, B a Borel subgroup and $N = [B, B]$ its unipotent radical, with the corresponding Lie algebras $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$. There is an open B–orbit $\mathbf{O} \subset \mathfrak{n}^\perp/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{b}$, consisting of vectors which are stabilized by the radical $N \subset B$, and such that all of their negative simple root components, with respect to the adjoint action of $H = B/N$, are non-zero. This orbit may also be described as the B–orbit of the sum of the projections of simple root generators f_i of any nilpotent subalgebra $\mathfrak{n}_-$, which is in generic position with $\mathfrak{b}$, onto $\mathfrak{g}/\mathfrak{b}$. The torus $H = B/N$ acts simply transitively on $\mathbf{O}$, so $\mathbf{O}$ is an H–torsor.

Suppose we are given a principal G–bundle $\mathcal{F}$ on a smooth curve X, or on a disc $D = \operatorname{Spec} \mathcal{O}$, or on a punctured disc $D^\times = \operatorname{Spec} \mathcal{K}$ (here we use the notation of Section 5.6), together with a connection ∇ (automatically flat) and a reduction $\mathcal{F}_B$ to the Borel subgroup B of G. Then we define the relative position of ∇ and $\mathcal{F}_B$ (i.e., the failure of ∇ to preserve $\mathcal{F}_B$) as follows. Locally, choose any flat connection ∇' on $\mathcal{F}$ preserving $\mathcal{F}_B$, and take the difference $\nabla - \nabla'$. It is easy to show that the resulting local sections of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega$ are independent of ∇', and define a global $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$–valued one-form on X, denoted by $\nabla/\mathcal{F}_B$.

Let X be as above. A *G–oper* on X is by definition a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal G–bundle $\mathcal{F}$ on X, ∇ is a connection on $\mathcal{F}$ and $\mathcal{F}_B$ is a B–reduction of $\mathcal{F}$, such that the one–form $\nabla/\mathcal{F}_B$ takes values in $\mathbf{O}_{\mathcal{F}_B} \subset (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$.

This definition is due to A. Beilinson and V. Drinfeld [BD1] (in the case when X is the punctured disc opers were introduced in [DS]). Note that $\mathbf{O}$ is $\mathbb{C}^\times$–invariant, so that $\mathbf{O} \otimes \Omega$ is a well-defined subset of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega$.

Equivalently, the above condition may be reformulated as saying that if we choose a local trivialization of $\mathcal{F}_B$ and a local coordinate t then the connection will be of the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + \mathbf{v}(t), \tag{10.1}$$

where each $\psi_i(t)$ is a nowhere vanishing function, and $\mathbf{v}(t)$ is a $\mathfrak{b}$–valued function. If we change the trivialization of $\mathcal{F}_B$, then this operator will get transformed by the corresponding gauge transformation. This observation allows us to describe opers on a disc $D = \operatorname{Spec} \mathcal{O}$ in a more explicit way.

Let us choose a coordinate t on D, i.e., an isomorphism $\mathcal{O} \simeq \mathbb{C}[[t]]$. Then the space $\operatorname{Op}_G(D)$ of G–opers on D is the quotient of the space of all operators of the form (10.1), where $\psi_i(t) \in \mathbb{C}[[t]], \psi_i(0) \neq 0, i = 1, \ldots, \ell$, and $\mathbf{v}(t) \in \mathfrak{b}[[t]]$, by the action of the group $B[[t]]$ by gauge transformations:

$$g \cdot (\partial_t + A(t)) = \partial_t + gA(t)g^{-1} - g^{-1}\partial_t g.$$

Since the B–orbit $\mathbf{O}$ is an H–torsor, we can use the H–action to make all functions $\psi_i(t)$ equal to 1. Thus, we obtain that $\operatorname{Op}_G(D)$ is equal to the quotient of the space $\widetilde{\operatorname{Op}}_G(D)$ of operators of the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} f_i + \mathbf{v}(t), \qquad \mathbf{v}(t) \in \mathfrak{b}[[t]], \tag{10.2}$$

by the action of the group $N[[t]]$ by gauge transformations.

If we choose another coordinate s, such that $t = \varphi(s)$, then the operator (10.2) will become

$$\nabla = \partial_s + \varphi'(s) \sum_{i=1}^{\ell} f_i + \varphi'(s) \cdot \mathbf{v}(\varphi(s)).$$

In order to bring it back to the form (10.2) we need to apply the gauge transformation by $\check{\rho}(\varphi'(s))$, where $\check{\rho} : \mathbb{C}^\times \to H$ is the one-parameter subgroup of H equal to the sum of the fundamental coweights of G. Choose a splitting $\imath : H \to B$ of the homomorphism $B \to H$. Then, considering $\check{\rho}$ as an element of the Lie algebra $\mathfrak{h} = \operatorname{Lie} H$, we have $[\check{\rho}, e_i] = e_i$ and $[\check{\rho}, f_i] = -f_i$. Therefore we find that

$$\check{\rho}(\varphi'(s)) \cdot \left(\partial_s + \varphi'(s) \sum_{i=1}^{\ell} f_i + \varphi'(s) \cdot \mathbf{v}(\varphi(s)) \right)$$

$$= \partial_s + \sum_{i=1}^{\ell} f_i + \varphi'(s) \check{\rho}(\varphi'(s)) \cdot \mathbf{v}(\varphi(s)) \cdot \check{\rho}(\varphi'(s))^{-1} - \check{\rho}\left(\frac{\varphi''(s)}{\varphi'(s)} \right). \tag{10.3}$$

Thus we obtain a well-defined action of the group $\operatorname{Aut} O$ on the space $\operatorname{Op}_G(\mathbb{D})$ of opers on the standard disc,

$$\mathbb{D} = \operatorname{Spec} \mathbb{C}[[t]].$$

We may therefore define $\operatorname{Op}_G(D)$ as the twist of $\operatorname{Op}_G(\mathbb{D})$ by the $\operatorname{Aut} O$–torsor $\mathcal{A}ut$ (see Section 5.6).

In particular, the above formulas imply the following result. Consider the H–bundle $\Omega^{\check{\rho}}$ on D, defined in the same way as in Section 5.5 (where we considered the ${}^L H$–bundle $\Omega^{-\rho}$).

Lemma 10.1. *The H–bundle $\mathcal{F}_H = \mathcal{F}_B \underset{B}{\times} H = \mathcal{F}_B/N$ is isomorphic to $\Omega^{\check{\rho}}$.*

Proof. It follows from formula (10.3) for the action of the changes of variables on opers that if we pass from a coordinate t on D to the coordinate s such that $t = \varphi(s)$, then we obtain a new trivialization of the H–bundle $\mathcal{F}_H$, which is related to the old one by the transformation $\check{\rho}(\varphi'(s))$. This precisely means that $\mathcal{F}_H \simeq \Omega^{\check{\rho}}$. □

10.2. **Canonical representatives.** In this section we find canonical representatives in the $N[[t]]$–gauge classes of connections of the form (10.2).

Lemma 10.2 ([DS]). *The action of* $N[[t]]$ *on* $\widetilde{\mathrm{Op}}_G(D)$ *is free.*

Proof. The operator $\operatorname{ad}\check{\rho}$ defines the principal gradation on $\mathfrak{b}$, with respect to which we have a direct sum decomposition $\mathfrak{b} = \oplus_{i \geq 0} \mathfrak{b}_i$. Let

$$p_{-1} = \sum_{i=1}^{\ell} f_i.$$

The operator $\operatorname{ad} p_{-1}$ acts from $\mathfrak{b}_{i+1}$ to $\mathfrak{b}_i$ injectively for all $i \geq 0$. Hence we can find for each $i \geq 0$ a subspace $V_i \subset \mathfrak{b}_i$, such that $\mathfrak{b}_i = [p_{-1}, \mathfrak{b}_{i+1}] \oplus V_i$. It is well-known that $V_i \neq 0$ if and only if i is an exponent of $\mathfrak{g}$, and in that case $\dim V_i$ is equal to the multiplicity of the exponent i. In particular, $V_0 = 0$. Let $V = \oplus_{i \in E} V_i \subset \mathfrak{n}$, where E is the set of exponents of $\mathfrak{g}$ counted with multiplicity.

We claim that each element of $\partial_t + p_{-1} + \mathbf{v}(t) \in \widetilde{\mathrm{Op}}_G(D)$ can be uniquely represented in the form

$$\partial_t + p_{-1} + \mathbf{v}(t) = \exp\left(\operatorname{ad} U\right) \cdot \left(\partial_t + p_{-1} + \mathbf{c}(t)\right), \tag{10.4}$$

where $U \in \mathfrak{n}[[t]]$ and $\mathbf{c}(t) \in V[[t]]$. To see that, we decompose with respect to the principal gradation: $U = \sum_{j \geq 0} U_j$, $\mathbf{v}(t) = \sum_{j \geq 0} \mathbf{v}_j(t)$, $\mathbf{c}(t) = \sum_{j \in E} \mathbf{c}_j(t)$. Equating the homogeneous components of degree j in both sides of (10.4), we obtain that $\mathbf{c}_i + [U_{i+1}, p_{-1}]$ is expressed in terms of $\mathbf{v}_i, \mathbf{c}_j, j < i$, and $U_j, j \leq i$. The injectivity of $\operatorname{ad} p_{-1}$ then allows us to determine uniquely $\mathbf{c}_i$ and U_{i+1}. Hence U and $\mathbf{c}$ satisfying equation (10.4) may be found uniquely by induction, and the lemma follows. □

There is a special choice of the transversal subspace $V = \oplus_{i \in E} V_i$ defined in the proof of Lemma 10.2. Namely, let p_1 be the unique element of $\mathfrak{n}$, such that $\{p_{-1}, 2\check{\rho}, p_1\}$ is an $\mathfrak{sl}_2$–triple. Let $V_{\mathrm{can}} = \oplus_{i \in E} V_{\mathrm{can},i}$ be the space of $\operatorname{ad} p_1$–invariants in $\mathfrak{n}$. Then p_1 spans $V_{\mathrm{can},1}$. Let p_j be a linear generator of V_{can,d_j} (if the multiplicity of d_j is greater than one, which happens only in the case $\mathfrak{g} = D_{2n}^{(1)}, d_j = 2n$, then we choose linearly independent vectors in V_{can,d_j}).

