A QUANTIZATION ON RIEMANN SURFACES WITH PROJECTIVE STRUCTURE

DAVID BEN-ZVI AND INDRANIL BISWAS

Abstract. Let X be a Riemann surface equipped with a projective structure. Let $\mathcal{L}$ be a square-root of the holomorphic cotangent bundle K_X. Consider the symplectic form on the complement of the zero section of $\mathcal{L}$ obtained by pulling back the symplectic form on K_X using the map $v \longmapsto v^{\otimes 2}$. We show that this symplectic form admits a natural quantization. This quantization also gives a quantization of the complement of the zero section in K_X equipped with the natural symplectic form.

1. Introduction

Let (M, ω) be a symplectic manifold. The commutative algebra $C(M)$ of all complex valued smooth functions on M is equipped with a Poisson bracket, defined by $\{f\,,g\} = \omega^{-1}(df\,,dg)$, where ω^{-1} is the bilinear form on T^*M obtained from ω. The Poisson bracket provides $C(M)$ with the structure of Lie algebra, satisfying the Leibniz identity $\{fg\,,k\} = f\{g\,,k\} + g\{f\,,k\}$.

Let $\mathcal{A}(M) := C(M)[[h]]$ be the space of all formal Taylor series

$$f := \sum_{j=0}^{\infty} h^j f_j$$

where $f_j \in C(M)$. A *quantization* of the symplectic form ω is an associative algebra operation on $\mathcal{A}(M)$ satisfying certain conditions [9], [1], [7]. For another element $g := \sum_{j=0}^{\infty} h^j g_j \in \mathcal{A}(M)$, if $f \star g = \sum_{j=0}^{\infty} h^j c_j$ is the multiplication, then the conditions say:

(1) each c_i is some polynomial (independent of f and g) in derivatives (of arbitrary order) of $\{f_j\}$ and $\{g_j\}$;
(2) $c_0 = f_0 g_0$;
(3) $f \star g - g \star f = \sqrt{-1}h\{f_0\,,g_0\} + h^2 k$, where $k \in \mathcal{A}(M)$ depends on f, g.

It is known that every symplectic manifold admits quantizations, [5], [6], [7]. However, there is no uniqueness of quantization.

Let X be a an arbitrary Riemann surface, K_X its holomorphic cotangent bundle, and Ω_0 the natural symplectic structure on the total space of K_X. In this paper we construct canonical quantizations of the complement $K'_X := K_X \setminus X$ of the zero

1991 *Mathematics Subject Classification.* 81S10, 53B10, 14H60, 14F05.
Key words and phrases. Projective structure, symplectic structure, Quantization, Moyal–Weyl algebra.

section in K_X, associated to projective structures on X. A projective structure on X is defined by giving a covering of X by holomorphic coordinate charts such that each transition function is a Möbius transformation. Every Riemann surface admits projective structures, and the space of all projective structures on X is an affine space for the vector space $H^0(X, K_X^{\otimes 2})$ of quadratic differentials on X.

The idea behind the construction is that any local construction on the projective line $\mathbb{CP}^1$ which is canonical enough in the sense that it is invariant under the action of the group $\mathrm{Aut}(\mathbb{CP}^1)$ of Möbius transformations, extends naturally to any Riemann surface equipped with a projective structure.

We fix a square-root $\mathcal{L}$ of the holomorphic cotangent bundle K_X of X. In other words, $\mathcal{L}$ is a holomorphic line bundle over X equipped with an isomorphism

$$\psi_0 \,:\, \mathcal{L}^{\otimes 2} \longrightarrow K_X\,.$$

Square-roots of K_X are also known as *theta characteristics*, and exist for any X. If X is a compact connected Riemann surface of genus g, then there are exactly 4^g theta characteristics on X.