According to Lemma 10.2, each G–oper may be represented by a unique operator $\nabla = \partial_t + p_{-1} + \mathbf{v}(t)$, where $\mathbf{v}(t) \in V_{\mathrm{can}}[[t]]$, so that we can write

$$\mathbf{v}(t) = \sum_{j=1}^{\ell} v_j(t) \cdot p_j.$$

Suppose now that $t = \varphi(s)$, where s is another coordinate on D such that $t = \varphi(s)$. With respect to the new coordinate s, ∇ becomes equal to $\partial_s + \widetilde{\mathbf{v}}(s)$, where $\widetilde{\mathbf{v}}(s)$ is expressed via $\mathbf{v}(t)$ and $\varphi(s)$ as in formula (10.3). By Lemma 10.2, there exists a unique operator $\partial_s + p_{-1} + \overline{\mathbf{v}}(s)$ with $\overline{\mathbf{v}}(s) \in V_{\mathrm{can}}[[s]]$ and $g \in B[[s]]$, such that

$$\partial_s + p_{-1} + \overline{\mathbf{v}}(s) = g \cdot \left(\partial_s + \widetilde{\mathbf{v}}(s)\right). \tag{10.5}$$

It is straightforward to find that

$$\begin{aligned} g &= \exp\left(\frac{1}{2}\frac{\varphi''}{\varphi'}\cdot p_1\right)\check{\rho}(\varphi), \\ \overline{v}_1(s) &= v_1(\varphi(s))\left(\varphi'\right)^2 - \frac{1}{2}\{\varphi, s\}, \\ \overline{v}_j(s) &= v_j(\varphi(s))\left(\varphi'\right)^{d_j+1}, \qquad j > 1, \end{aligned}$$

where

$$\{\varphi, s\} = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2$$

is the Schwarzian derivative.

These formulas mean that under changes of variables, v_1 transforms as a projective connection, and $v_j, j > 1$, transforms as a $(d_j + 1)$–differential on D. Thus, we obtain an isomorphism

$$\operatorname{Op}_G(D) \simeq \mathcal{P}roj \times \bigoplus_{j=2}^{\ell} \omega^{\otimes(d_j+1)}, \tag{10.6}$$

where $\omega^{\otimes n}$ is the space of n–differentials on D and $\mathcal{P}roj$ is the $\omega^{\otimes 2}$–torsor of projective connections on D.

10.3. **Miura opers.**

A *Miura G–oper* on X, which is a smooth curve, or D, or $D^\times$, is by definition a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$, where $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is a G–oper on X and $\mathcal{F}'_B$ is another B–reduction of $\mathcal{F}$ which is preserved by ∇.

Consider the space of Miura G–opers on the disc D. A B–reduction of $\mathcal{F}$ which is preserved by the connection ∇ is uniquely determined by a B–reduction of the fiber $\mathcal{F}_0$ of $\mathcal{F}$ at the origin $0 \in D$ (recall that the underlying G–bundles of all G–opers are isomorphic to each other). The set of such reductions is the $\mathcal{F}_0$–twist $(G/B)_{\mathcal{F}_0}$ of the flag manifold G/B. Therefore the natural forgetful morphism from the space of all Miura opers on D to $\operatorname{Op}_G(D)$ is a G/B–bundle.

A Miura G–oper is called *generic* if the B–reductions $\mathcal{F}_B$ and $\mathcal{F}'_B$ are in generic relative position. We denote the space of generic Miura opers on D by $\operatorname{MOp}_G(D)$. We have a natural forgetful morphism $\operatorname{MOp}_G(D) \to \operatorname{Op}_G(D)$. The group $N_{\mathcal{F}_{B,0}}$, where $\mathcal{F}_{B,0}$ is the fiber of $\mathcal{F}_B$ at 0, acts on $(G/B)_{\mathcal{F}_0}$, and the subset of generic reductions is the open $N_{\mathcal{F}_{B,0}}$–orbit of $(G/B)_{\mathcal{F}_0}$. This orbit is in fact an $N_{\mathcal{F}_{B,0}}$–torsor. Therefore we obtain that the morphism $\operatorname{MOp}_G(D) \to \operatorname{Op}_G(D)$ is a principal $N_{\mathcal{F}_{B,0}}$–bundle.

Now we identify $\operatorname{MOp}_G(D)$ with the space of H–connections. Consider the H–bundles $\mathcal{F}_H = \mathcal{F}_B/N$ and $\mathcal{F}'_H = \mathcal{F}'_B/N$ corresponding to a generic Miura oper $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ on X. If $\mathcal{P}$ is an H–bundle, then applying to it the automorphism w_0 of H, corresponding to the longest element of the Weyl group of G, we obtain a new H–bundle which we denote by $w_0^*(\mathcal{F}_H)$.

Lemma 10.3. *For a generic Miura oper $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ the H–bundle $\mathcal{F}'_H$ is isomorphic to $w_0^*(\mathcal{F}_H)$.*

Proof. Consider the vector bundles $\mathfrak{g}_{\mathcal{F}} = \mathcal{F} \underset{G}{\times} \mathfrak{g}$, $\mathfrak{b}_{\mathcal{F}_B} = \mathcal{F}_B \underset{B}{\times} \mathfrak{b}$ and $\mathfrak{b}_{\mathcal{F}'_B} = \mathcal{F}'_B \underset{B}{\times} \mathfrak{b}$. We have the inclusions $\mathfrak{b}_{\mathcal{F}_B}, \mathfrak{b}_{\mathcal{F}'_B} \subset \mathfrak{g}_{\mathcal{F}}$ which are in generic position. Therefore the intersection $b_{\mathcal{F}_B} \cap b_{\mathcal{F}'_B}$ is isomorphic to $\mathfrak{b}_{\mathcal{F}_B}/[\mathfrak{b}_{\mathcal{F}_B}, \mathfrak{b}_{\mathcal{F}_B}]$, which is the trivial vector bundle with the fiber $\mathfrak{h}$. It naturally acts on the bundle $\mathfrak{g}_{\mathcal{F}}$ and under this action $\mathfrak{g}_{\mathcal{F}}$ decomposes into a direct sum of $\mathfrak{h}$ and the line subbundles $\mathfrak{g}_{F,\alpha}, \alpha \in \Delta$. Furthermore, $\mathfrak{b}_{\mathcal{F}_B} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{F,\alpha}, \mathfrak{b}_{\mathcal{F}'_B} = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{F,w_0(\alpha)}$. Since the action of B on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ factors through $H = B/N$, we find that

$$\mathcal{F}_H \underset{H}{\times} \bigoplus_{i=1}^{\ell} \mathbb{C}_{\alpha_i} \simeq \bigoplus_{i=1}^{\ell} \mathfrak{g}_{\mathcal{F},\alpha_i}, \qquad \mathcal{F}'_H \underset{H}{\times} \bigoplus_{i=1}^{\ell} \mathbb{C}_{\alpha_i} \simeq \bigoplus_{i=1}^{\ell} \mathfrak{g}_{\mathcal{F},w_0(\alpha_i)}.$$

Therefore we obtain that

$$\mathcal{F}_H \underset{H}{\times} \mathbb{C}_{\alpha_i} \simeq \mathcal{F}'_H \underset{H}{\times} \mathbb{C}_{w_0(\alpha_i)}, \qquad i = 1, \ldots, \ell.$$

Since G is of adjoint type by our assumption, the above associated line bundles completely determine $\mathcal{F}_H$ and $\mathcal{F}'_H$, and the above isomorphisms imply that $\mathcal{F}'_H \simeq w_0^*(\mathcal{F}_H)$. □

Since the B–bundle $\mathcal{F}'_B$ is preserved by the oper connection ∇, we obtain a connection $\overline{\nabla}$ on $\mathcal{F}'_H$ and hence on $\mathcal{F}_H \simeq \Omega^{\check{\rho}}$. Therefore we obtain a morphism β from the variety $\mathrm{MOp}_G(D)_{\mathrm{gen}}$ of generic Miura opers on D to the variety of connections $\mathrm{Conn}(\Omega^{\check{\rho}})_D$ on the H–bundle $\Omega^{\check{\rho}}$ on U.

Proposition 10.4. *The map $\beta : \mathrm{MOp}_G(D) \to \mathrm{Conn}(\Omega^{\check{\rho}})_D$ is an isomorphism.*

Proof. We define a map τ in the opposite direction. Suppose we are given a connection $\overline{\nabla}$ on the H–bundle $\Omega^{\check{\rho}}$ on D. We associate to it a generic Miura oper as follows. Let us choose a splitting $H \to B$ of the homomorphism $B \to H$ and set $\mathcal{F} = \Omega^{\check{\rho}} \underset{H}{\times} G, \mathcal{F}_B = \Omega^{\check{\rho}} \underset{H}{\times} B$, where we consider the adjoint action of H on G and on B obtained through the above splitting. The choice of the splitting also gives us the opposite Borel subgroup B_-, which is the unique Borel subgroup in generic position with B containing H. Then we set $\mathcal{F}'_B = \Omega^{\check{\rho}} \underset{H}{\times} B_-$.

Observe that the space of connections on $\mathcal{F}$ is isomorphic to the direct product

$$\mathrm{Conn}(\Omega^{\check{\rho}})_D \times \bigoplus_{\alpha \in \Delta} \omega^{\alpha(\check{\rho})+1}.$$

Its subspace corresponding to negative simple roots is isomorphic to $\left(\bigoplus_{i=1}^{\ell} \mathfrak{g}_{-\alpha_i}\right) \otimes \mathcal{O}$. Having chosen a basis element f_i of $\mathfrak{g}_{-\alpha_i}$ for each $i = 1, \ldots, \ell$, we now construct an element $p_{-1} = \sum_{i=1}^{\ell} f_i$ of this space. Now we set $\nabla = \overline{\nabla} + p_{-1}$. By construction, ∇ has the correct relative position with the B–reduction $\mathcal{F}_B$ and preserves the B–reduction $\mathcal{F}'_B$. Therefore the quadruple $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ is a generic Miura oper on D. We set $\tau(\overline{\nabla}) = (\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$.

This map is independent of the choice of splitting $H \to B$ and of the generators $f_i, i = 1, \ldots, \ell$. Indeed, changing the splitting $H \to B$ amounts to a conjugation of the old splitting by an element of N. This is equivalent to applying to ∇ the gauge transformation by this element. Therefore it will not change the underlying Miura oper structure. Likewise, rescaling of the generators f_i may be achieved by a gauge transformation by a constant element of H, and this again does not change the Miura oper structure. It is clear from the construction that β and τ are mutually inverse isomorphisms. $\square$

11. IDENTIFICATION WITH THE $\mathcal{W}$–ALGEBRA AND THE CENTER

Under the isomorphism of Proposition 10.4, the natural forgetful morphism

$$\mathrm{MOp}_G(D)_{\mathrm{gen}} \to \mathrm{Op}_G(D)$$

becomes a map

$$\mathrm{Conn}(\Omega^{\check{\rho}})_D \to \mathrm{Op}_G(D).$$

We call this map the *Miura transformation.*

From now on we will work with the objects attached to the standard, coordinatized, disc $\mathbb{D} = \operatorname{Spec} \mathbb{C}[[t]]$, keeping track of the action of the Lie algebra $\operatorname{Der} O$ and the group $\operatorname{Aut} O$. As explained above, this is equivalent to working with all possible discs (not equipped with a preferred coordinate) at the same time.

The Miura transformation $\mu : \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}} \to \mathrm{Op}_G(\mathbb{D})$ gives rise to a homomorphism of the corresponding rings of functions $\widetilde{\mu} : \operatorname{Fun} \mathrm{Op}_G(\mathbb{D}) \to \operatorname{Fun} \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$. We will now identify $\operatorname{Fun} \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$ with $\pi_0(\mathfrak{g})_{\nu_0}$ and the image of $\widetilde{\mu}$ with the intersection of kernels of the screening operators. This will give us an identification of $\operatorname{Fun} \mathrm{Op}_G(\mathbb{D})$ with $\mathbf{W}_{\nu_0}(\mathfrak{g})$.

11.1. **Screening operators.** Since μ is an $N_{\mathcal{F}_{B,0}}$–bundle, we obtain that the image of the homomorphism $\widetilde{\mu}$ is equal to the space of $N_{\mathcal{F}_{B,0}}$–invariants of $\operatorname{Fun} \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$, and hence to the space of $\mathfrak{n}_{\mathcal{F}_{B,0}}$–invariants of $\operatorname{Fun} \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$. We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{b}$ and a trivialization of $\mathcal{F}_B$. Then we identify $\operatorname{Fun} \mathrm{Op}_G(\mathbb{D})$ and $\operatorname{Fun} \mathrm{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$ with rings of polynomials in infinitely many variables. Each space has an action of $\operatorname{Der} O$ and $\widetilde{\mu}$ is a $\operatorname{Der} O$–equivariant homomorphism between these rings. We will describe the image of $\widetilde{\mu}$ as the intersection of the kernels of certain linear operators which will turn out to be nothing but the screening operators used in the definition of the classical $\mathcal{W}$–algebra.

Using our trivialization of $\mathcal{F}_B$, we write each oper in the form

$$\partial_t + p_{-1} + \sum_{i=1}^{\ell} v_i(t) \cdot \mathbf{c}_i, \qquad v_i(t) \in \mathbb{C}[[t]]$$

(see Section 10.2), where

$$v_i(t) = \sum_{n<-d_i} v_{i,n} t^{-n-d_i-1}.$$

Thus,

$$\operatorname{Fun} \mathrm{Op}_G(\mathbb{D}) = \mathbb{C}[v_{i,n_i}]_{i=1,\dots,\ell; n_i<-d_i}.$$

Likewise, we write each generic Miura oper as

$$\partial_t + p_{-1} + \mathbf{u}(t), \qquad \mathbf{u}(t) \in \mathfrak{h}[[t]]. \tag{11.1}$$

Set $u_i(t) = \alpha_i(\mathbf{u}(t)), i = 1, \ldots, \ell$, and

$$u_i(t) = \sum_{n<0} u_{i,n} t^{-n-1}.$$

Then

$$\operatorname{Fun} \operatorname{MOp}_G(\mathbb{D})_{\mathrm{gen}} = \operatorname{Fun} \operatorname{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}} = \mathbb{C}[u_{i,n}]_{i=1,\ldots,\ell;n<0}.$$

Hence the Miura transformation gives rise to a homomorphism

$$\widetilde{\mu} : \mathbb{C}[v_{i,n_i}]_{i=1,\ldots,\ell;n_i<-d_i} \to \mathbb{C}[u_{i,n}]_{i=1,\ldots,\ell;n<0}. \tag{11.2}$$

Example. We compute the Miura transformation μ in the case when $\mathfrak{g} = \mathfrak{sl}_2$. In this case an oper has the form

$$\partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix},$$

and a generic Miura oper has the form

$$\partial_t + \begin{pmatrix} \frac{1}{2}u(t) & 0 \\ 1 & -\frac{1}{2}u(t) \end{pmatrix}.$$

To compute μ, we need to find an element of $N[[t]]$ such that the corresponding gauge transformation brings the Miura oper into the oper form. We find that

$$\begin{pmatrix} 1 & -\frac{1}{2}u(t) \\ 0 & 1 \end{pmatrix} \left(\partial_t + \begin{pmatrix} \frac{1}{2}u(t) & 0 \\ 1 & -\frac{1}{2}u(t) \end{pmatrix} \right) \begin{pmatrix} 1 & \frac{1}{2}u(t) \\ 0 & 1 \end{pmatrix} = \partial_t + \begin{pmatrix} 0 & \frac{1}{4}u(t)^2 + \frac{1}{2}\partial_t u(t) \\ 1 & 0 \end{pmatrix}.$$

Therefore we obtain

$$\mu(u(t)) = v(t) = \frac{1}{4}u(t)^2 + \frac{1}{2}\partial_t u(t),$$

which may also be written in the form

$$\partial_t^2 - v(t) = \left(\partial_t + \frac{1}{2}u(t)\right)\left(\partial_t - \frac{1}{2}u(t)\right).$$

It is this transformation that was originally introduced by R. Miura as the map intertwining the flows of the KdV hierarchy and the mKdV hierarchy.

In the case when $\mathfrak{g} = \mathfrak{sl}_2$ opers are projective connections and Miura opers are affine connections (see, e.g., [FB], Ch. 9). The Miura transformation is nothing but the natural map from affine connections to projective connections. ☐

The homomorphism (11.2) is $\operatorname{Der} O$–equivariant. The action of $\operatorname{Der} O$ on $\operatorname{Fun} \operatorname{Op}_G(\mathbb{D})$ is obtained from formulas (10.5), and the action on $\operatorname{Fun} \operatorname{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$ is determined as follows. If $\overline{\nabla} = \partial_t + \mathbf{u}(t)$ is a point of $\operatorname{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$, and s is a new coordinate such that $t = \varphi(s)$, then the same connection will appear as $\partial_s + \widetilde{\mathbf{u}}(s)$, where

$$\widetilde{\mathbf{u}}(s) = \varphi' \cdot \mathbf{u}(\varphi(s)) - \check{\rho}\frac{\varphi''}{\varphi'}. \tag{11.3}$$

This translates into the following formulas for the action of the generators $L_n = -t^{n+1}\partial_t$ on the $u_{i,m}$'s:

$$
\begin{aligned}
L_n \cdot u_{i,m} &= -m u_{i,n+m}, \qquad -1 \leq n < -m, \\
L_n \cdot u_{i,-n} &= -n, \qquad n > 0, \\
L_n \cdot u_{i,m} &= 0, \qquad n > -m.
\end{aligned}
\tag{11.4}
$$

Having chosen a trivialization of $\mathcal{F}_B$, we identify the twist $\mathfrak{n}_{\mathcal{F}_{B,0}}$ with $\mathfrak{n}$. Now we choose the generators $e_i, i = 1, \ldots, \ell$, of $\mathfrak{n}$ with respect to the action of $\mathfrak{h}$ on $\mathfrak{n}$ in such a way that for each i the generator e_i and the previously chosen f_i satisfy the standard relations of $\mathfrak{sl}_2$. The $N_{\mathcal{F}_{B,0}}$–action on $\operatorname{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$ then gives rise to an infinitesimal action of e_i on $\operatorname{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$. We will now compute the corresponding derivation on $\operatorname{Fun}\operatorname{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$.