Let S denote the complex surface defined by the complement of the zero section of $\mathcal{L}$ in the total space of $\mathcal{L}$. In other words, S consists of all nonzero vectors in $\mathcal{L}$. Consider the holomorphic map

$$\psi \,:\, S \longrightarrow K_X \tag{1.1}$$

defined by $\psi(v) = \psi_0(v \otimes v)$. The map ψ is an unramified double cover of K'_X with the nontrivial deck transformation of S acting as $v \longmapsto -v$. We take

$$\Omega := \frac{\psi^*\Omega_0}{2} \tag{1.2}$$

as the symplectic form on S, and note it is invariant under the involution of S.

If we set X to be $\mathbb{CP}^1$, then the symplectic manifold S is isomorphic to $\mathbb{C}^2 \setminus \{0\}$ equipped with the standard symplectic structure. The Moyal–Weyl quantization of $\mathbb{C}^2$ is invariant under the action of $\mathrm{SL}(2, \mathbb{C})$. This enables us to construct a quantization of the surface S associated to an arbitrary X from a projective structure on X. We also show how to recover the projective structure on X from the associated quantization. Thus for X compact and connected, we obtain a $3g - 3$ dimensional family of quantizations of S.

The quantizations of Ω constructed here are invariant under the involution of S. Therefore, such a quantization descends to a quantization of the restriction of the symplectic form $\Omega_0/2$ to the Zariski open subset K'_X of K_X. This quantization is singular near the boundary. In other words, it does not extend across the zero section of K_X.

Let us compare this construction with the approach to quantization via symplectic connections. As noted in [1], if we are given a *flat* connection on the tangent bundle of a symplectic manifold M, preserving the symplectic form, then it is straightforward to write down a Moyal–type quantization of $C(M)$. This is done by using the resulting affine symplectic charts on M to transfer the quantization from affine space. The Fedosov approach to quantization [7] starts with a symplectic form which is not flat and iteratively adjusts it to produce a flat, but nonlinear, connection, and resulting quantization.

In the case of the symplectic manifold $M = K_X$, the tangent bundle of K_X restricted to X is the holomorphic rank two bundle $K_X \bigoplus K_X^{-1}$. This bundle does not admit any holomorphic connection along X. Our construction is based on the observation that we do have flat connections on an appropriate "twisted square–root" of this bundle. Namely we take a square–root $\mathcal{L}$ of K_X, and construct the unique nontrivial extension $\mathcal{V} = J^1(\mathcal{L}^{-1})$ of $\mathcal{L}$ by $\mathcal{L}^{-1}$. The bundle $\mathcal{V}$ *does* admit flat connections, known as projective connections, and these are in natural bijection with projective structures on X. Moreover these connections preserve a natural symplectic form on the complement of the zero section. Using these connections to quantize the space S reproduces our quantization described above.

Thus by substituting projective connections for flat symplectic connections, we obtain a short alternative to the Fedosov quantization procedure for the surface S. It would be interesting to make use of the higher–order analogues of projective structures on X to probe further into the space of quantizations on the cotangent bundle.

We refer to [3] for many interesting examples of quantization of Kähler structures.

2. Equivalence of symplectic structures

A *projective atlas* on X is defined by giving a covering of X by holomorphic coordinate charts, say $\{U_\alpha, \phi_\alpha\}_{\alpha\in I}$, where ϕ_α is a biholomorphism from an open subset U_α of X to an open subset of $\mathbb{C}$, such that for every pair $\alpha, \beta \in I$, the map $\phi_\beta \circ \phi_\alpha^{-1}$ is the restriction of a Möbius transformation to the image $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}$ [8]. Recall that a Möbius transformation is a function of the form $z \longmapsto (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

Another such projective atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in I'}$ is called *equivalent* to $\{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ if their union $\{U_\alpha, \phi_\alpha\}_{\alpha\in I\cup I'}$ is again a part of a data for a projective atlas. A *projective structure* on X is an equivalence class of projective atlases.

Let $\mathcal{P}(X)$ denote the space of all projective structures on X.