The action of e_i is given by the infinitesimal gauge transformation

$$
\delta \mathbf{u}(t) = [x_i(t) \cdot e_i, \partial_t + p_{-1} + \mathbf{u}(t)],
\tag{11.5}
$$

where $x_i(t) \in \mathbb{C}[[t]]$ is such that $x_i(0) = 1$, and the right hand side of formula (11.5) belongs to $\mathfrak{h}[[t]]$. It turns out that these conditions determine $x_i(t)$ uniquely. Indeed, the right hand side of (11.5) reads:

$$
x_i(t) \cdot \check{\alpha}_i - u_i(t) x_i(t) \cdot e_i - \partial_t x_i(t) \cdot e_i.
$$

Therefore it belongs to $\mathfrak{h}[[t]]$ if and only if

$$
\partial_t x_i(t) = -u_i(t) x_i(t).
\tag{11.6}
$$

If we write

$$
x_i(t) = \sum_{n \leq 0} x_{i,n} t^{-n},
$$

and substitute it into formula (11.6), we obtain that the coefficients $x_{i,n}$ satisfy the following recurrence relation:

$$
n x_{i,n} = \sum_{k+m=n; k<0; m\leq 0} u_{i,k} x_{i,m}, \qquad n < 0.
$$

We find from this formula that

$$
\sum_{n \leq 0} x_{i,n} t^{-n} = \exp\left(- \sum_{m>0} \frac{u_{i,-m}}{m} t^m \right).
\tag{11.7}
$$

Now we obtain that

$$
\delta u_j(t) = \alpha_j(\delta \mathbf{u}(t)) = a_{ij} x_i(t),
$$

where (a_{ij}) is the Cartan matrix of $\mathfrak{g}$. In other words, the operator e_i acts on the algebra $\operatorname{Fun}\operatorname{Conn}(\Omega^{\check{\rho}})_D = \mathbb{C}[u_{i,n}]$ by the derivation

$$
\sum_{j=1}^{\ell} a_{ij} \sum_{n \geq 0} x_{i,n} \frac{\partial}{\partial u_{i,-n-1}},
\tag{11.8}
$$

where $x_{i,n}$ are given by formula (11.7). Thus, we obtain the following characterization of $\operatorname{Fun}\operatorname{Op}_G(D)$ inside $\operatorname{Fun}\operatorname{Conn}(\Omega^{\check{\rho}})_D$.

Proposition 11.1. *The image of* $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D})$ *in* $\operatorname{Fun}\operatorname{Conn}(\Omega^{\check\rho})_{\mathbb{D}}$ *under the Miura map* $\widetilde\mu$ *is equal to the intersection of kernels of the operators given by formula* (11.8) *for* $i = 1, \ldots, \ell$.

11.2. **Back to the $\mathcal{W}$–algebra.** Comparing formula (11.8) with formula (9.8) we find that if we replace $\mathbf{b}'_{i,n}$ by $u_{i,n}$ in formula (9.8) for the screening operator $\mathbf{V}_i[1]$, then we obtain formula (11.8). Therefore the intersection of the kernels of the operators (11.8) is equal to the intersection of the kernels of the operators $\mathbf{V}_i[1], i = 1, \ldots, \ell$. But the latter is the classical $\mathcal{W}$–algebra $\mathbf{W}_{\nu_0}(\mathfrak{g})$. Hence we obtain a commutative diagram

$$\begin{tikzcd}
\pi_0(\mathfrak{g})_{\nu_0} \arrow[r, "\sim"] & \operatorname{Fun}\operatorname{Conn}(\Omega^{\check\rho})_{\mathbb{D}} \\
\mathbf{W}_{\nu_0}(\mathfrak{g}) \arrow[u] \arrow[r, "\sim"] & \operatorname{Fun}\operatorname{Op}_G(\mathbb{D}) \arrow[u]
\end{tikzcd} \tag{11.9}$$

where the top arrow is an isomorphism of algebras given on generators by the assignment $\mathbf{b}'_{i,n} \mapsto u_{i,n}$.

In particular, we obtain from the above isomorphisms vertex Poisson algebra structures on $\mathbf{W}_{\nu_0}(\mathfrak{g})$ and $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D})$ (the latter structure may alternatively be defined by means of the Drinfeld-Sokolov reduction, as we show below). As before, these structures depend on the choice of inner product ν_0, which we will sometimes use as a subscript to indicate which Poisson structure we consider.

The above diagram gives us a geometric interpretation of the screening operators $\mathbf{V}_i[1]$: they correspond to the action of the generators e_i of $\mathfrak{n}$ on $\operatorname{Fun}\operatorname{Conn}(\Omega^{\check\rho})_{\mathbb{D}}$.

As a byproduct, we obtain a character formula for $\mathbf{W}_{\nu_0}(\mathfrak{g})$, as we promised in Section 9.4. Indeed, the action of N on $\operatorname{Conn}(\Omega^{\check\rho})_{\mathbb{D}}$ is free, so that

$$H^i(\mathfrak{n}, \operatorname{Fun}\operatorname{Conn}(\Omega^{\check\rho})_{\mathbb{D}}) = 0, \qquad i > 0.$$

Therefore the character of $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D})$ is equal to the character of the Chavalley complex computing the cohomology of $\mathfrak{n}$ with coefficients in $\operatorname{Fun}\operatorname{Conn}(\Omega^{\check\rho})_{\mathbb{D}}$. The latter is equal to the character of $\operatorname{Fun}\operatorname{Conn}(\Omega^{\check\rho})_{\mathbb{D}}$, i.e., $\prod_{n>0}(1-q^n)^{-\ell}$, times the character of $\bigwedge^\bullet \mathfrak{n}^*$. But since $\deg e_i = -1$, we obtain that that $\deg e_\alpha = -\alpha(\check\rho)$, and so

$$\operatorname{ch} \bigwedge{}^\bullet \mathfrak{n}^* = \prod_{\alpha\in\Delta_+}(1 - q^{\alpha(\check\rho)}) = \prod_{i=1}^{\ell}\prod_{n_i=1}^{d_i}(1 - q^{n_i}).$$

Thus, we obtain that the character of $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D}) \simeq \mathbf{W}_{\nu_0}(\mathfrak{g})$ is given by the right hand side of formula (9.6).

Comparing formulas (11.4) and (9.11), we find that the top isomorphism in (11.9) is $\operatorname{Der} O$–equivariant. Since the screening operators commute with the $\operatorname{Der} O$–action, we find that the bottom isomorphism is also $\operatorname{Der} O$–equivariant. Thus we obtain the following

Theorem 11.2. *The classical $\mathcal{W}$–algebra* $\mathbf{W}_{\nu_0}(\mathfrak{g})$ *is* $\operatorname{Der} O$*–equivariantly isomorphic to the algebra* $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D})$ *of functions on the space of opers on* $\mathbb{D} = \operatorname{Spec}\mathbb{C}[[t]]$.

Combining Theorem 11.2 with Theorem 9.8 and Proposition 9.9 we come to the following result, where by ${}^LG$ we understand the group of inner automorphisms of ${}^L\mathfrak{g}$.

Theorem 11.3. *There is an isomorphism* $\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \simeq \operatorname{Fun} \operatorname{Op}_{{}^L G}(\mathbb{D})_{\kappa_0^\vee}$ *which preserves the vertex Poisson structures and the* $\operatorname{Der} \mathcal{O}$*–module structures on both sides. Moreover, it fits into a commutative diagram of vertex Poisson algebras equipped with* $\operatorname{Der} \mathcal{O}$*–action:*

$$\begin{tikzcd}
\pi_0(\mathfrak{g})_{\kappa_0} \arrow[r, "\sim"] & \operatorname{Fun} \operatorname{Conn}(\Omega^\rho)_{\mathbb{D},\kappa_0^\vee} \\
\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \arrow[r, "\sim"] \arrow[u] & \operatorname{Fun} \operatorname{Op}_{{}^L G}(\mathbb{D})_{\kappa_0^\vee} \arrow[u]
\end{tikzcd} \tag{11.10}$$

where the upper arrow is given on the generators by the assignment $b_{i,n} \mapsto -u_{i,n}$.

This theorem is consistent with the previously given descriptions of the transformation properties of the $b_i(t)$'s and the $u_i(t)$'s. Indeed, according to the computations of Section 5.5, the $b_i(t)$'s transform as components of a connection on the ${}^L H$–bundle $\Omega^{-\rho}$ on the standard disc $\mathbb{D} = \operatorname{Spec} \mathbb{C}[[t]]$. On the other hand, by formula (10.3), the $u_i(t)$'s transform as components of a connection on the dual ${}^L H$–bundle Ω^ρ on $\mathbb{D}$. The map $b_i(t) \mapsto -u_i(t)$ sends the former to the latter.

Given any disc D, we may consider the twists of our algebras by the $\operatorname{Aut} \mathcal{O}$–torsor $\mathcal{A}ut$, defined as in Section 5.6. We will mark them by the subscript D. Then we obtain that

$$\mathbf{W}_{\nu_0}(\mathfrak{g})_D \simeq \operatorname{Fun} \operatorname{Op}_G(D)_{\nu_0},$$
$$\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0,D} \simeq \operatorname{Fun} \operatorname{Op}_{{}^L G}(D)_{\kappa_0^\vee}.$$

11.3. The associated graded spaces.

In this section we describe the isomorphism $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun} \operatorname{Op}_{{}^L G}(\mathbb{D})$ of Theorem 11.3 at the level of associated graded spaces. Note that both algebras $\mathfrak{z}(\widehat{\mathfrak{g}})$ and $\operatorname{Fun} \operatorname{Op}_{{}^L G}(\mathbb{D})$ have natural filtrations.