Note that associating the Möbius transformation $\phi_\beta\circ\phi_\alpha^{-1}$ to each pair (α, β) we get a 1-cocycle with values on $PGL(2, \mathbb{C})$. From [8, page 180, Lemma 26] we know that the cohomology class $c \in H^1(X, PGL(2, \mathbb{C}))$ defined by this cocycle lifts to an element $\overline{c} \in H^1(X, \mathrm{SL}(2, \mathbb{C}))$ for the projection

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{SL}(2, \mathbb{C}) \longrightarrow PGL(2, \mathbb{C}) \longrightarrow 0\,.$$

Furthermore, the space of all such lifts $\overline{c}$ of c is in bijective correspondence with the space of all theta characteristics on X [8].

Since we have fixed a theta characteristic $\mathcal{L}$, any element the space $\mathcal{P}(X)$ of all projective structures on X gives an element in $H^1(X,\, \mathrm{SL}(2,\mathbb{C}))$.

We will describe the relationship between $\mathcal{L}$ and the lifts of cohomology classes to $H^1(X,\, \mathrm{SL}(2,\mathbb{C}))$.

Let V be a complex vector space of dimension two. Let $\mathbb{P}(V)$ denote the projective line consisting of all one dimensional subspaces of V. The tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ over $\mathbb{P}(V)$ will be denoted by L. For any complex vector space W, the trivial vector bundle over $\mathbb{P}(V)$ with W as fiber will be denoted by $\underline{W}$. By $\mathrm{SL}(V)$ we will mean the group of all automorphisms of V that act trivially on $\bigwedge^2 V$.

Fix once and for all a symplectic structure on V, that is a nonzero element $\omega \in \bigwedge^2 V^*$. Note that using ω, a canonical identification

$$L^{\otimes 2} \cong K_{\mathbb{P}(V)} \tag{2.1}$$

is obtained. To see this consider the universal exact sequence

$$0 \longrightarrow L \stackrel{i}{\longrightarrow} \underline{V} \longrightarrow L^* \otimes \bigwedge^2 \underline{V} \longrightarrow 0 \tag{2.2}$$

over $\mathbb{P}(V)$. The canonical bundle $K_{\mathbb{P}(V)}$ is $\mathrm{Hom}(L^* \bigotimes \bigwedge^2 \underline{V}, L)$. Now, since ω trivializes $\underline{\bigwedge^2 V}$, the line bundle $L^{\otimes 2}$ gets identified with $K_{\mathbb{P}(V)}$.

Consider a datum $\{U_\alpha, \phi_\alpha, G_{\alpha,\beta}\}_{\alpha,\beta\in I}$, where the maps

$$\phi_\alpha : U_\alpha \longrightarrow \mathbb{P}(V) \tag{2.3}$$

are holomorphic coordinate charts on X with $\bigcup_\alpha U_\alpha = X$, and $G_{\alpha,\beta} \in \mathrm{SL}(V)$, satisfying the following three conditions:

(1) $\phi_\alpha \circ \phi_\beta^{-1}$ coincides with the action of $G_{\alpha,\beta}$ on $\mathbb{P}(V)$;
(2) $G_{\beta,\alpha} = G_{\alpha,\beta}^{-1}$;
(3) $G_{\alpha,\beta} G_{\beta,\gamma} G_{\gamma,\alpha} = \mathrm{Id}_V$ for every triple $\alpha, \beta, \gamma \in I$.

two such pairs of datum will be called *equivalent* if their union is again a part of such a datum. The space of such equivalence classes, which we will call $\mathrm{SL}(V)$-*structures* on X, is clearly the space of all possible lifts $\overline{c} \in H^1(X,\, \mathrm{SL}(2,\mathbb{C}))$, where c runs through $\mathcal{P}(X)$.