The filtration on $\mathfrak{z}(\widehat{\mathfrak{g}})$ is induced by the Poincaré–Birkhoff–Witt filtration on the universal enveloping algebra $U(\widehat{\mathfrak{g}}_{\kappa_c})$, see Section 9.2. By Theorem 9.6, $\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) = (\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}$. But

$$\operatorname{gr} V_{\kappa_c}(\mathfrak{g}) = \operatorname{Sym} \mathfrak{g}((t))/\mathfrak{g}[[t]] \simeq \operatorname{Fun} \mathfrak{g}^*[[t]]dt,$$

independently of the choice of coordinate t and inner product on $\mathfrak{g}$. In Proposition 9.3 we gave a description of $(\operatorname{gr} V_{\kappa_c}(\mathfrak{g}))^{\mathfrak{g}[[t]]}$. The coordinate-independent version of this description is as follows. Let $C_{\mathfrak{g}}^\vee = \operatorname{Spec}\, (\operatorname{Fun} \mathfrak{g})^G$ and

$$C_{\mathfrak{g},\Omega}^\vee = \Omega \underset{\mathbb{C}^\times}{\times} C_{\mathfrak{g}}^\vee,$$

where $\Omega = \mathbb{C}[[t]]dt$ is the topological module of differentials on $\mathbb{D} = \operatorname{Spec} \mathbb{C}[[t]]$. Note that a choice of homogeneous generators $P_i, i = 1, \ldots, \ell$, of $(\operatorname{Fun} \mathfrak{g})^G$ gives us an identification

$$C_{\mathfrak{g},\Omega}^\vee \simeq \bigoplus_{i=1}^{\ell} \Omega^{\otimes (d_i+1)}.$$

By Proposition 9.3 we have a canonical isomorphism

$$\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun} C_{\mathfrak{g},\Omega}^\vee. \tag{11.11}$$

Next we consider the map

$$\alpha : \mathfrak{z}(\widehat{\mathfrak{g}}) \to \pi_0(\mathfrak{g})$$

which is equal to the restriction of the embedding $V_{\kappa_c}(\mathfrak{g}) \to W_{0,\kappa_c}$ to $\mathfrak{z}(\widehat{\mathfrak{g}})$. In the proof of Proposition 5.2 we described a filtration on W_{0,κ_c} compatible with the PBW filtration on $V_{\kappa_c}(\mathfrak{g})$. This implies that the map $\mathfrak{z}(\widehat{\mathfrak{g}}) \to \pi_0(\mathfrak{g})$ is also compatible with filtrations. According to Section 5.5,

$$\pi_0(\mathfrak{g}) = \operatorname{Fun} \operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}}.$$

The space $\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}}$ of connections on the ${}^L H$–bundle $\Omega^{-\rho}$ is an affine space over the vector space ${}^L\mathfrak{h} \otimes \Omega = \mathfrak{h}^* \otimes \Omega$. Therefore we find that

$$\operatorname{gr} \pi_0(\mathfrak{g}) = \operatorname{Fun} \mathfrak{h}^* \otimes \Omega. \tag{11.12}$$

We have the Harish-Chandra isomorphism $(\operatorname{Fun} \mathfrak{g}^*)^G \simeq (\operatorname{Fun} \mathfrak{h}^*)^W$, where W is the Weyl group of $\mathfrak{g}$, and hence an embedding $(\operatorname{Fun} \mathfrak{g}^*)^G \to \operatorname{Fun} \mathfrak{h}^*$. This embedding gives rise to an embedding

$$\operatorname{Fun} C^{\vee}_{\mathfrak{g},\Omega} \to \operatorname{Fun} \mathfrak{h}^* \otimes \Omega.$$

It follows from the proof of Proposition 5.2 that this is precisely the map

$$\operatorname{gr} \alpha : \operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) \to \operatorname{gr} \pi_0(\mathfrak{g})$$

under the identifications (11.11) and (11.12).

Let

$$C_{\mathfrak{g}} = \mathfrak{g}/G = \operatorname{Spec}(\operatorname{Fun} \mathfrak{g})^G$$

and

$$C_{\mathfrak{g},\Omega} = \Omega \underset{\mathbb{C}^\times}{\times} C_{\mathfrak{g}}.$$

Recall the canonical form of the opers described in the isomorphism (10.6). Since $\mathcal{P}roj$ is an affine space modeled on the space of quadratic differentials, we find that $\operatorname{Op}_{{}^L G}(\mathbb{D})$ is an affine space modeled on $C_{\mathfrak{g},\Omega}$. Hence $\operatorname{Fun} \operatorname{Op}_{{}^L G}(\mathbb{D})$ has a natural filtration, and we obtain an identification

$$\operatorname{gr} \operatorname{Fun} \operatorname{Op}_{{}^L G}(\mathbb{D}) \simeq \operatorname{Fun} C_{{}^L \mathfrak{g},\Omega} \tag{11.13}$$

(this identification is explained independently of the canonical form (10.6) in [BD1], §§ 3.1.12–3.1.14).

On the other hand, Proposition 10.4 implies that the space $\operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}}$ is isomorphic to the space of connections $\operatorname{Conn}(\Omega^{\check{\rho}})_{\mathbb{D}}$, which is an affine space over the vector space ${}^L\mathfrak{h} \otimes \Omega$. Therefore

$$\operatorname{gr} \operatorname{Fun} \operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}} \simeq \operatorname{Fun} {}^L\mathfrak{h} \otimes \Omega. \tag{11.14}$$

We have the Harish-Chandra isomorphism $(\operatorname{Fun} \mathfrak{g})^G \simeq (\operatorname{Fun} \mathfrak{h})^W$, where W is the Weyl group of $\mathfrak{g}$, and hence an embedding $(\operatorname{Fun} \mathfrak{g})^G \to \operatorname{Fun} \mathfrak{h}$. This embedding gives rise to an embedding

$$\operatorname{Fun} C_{\mathfrak{g},\Omega} \to \operatorname{Fun} \mathfrak{h} \otimes \Omega.$$

The explicit construction of the Miura transformation $\operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}} \to \operatorname{Op}_{{}^L G}(\mathbb{D})$ given in the proof of Proposition 10.4 implies that the map

$$\operatorname{Fun} \operatorname{Op}_{{}^L G}(\mathbb{D}) \to \operatorname{Fun} \operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}}$$

preserves filtrations. Furthermore, with respect to the identifications (11.13) and (11.14), its associated graded map is nothing but the homomorphism

$$\operatorname{Fun} C_{{}^L\mathfrak{g},\Omega} \to \operatorname{Fun} {}^L\mathfrak{h} \otimes \Omega.$$

Now recall that our isomorphism $\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun} \operatorname{Op}_{^LG}(\mathbb{D})$ fits into the commutative diagram (11.10). All maps in this diagram preserve the filtrations, and we have described above the maps of the associated graded induced by the vertical arrows in (11.10). Moreover, we have

$$\operatorname{gr} \pi_0(\mathfrak{g}) \simeq \operatorname{Fun} \mathfrak{h}^* \otimes \Omega = \operatorname{Fun} {}^L\mathfrak{h} \otimes \Omega \simeq \operatorname{gr} \operatorname{Conn}(\Omega^\rho)_{\mathbb{D}},$$

and

$$\operatorname{gr} \mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun} C^\vee_{\mathfrak{g},\Omega} = \operatorname{Fun} C_{{}^L\mathfrak{g},\Omega} \simeq \operatorname{gr} \operatorname{Fun} \operatorname{Op}_{^LG}(\mathbb{D}),$$

where we used the canonical isomorphisms

$$(\operatorname{Fun} \mathfrak{g}^*)^G = (\operatorname{Fun} \mathfrak{h}^*)^W = (\operatorname{Fun} {}^L\mathfrak{h})^W = (\operatorname{Fun} {}^L\mathfrak{g})^{^LG}.$$

According to Theorem 11.3, under these identifications the map

$$\operatorname{gr} \pi_0(\mathfrak{g}) \to \operatorname{gr} \operatorname{Conn}(\Omega^\rho)_{\mathbb{D}}$$

induced by the upper vertical arrow in the diagram (11.10) is equal to the the operator $(-1)^{\deg}$ which takes the value $(-1)^n$ on elements of degree n. Therefore we obtain that the same is true for the associated graded of our isomorphism

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \simeq \operatorname{Fun} \operatorname{Op}_{^LG}(\mathbb{D}). \tag{11.15}$$

Thus we obtain the following

Theorem 11.4. *The isomorphism* (11.15) *preserves filtrations. The corresponding associated graded spaces are both isomorphic to* $\operatorname{Fun} C^\vee_{\mathfrak{g},\Omega}$. *The corresponding isomorphism of the associated graded spaces is equal to* $(-1)^{\deg}$.

12. THE CENTER OF THE COMPLETED UNIVERSAL ENVELOPING ALGEBRA OF $\widehat{\mathfrak{g}}_{\kappa_c}$

Let $U_\kappa(\widehat{\mathfrak{g}})$ be the quotient of the universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa)$ of $\widehat{\mathfrak{g}}_\kappa$ by the ideal generated by $(K-1)$. Define its completion $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ as follows:

$$\widetilde{U}_\kappa(\widehat{\mathfrak{g}}) = \varprojlim U_\kappa(\widehat{\mathfrak{g}})/U_\kappa(\widehat{\mathfrak{g}}) \cdot (\mathfrak{g} \otimes t^N \mathbb{C}[[t]]).$$

It is clear that $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ is a topological algebra which acts on all smooth representations of $\widehat{\mathfrak{g}}_\kappa$ (i.e., such that any vector is annihilated by $\mathfrak{g} \otimes t^N \mathbb{C}[[t]]$ for sufficiently large N), on which K acts as the identity. In this section we describe the center $Z(\widehat{\mathfrak{g}})$ of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ and show that it is isomorphic to the topological algebra of functions on the space $\operatorname{Op}_{^LG}(\mathbb{D}^\times)$ of ${}^LG$–opers on the punctured standard disc $\mathbb{D}$.

12.1. **Enveloping algebra of a vertex algebra.** In [FB], § 4.3.1, we attached to any vertex algebra V a complete topological associative algebra $\widetilde{U}(V)$. Moreover, the assignment $V \mapsto \widetilde{U}(V)$ gives rise to a functor from the category of vertex algebras to the category of complete topological associative algebras. Let us recall this construction.

First we define a Lie algebra

$$U(V) = (V \otimes \mathbb{C}((t)))/\operatorname{Im}(T \otimes 1 + 1 \otimes \partial_t).$$

It is topologically spanned by elements $A_{[n]} = A \otimes t^n, A \in V, n \in \mathbb{Z}$, with the commutation relations

$$[A_{[m]}, B_{[k]}] = \sum_{n \geq 0} \binom{m}{n} (A_{(n)} \cdot B)_{[m+k-n]}. \tag{12.1}$$

Next, let $U(U(V))$ the universal enveloping algebra of the Lie algebra $U(V)$. Define its completion

$$\widetilde{U}(U(V)) = \varprojlim U(U(V))/I_N,$$

where I_N is the left ideal generated by $A_{[n]}, A \in V, n > N$. Then $\widetilde{U}(V)$ is by definition the quotient of $\widetilde{U}(U(V))$ by the two-sided ideal generated by the Fourier coefficients of the series

$$Y[A_{(-1)}B, z] - {:}Y[A, z]Y[B, w]{:}, \qquad A, B \in V,$$

where $Y[A, z] = \sum_{n \in \mathbb{Z}} A_{[n]} z^{-n-1}$, and the normal ordering is defined in the same way as for the vertex operators $Y(A, z)$.

In the case when $V = V_\kappa(\mathfrak{g})$, each $A \in V_\kappa(\mathfrak{g})$ is a linear combination of monomials in $J^a_n, n < 0$, applied to the vacuum vector v_κ. We have a linear map $U(V_\kappa(\mathfrak{g})) \to \widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ sending $(J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_\kappa)_{[k]}$ to

$$\begin{aligned} &\operatorname{Res}_{z=0} Y(J^{a_1}_{n_1} \dots J^{a_m}_{n_m} v_\kappa, z) z^k dz \\ &\qquad = \frac{1}{(-n_1-1)!} \cdots \frac{1}{(-n_m-1)!} \operatorname{Res}_{z=0} {:}\partial_z^{-n_1-1} J^{a_1}(z) \dots \partial_z^{-n_m-1} J^{a_m}(z){:}\, z^k dz, \end{aligned}$$

where as before $J^a(z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}$. By [FB], Prop. 4.2.2, this map is a homomorphism of Lie algebras. Hence we obtain a homomorphism from the universal enveloping algebra of $U(V_\kappa(\mathfrak{g}))$ to the completed enveloping algebra $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ defined above. By conituity, this map extends to the completion of $U(V_\kappa(\mathfrak{g}))$. According to [FB], Lemma 4.3.2, this map gives rise to an isomorphism

$$\widetilde{U}(V_\kappa(\mathfrak{g})) \simeq \widetilde{U}_\kappa(\widehat{\mathfrak{g}}).$$

12.2. **From $\mathfrak{z}(\widehat{\mathfrak{g}})$ to the center.** Let $B \in \mathfrak{z}(\widehat{\mathfrak{g}}) \subset V_{\kappa_c}(\mathfrak{g})$. Then $\mathfrak{g}[[t]] \cdot B = 0$ and formula (12.1) for the commutation relations implies that all Fourier coefficients $B_{[k]}$ of the vertex operator $Y(B, z)$ commute with the entire affine algebra $\widehat{\mathfrak{g}}_{\kappa_c}$. Thus we obtain an injective map $U(\mathfrak{z}(\widehat{\mathfrak{g}})) \to Z(\widehat{\mathfrak{g}})$. Moreover, applying the enveloping algebra functor $\widetilde{U}$ to $\mathfrak{z}(\widehat{\mathfrak{g}})$ we obtain an injective homomorphism $\widetilde{U}(\mathfrak{z}(\widehat{\mathfrak{g}})) \to Z(\widehat{\mathfrak{g}})$.

The algebra $\widetilde{U}(\mathfrak{z}(\widehat{\mathfrak{g}}))$ is a completion of a polynomial algebra. Indeed, recall from Section 9.4 that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[S_i^{(n)}]_{i=1,\dots,\ell; n \geq 0},$$

where $S_i^{(n)} = T^n S_i$. Let $S_{i,[n]}, n \in \mathbb{Z}$, be the Fourier coefficients of the vertex operator $Y(S_i, z)$, considered as elements of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$. Then $\widetilde{U}(\mathfrak{z}(\widehat{\mathfrak{g}}))$ is the completion of the polynomial algebra $\mathbb{C}[S_{i,[n]}]_{i=1,\ldots,\ell;n\in\mathbb{Z}}$ with respect to the topology in which the basis of open neighborhoods of 0 is formed by the ideals generated by $S_{i,[n]}, i = 1,\ldots,\ell; n > N$, for $N \in \mathbb{Z}$. Thus, the completed polynomial algebra $\widetilde{U}(\mathfrak{z}(\widehat{\mathfrak{g}}))$ embeds into the center $Z(\widehat{\mathfrak{g}})$ of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$.

Proposition 12.1. *$Z(\widehat{\mathfrak{g}})$ is equal to $\widetilde{U}(\mathfrak{z}(\widehat{\mathfrak{g}}))$.*

The proof is given in [BD1], Theorem 3.7.7. It is based on the description of the associated graded of $Z(\widehat{\mathfrak{g}})$ with respect to the PBW filtration, which is obtained by an argument similar to the one used in the proof of Proposition 9.3. We obtain then that the associated graded of $Z(\widehat{\mathfrak{g}})$ is equal to the completed polynomial algebra in $P_{i,n}, i = 1,\ldots,\ell; n \in \mathbb{Z}$, which are the Fourier coefficients of the series $P_i(J^a(z))$ (see Section 9.2 for the definition of the P_i's). But by construction $P_{i,n}$ is the symbol of $S_{i,[-n-1]}$. Therefore $Z(\widehat{\mathfrak{g}})$ cannot contain anything but elements of the completed polynomial algebra in the $S_{i,[n]}$'s, hence the result.

12.3. Identification of the center with $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D}^\times)$.

Recall the $\operatorname{Aut} O$–equivariant isomorphism of vertex Poisson algebras

$$\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0} \simeq \operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})_{\kappa_0^\vee} \tag{12.2}$$

established in Theorem 11.3. For a vertex Poisson algebra P, the space $U(P)$ (also denoted by $\operatorname{Lie}(P)$) has a natural Lie algebra structure (see [FB], § 16.1.7), and therefore the commutative algebra $\widetilde{U}(P)$ carries a Poisson algebra structure. In particular, $\widetilde{U}(\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0})$ is a Poisson algebra. Under the identification $\widetilde{U}(\mathfrak{z}(\widehat{\mathfrak{g}})_{\kappa_0}) \simeq Z(\widehat{\mathfrak{g}})$ the corresponding Poisson structure on $Z(\widehat{\mathfrak{g}})_{\kappa_0}$ may be described as follows. Consider $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ as the one-parameter family A_ϵ of associative algebras depending on the parameter $\epsilon = (\kappa - \kappa_c)/\kappa_0$. Then $Z(\widehat{\mathfrak{g}})$ is the center of A_0. Given $x, y \in Z(\widehat{\mathfrak{g}})$, let $\widetilde{x}, \widetilde{y}$ be their liftings to A_ϵ. Then the Poisson bracket $\{x, y\}$ is defined as the ϵ–linear term in the commutator $[\widetilde{x}, \widetilde{y}]$ considered as a function of ϵ (it is independent of the choice of the liftings). The fact that this is indeed a Poisson structure was observed by V. Drinfeld following the work [H] of T. Hayashi. We denote the center $Z(\widehat{\mathfrak{g}})$ equipped with this Poisson structure by $Z(\widehat{\mathfrak{g}})_{\kappa_0}$.

Likewise, the vertex Poisson algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})_{\kappa_0^\vee}$ gives rise to a topological Poisson algebra $\widetilde{U}(\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})_{\kappa_0^\vee})$. Then the isomorphism (12.2) gives rise to an isomorphism of Poisson algebras

$$Z(\widehat{\mathfrak{g}})_{\kappa_0} \simeq \widetilde{U}(\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})_{\kappa_0^\vee}).$$

Moreover, this isomorphism is $\operatorname{Aut} O$–equivariant.

Lemma 12.2. *The algebra $\widetilde{U}(\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})_{\kappa_0^\vee})$ is canonically isomorphic to the topological algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D}^\times)$ of functions on the space of ^LG–opers on the punctured disc $\mathbb{D}^\times = \operatorname{Spec}\mathbb{C}((t))$.*

Proof. As explained in Section 11.1, the algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})$ is isomorphic to the polynomial algebra $\mathbb{C}[v_{i,n_i}]_{i=1,\dots,\ell;n_i<-d_i}$, where $v_{i,n}$ are the coefficients of the oper connection

$$\nabla = \partial_t + p_{-1} + \sum_{i=1}^{\ell} v_i(t)\mathbf{c}_i, \tag{12.3}$$

where $v_i(t) = \sum_{n<-d_i} v_{i,n} t^{-n-d_i-1}$. Then the above construction of the functor $\widetilde{U}$ implies that the algebra $\widetilde{U}(\operatorname{Op}_{^LG}(\mathbb{D})_{\kappa_0^\vee})$ is the completion of the polynomial algebra in the variables $v_{i,n}, i = 1,\dots,\ell; n \in \mathbb{Z}$, with respect to the topology in which the base of open neighborhoods of 0 is formed by the ideals generated by $v_{i,n}, n < N$. But this is precisely the algebra of functions on the space of opers of the form (12.3), where $v_i(t) = \sum_{n\in\mathbb{Z}} v_{i,n} t^{-n-d_i-1} \in \mathbb{C}((t))$. □

Since $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D})_{\kappa_0^\vee}$ is a vertex Poisson algebra, we obtain that $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D}^\times)$ is a Poisson algebra. We will use the subscript $\kappa_0^\vee$ to indicate the dependence of this Poisson structure on the inner product $\kappa_0^\vee$ on $^L\mathfrak{g}$. We will give another definition of this Poisson structure in the next section. Thus we obtain the following result which was originally conjectured by V. Drinfeld.

Theorem 12.3. *The center $Z(\widehat{\mathfrak{g}})_{\kappa_0}$ is isomorphic, as a Poisson algebra, to the Poisson algebra $\operatorname{Fun}\operatorname{Op}_{^LG}(\mathbb{D}^\times)_{\kappa_0^\vee}$. Moreover, this isomorphism is $\operatorname{Aut} O$–equivariant.*

12.4. The Poisson structure on $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D}^\times)_{\nu_0}$.

The Poisson algebra of functions on the space $\operatorname{Op}_G(\mathbb{D}^\times)$ of opers on the punctured disc $\mathbb{D}^\times$ may be obtained by hamiltonian reduction called the Drinfeld-Sokolov reduction [DS].

We start with the Poisson manifold $\operatorname{Conn}_{\mathfrak{g}}$ of connections on the trivial G–bundle on $\mathbb{D}^\times$, i.e., operators of the form $\nabla = \partial_t + A(t)$, where $A(t) \in \mathfrak{g}((t))$. The Poisson structure on this manifold comes from its identification with a hyperplane in the dual space to the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_{\nu_0}$. Indeed, the topological dual space to $\widehat{\mathfrak{g}}_{\nu_0}$ may be identified with the space of all λ–connections on the trivial bundle on $\mathbb{D}^\times$, see [FB], § 16.4. Namely, we split $\widehat{\mathfrak{g}}_{\nu_0} = \mathfrak{g}((t)) \oplus \mathbb{C}K$ as a vector space. Then a λ–connection $\partial_t + A(t)$ gives rise to a linear functional on $\widehat{\mathfrak{g}}_{\nu_0}$ which takes value

$$\lambda b + \operatorname{Res} \nu_0(A(t), B(t))dt$$

on $(B(t) + bK) \in \mathfrak{g}((t)) \oplus \mathbb{C}K = \widehat{\mathfrak{g}}_{\nu_0}$. Note that under this identification the action of $\operatorname{Aut} O$ by changes of coordinates and the coadjoint (resp., gauge) action of $G((t))$ on the space of λ–connections (resp., the dual space to $\widehat{\mathfrak{g}}_{\nu_0}$) agree. The space $\operatorname{Conn}_{\mathfrak{g}}$ is now identified with the hyperplane in $\widehat{\mathfrak{g}}^*_{\nu_0}$ which consists of those functionals which take value 1 on the central element K.

The dual space $\widehat{\mathfrak{g}}^*_{\nu_0}$ carries a canonical Poisson structure called the Kirillov–Kostant structure (see, e.g., [FB], § 16.4.1 for more details). Because K is a central element, this Poisson structure restricts to the hyperplane which we have identified with $\operatorname{Conn}_{\mathfrak{g}}$. Therefore we obtain a Poisson structure on the $\operatorname{Conn}_{\mathfrak{g}}$.

The group $N((t))$ acts on $\operatorname{Conn}_{\mathfrak{g}}$ by gauge transformations. This action corresponds to the coadjoint action of $N((t))$ on $\widehat{\mathfrak{g}}^*_{\nu_0}$ and is hamiltonian, the moment map being the surjection $m : \operatorname{Conn}_{\mathfrak{g}} \to \mathfrak{n}((t))^*$ dual to the embedding $\mathfrak{n}((t)) \to \widehat{\mathfrak{g}}_{\nu_0}$. We pick a one–point coadjoint $N((t))$–orbit in $\mathfrak{n}((t))^*$ represented by the linear functional ψ which is equal to the composition $\mathfrak{n}((t)) \to \mathfrak{n}/[\mathfrak{n},\mathfrak{n}]((t)) = \bigoplus_{i=1}^{\ell} \mathfrak{g}_{\alpha_i}((t))$ and the functional

$$(x_i(t))_{i=1}^{\ell} \mapsto \sum_{i=1}^{\ell} \operatorname{Res} x_i(t) dt.$$

One shows in the same way as in the proof of Lemma 10.2 that the action of $N((t))$ on $m^{-1}(\psi)$ is free. Moreover, the quotient $m^{-1}(\psi)/N((t))$, which is the Poisson reduced manifold, is canonically identified with the space of G–opers on $\mathbb{D}^\times$. Therefore we obtain a Poisson structure on the topological algebra of functions $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D}^\times)$. On the other hand we have defined a Poisson algebra structure on $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D}^\times)_{\nu_0}$ using its identification with $\widetilde{U}(\operatorname{Fun}\operatorname{Op}_G(\mathbb{D})_{\nu_0})$, the isomorphism $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D})_{\nu_0} \simeq \mathbf{W}_{\nu_0}(\mathfrak{g})_{\nu_0}$ obtained in Theorem 11.2, and the vertex Poisson algebra structure on $\mathbf{W}_{\nu_0}(\mathfrak{g})$.

Lemma 12.4. *The two Poisson structures coincide.*

Proof. In [FB], §§ 15.4 and 16.8, we defined a complex $C^\bullet_\infty(\mathfrak{g})$ and showed that its 0th cohomology is a vertex Poisson algebra canonically isomorphic to $\mathbf{W}_{\nu_0}(\mathfrak{g})$ (and all other cohomologies vanish). It is clear from the construction that if we apply the functor $\widetilde{U}$ to the complex $C^\bullet_\infty(\mathfrak{g})$ we obtain the BRST complex of the Drinfeld-Sokolov reduction described above. Since the functor $\widetilde{U}$ is left exact, we obtain that the 0th cohomology of $\widetilde{U}(C^\bullet_\infty(\mathfrak{g}))$, i.e., the Poisson algebra $\operatorname{Fun}\operatorname{Op}_G(\mathbb{D}^\times)_{\nu_0}$ is isomorphic to the Poisson algebra $\widetilde{U}(\mathbf{W}_{\nu_0}(\mathfrak{g}))$ which is what we needed to prove. □

12.5. **The Miura transformation as the Harish-Chandra homomorphism.** The Harish-Chandra homomorphism is the homomorphism from the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra to the algebra $\operatorname{Fun}\mathfrak{h}^*$ of polynomials on $\mathfrak{h}^*$. It identifies $Z(\mathfrak{g})$ with the algebra $(\operatorname{Fun}\mathfrak{h}^*)^W$ of W–invariant polynomials on $\mathfrak{h}^*$. To construct this homomorphism, one needs to assign a central character to each $\lambda \in \mathfrak{h}^*$. This central character is just the character of the center on the Verma module $M_{\lambda-\rho}$.

In the affine case, we construct a similar homomorphism from the center $Z(\widehat{\mathfrak{g}})$ of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ to the topological algebra $\operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}^\times}$ of functions on the space of connections on the ${}^L H$–bundle $\Omega^{-\rho}$ on $\mathbb{D}^\times$. According Proposition 5.5, points of $\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}^\times}$ parameterize Wakimoto modules of critical level. Thus, for each $\chi \in \operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}^\times}$ we have the Wakimoto module W_χ of critical level. The following theorem describes the affine analogue of the Harish-Chandra homomorphism.

Note that $\operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}^\times}$ is the completion of the polynomial algebra in $b_{i,n}, i = 1,\ldots,\ell; n \in \mathbb{Z}$, with respect to the topology in which the base of open neighborhoods of 0 is formed by the ideals generated by $b_{i,n}, n < N$. In the same way as in the proof of Lemma 12.2 we show that it is isomorphic to $\widetilde{U}(\operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}})$. We also define the topological algebra $\operatorname{Fun}\operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}^\times}$ as $\widetilde{U}(\operatorname{Fun}\operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}})$. It is the completion of the polynomial algebra in $u_{i,n}, i = 1,\ldots,\ell; n \in \mathbb{Z}$, with respect to the topology in which the base of open neighborhoods of 0 is formed by the ideals generated by $u_{i,n}, n < N$. The isomorphism of Proposition 10.4 gives rise to an isomorphism of the topological algebras

$$\operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}^\times} \to \operatorname{Fun}\operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}^\times},$$

under which $b_{i,n} \mapsto -u_{i,n}$.

Theorem 12.5. *The center* $Z(\widehat{\mathfrak{g}})$ *acts by a central character on each Wakimoto module* W_χ. *The corresponding homomorphism of algebras* $Z(\widehat{\mathfrak{g}}) \to \operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}^\times}$ *fits into a commutative diagram*

$$\begin{tikzcd}
\operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}^\times} \arrow[r, "\sim"] & \operatorname{Fun}\operatorname{Conn}(\Omega^{\rho})_{\mathbb{D}^\times} \\
Z(\widehat{\mathfrak{g}}) \arrow[u] \arrow[r, "\sim"] & \operatorname{Fun}\operatorname{Op}_{{}^L G}(\mathbb{D}^\times) \arrow[u]
\end{tikzcd} \tag{12.4}$$

where the upper arrow is given on the generators by the assignment $b_{i,n} \mapsto -u_{i,n}$.

Proof. The action of $U(V_{\kappa_c}(\mathfrak{g}))$, and hence of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$, on W_χ is obtained through the homomorphism of vertex algebra $V_{\kappa_c}(\mathfrak{g}) \to M_{\mathfrak{g}} \otimes \pi_0(\mathfrak{g})$. In particular, the action of $Z(\widehat{\mathfrak{g}})$ on W_χ is obtained through the homomorphism of commutative vertex algebras $\mathfrak{z}(\widehat{\mathfrak{g}}) \to \pi_0(\mathfrak{g})$. But $\pi_0(\mathfrak{g}) = \operatorname{Fun}\operatorname{Conn}(\Omega^{-\rho})_{\mathbb{D}}$. Therefore the statement of the theorem follows from Theorem 11.3 by applying the functor of enveloping algebras $V \mapsto \widetilde{U}(V)$ introduced in Section 12.1. $\square$

Thus, we see that the affine analogue of the Harish-Chandra homomorphism is nothing but the Miura transformation for the Langlands dual group. In particular, its image (which in the finite-dimensional case consists of all W–invariant polynomials) is equal to the intersection of the kernels of the screening operators.

12.6. **Compatibility with the finite-dimensional Harish-Chandra homomorphism.** In this section we describe a certain compatibility between the Miura transformation and the Harish-Chandra homomorphism. In particular, we will establish Proposition 12.7 which is used in [BD1].

We have a natural $\mathbb{Z}$–gradation on $Z(\widehat{\mathfrak{g}})$ by the operator $L_0 = -t\partial_t \in \operatorname{Der} O$. Let us denote by the upper script 0 the degree 0 part and by the upper script < 0 be the span of all elements of negative degrees in any $\mathbb{Z}$–graded object. Consider the Wakimoto modules W_χ, where

$$\chi = \partial_t + \frac{\lambda - \rho}{t}, \qquad \lambda \in \mathfrak{h}^*.$$

These Wakimoto modules are $\mathbb{Z}$–graded, with the degree 0 part W_χ^0 being the subspace $\mathbb{C}[a^*_{\alpha,0}|0\rangle]_{\alpha\in\Delta_+} \subset M_{\mathfrak{g}} \subset W_\chi$. The Lie algebra $\mathfrak{g}$ preserves this subspace, and it follows immediately from our construction of Wakimoto modules that as a $\mathfrak{g}$–module, W_χ^0 is isomorphic to the contragredient Verma module $M^*_{\lambda-\rho}$. The algebra

$$Z' = Z(\widehat{\mathfrak{g}})^0 / (Z(\widehat{\mathfrak{g}}) \cdot Z(\widehat{\mathfrak{g}})^{<0})^0$$

naturally acts on W_χ^0 and commutes with $\mathfrak{g}$. Therefore varying $\lambda \in \mathfrak{h}^*$ we obtain a homomorphism

$$Z' \to (\operatorname{Fun}\mathfrak{h}^*)^W, \tag{12.5}$$

which factors through the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to (\operatorname{Fun}\mathfrak{h}^*)^W$.

On the other hand, let $\mathrm{Op}_{^LG}(\mathbb{D}^\times)_{\leq 1}$ be the variety of *opers with regular singularity*, whose points are $B[[t]]$–gauge equivalence classes of operators of the form

$$\partial_t + \frac{1}{t}\left(p_{-1} + \mathbf{v}(t)\right), \qquad \mathbf{v}(t) \in {}^L\mathfrak{b}[[t]] \tag{12.6}$$

(see [BD1], § 3.8.8). According to [BD1], Prop. 3.8.9, the natural map $\mathrm{Op}_{^LG}(\mathbb{D}^\times)_{\leq 1} \to \mathrm{Op}_{^LG}(\mathbb{D}^\times)$, assigning to a $B[[t]]$–equivalence class the corresponding $B((t))$–equivalence class, is injective. We will therefore view $\mathrm{Op}_{^LG}(\mathbb{D}^\times)_{\leq 1}$ as a subvariety of $\mathrm{Op}_{^LG}(\mathbb{D}^\times)$.

Likewise we introduce the subvariety $\mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times,\leq 1}$ of connections with regular singularity on the ${}^LH$–bundle Ω^ρ on $\mathbb{D}^\times$, whose points are operators of the form

$$\partial_t + \frac{1}{t}(\rho + \mathbf{u}(t)), \qquad \mathbf{u}(t) \in {}^L\mathfrak{h}[[t]]. \tag{12.7}$$

The Miura transformation then restricts to a morphism

$$\mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times,\leq 1} \to \mathrm{Op}_{^LG}(\mathbb{D}^\times)_{\leq 1}$$

sending a connection $\overline{\nabla}$ of the form (12.7) to an oper on $\mathbb{D}^\times$ with the connection operator $\nabla = \overline{\nabla} + p_{-1}$.

We have a $\mathbb{Z}$–gradation on both $\mathrm{Fun}\,\mathrm{Op}_{^LG}(\mathbb{D}^\times)$ and $\mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times}$ induced by the operator $L_0 = -t\partial_t \in \mathrm{Der}\,\mathcal{O}$. This gradation descends to $\mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times,\leq 1}$ and $\mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times,\leq 1}$ where it takes only non-negative values.

Define residue maps

$$\begin{aligned} \mathrm{Res}:\ & \mathrm{Op}_{^LG}(\mathbb{D}^\times)_{\leq 1} \to \mathrm{Spec}\,(\mathrm{Fun}\,{}^L\mathfrak{g})^{^LG}, \\ \mathrm{Res}:\ & \mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times,\leq 1}) \to \mathrm{Spec}\,{}^L\mathfrak{h}, \end{aligned}$$

sending an oper (12.6) (considered as a $B[[t]]$–equivalence class) to $p_{-1} + \mathbf{v}(0)$ (resp., a Miura oper (12.7) to $\rho + \mathbf{u}(0)$). They give rise to homomorphisms

$$\begin{aligned} (\mathrm{Fun}\,{}^L\mathfrak{g})^{^LG} &\to (\mathrm{Fun}\,\mathrm{Op}_{^LG}(\mathbb{D}^\times)_{\leq 1})^0, \\ \mathrm{Fun}\,{}^L\mathfrak{h} &\to (\mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times,\leq 1})^0. \end{aligned}$$

Lemma 12.6. *These homomorphisms are isomorphisms and they fit into the commutative diagram*

$$\begin{tikzcd}
(\mathrm{Fun}\,{}^L\mathfrak{h}) \arrow[r, "\sim"] & (\mathrm{Fun}\,\mathrm{Conn}(\Omega^\rho)_{\mathbb{D}^\times,\leq 1})^0 \\
(\mathrm{Fun}\,{}^L\mathfrak{g})^{^LG} \arrow[r, "\sim"] \arrow[u] & (\mathrm{Fun}\,\mathrm{Op}_{^LG}(\mathbb{D}^\times)_{\leq 1})^0 \arrow[u]
\end{tikzcd}$$

where the left vertical arrow is the Harish-Chandra homomorphism.

Proof. Note that

$$\partial_t + p_{-1} + \frac{1}{t}(\rho + \mathbf{u}(t)) = \rho(t)^{-1}\left(\partial_t + \frac{p_{-1}}{t} + \frac{1}{t}\mathbf{u}(t)\right)\rho(t).$$

But the image of $p_{-1} + \mathbf{u}(0)$ in ${}^L\mathfrak{g}/{}^LG$ is the same as the image of $\mathbf{u}(0) \in {}^L\mathfrak{h}$ in ${}^L\mathfrak{h}/W$. This implies that the above diagram is commutative. It is clear from the definition that the upper horizontal arrow is an isomorphism. To see that the lower horizontal arrow is an isomorphism we pass to the associated graded spaces where it becomes obvious (see Section 9.3). □

Since the ideal of $\mathrm{Op}_{^L G}(\mathbb{D}^\times)_{\leq 1}$ in $\operatorname{Fun}\mathrm{Op}_{^L G}(\mathbb{D}^\times)$ is equal to $\operatorname{Fun}\mathrm{Op}_{^L G}(\mathbb{D}^\times)^{<0}$ we obtain that the isomorphism of Theorem 12.3 gives rise to an isomorphism of algebras

$$Z' \simeq (\operatorname{Fun}\mathrm{Op}_{^L G}(\mathbb{D}^\times)_{\leq 1})^0. \tag{12.8}$$

Then Lemma 12.6 implies that the map (12.5) is an isomorphism. Recall that ${}^L\mathfrak{h} = \mathfrak{h}^*$ and that in the isomorphism of Theorem 12.5 we have $b_{i,n} \mapsto -u_{i,n}$. This implies the following

Proposition 12.7. *There is a commutative diagram of isomorphisms*

$$\begin{tikzcd}
Z' \arrow[r, "\sim"] \arrow[d] & (\operatorname{Fun}\mathrm{Op}_{^L G}(\mathbb{D}^\times)_{\leq 1})^0 \\
(\operatorname{Fun}\mathfrak{h}^*)^W \arrow[r, "\sim"] & (\operatorname{Fun}\mathfrak{h}^*)^W \arrow[u]
\end{tikzcd}$$

where the lower horizontal arrow is given by $f \mapsto f^-$, $f^-(\lambda) = f(-\lambda)$.

Note that this is the statement of Theorem 3.8.17 of [BD1].

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