The action of $\mathrm{SL}(V)$ on $\mathbb{P}(V)$ lifts to the exact sequence (2.2). This gives an exact sequence

$$0 \longrightarrow \mathcal{L}' \stackrel{\iota}{\longrightarrow} \mathcal{V} \longrightarrow (\mathcal{L}')^* \otimes \bigwedge^2 \mathcal{V} \longrightarrow 0 \tag{2.4}$$

over X. Given any datum $P := \{U_\alpha, \phi_\alpha, G_{\alpha,\beta}\}_{\alpha,\beta\in I}$ as before, consider the pull-back to U_α, using the map ϕ_α in (2.3), of the exact sequence (2.2). For another such coordinate function (U_β, ϕ_β), the two exact sequences on $U_\alpha \bigcap U_\beta$ are identified using the action of $G_{\alpha,\beta} \in \mathrm{SL}(V)$ on (2.2). The cocycle condition on $G_{\alpha,\beta}$ ensures that these individual exact sequences on U_α, $\alpha \in I$, patch together to give an exact sequence as in (2.4). The above homomorphism ι is obtained from i in (2.2).

The isomorphism (2.1) is equivariant for the action of $\mathrm{SL}(V)$ on L and $K_{\mathbb{P}(V)}$ respectively. Therefore, it induces a canonical identification between K_X and $(\mathcal{L}')^{\otimes 2}$. The earlier mentioned map from $\mathrm{SL}(V)$-structures on X to theta characteristics is defined by the above construction of the line bundle $\mathcal{L}'$ from an $\mathrm{SL}(V)$-structure P.

As mentioned earlier, $\mathcal{P}(X)$ is identified with the space of all $\mathrm{SL}(V)$-structures on X such that the associated theta characteristic is the chosen one, namely $\mathcal{L}$.

The natural flat connection on the trivial vector bundle $\underline{V}$ in (2.2) is invariant under the action of $\mathrm{SL}(V)$ on $\underline{V}$. Therefore, given a projective structure P on X, the vector bundle $\mathcal{V}$ constructed from P gets a flat connection. This connection on $\mathcal{V}$ will be denoted by ∇^P.

For any datum $\{U_\alpha, \phi_\alpha, G_{\alpha,\beta}\}_{\alpha,\beta\in I}$ defining P, consider the trivial vector bundle on each U_α with fiber V. These vector bundles patch together over any $U_\alpha \bigcup U_\beta$ using the action of $G_{\alpha,\beta}$ on V as the clutching function. The resulting flat vector bundle over X is clearly $(\mathcal{V}, \nabla^P)$.

The symplectic structure on V defined by ω gives a symplectic structure on the fibers of $\mathcal{V}$. Consequently, we have

$$\underline{\omega} \in H^0(\mathcal{V}, \wedge^2\pi^*\mathcal{V}^*), \tag{2.5}$$

where π denotes the projection of $\mathcal{V}$ to X.

Using the connection ∇^P, the above defined $\underline{\omega}$ gives a 2-form on the total space of $\mathcal{V}$. Indeed, ∇^P defines a projection

$$\Gamma_P : T\mathcal{V} \longrightarrow \pi^*\mathcal{V}$$

with the horizontal distribution for ∇^P as the kernel. Now,

$$\Omega_P := \Gamma_P^*\underline{\omega}$$

is a 2-form on $\mathcal{V}$, where $\underline{\omega}$ is defined in (2.5). From the definition of $\underline{\omega}$ and the condition that ∇^P is flat it follows that Ω_P is closed.

Recall that S denotes the surface consisting of nonzero vectors in $\mathcal{L}$. Let τ denote the inclusion of S in $\mathcal{V}$ defined by the homomorphism ι in (2.4).

Lemma 2.6. *The 2-form $\tau^*\Omega_P$ on S coincides with the symplectic form Ω constructed in (1.2).*

Proof. We will first prove this for $X = \mathbb{P}(V)$. In this special case $\mathcal{V}$ is the trivial vector bundle with V as the fiber. Let $f : \mathcal{V} = \mathbb{P}(V) \times V \longrightarrow V$ be the projection. Clearly, $\underline{\omega}$ in (2.5) is $f^*\omega$.

Furthermore, in this special case $\mathcal{L}$ is the tautological line bundle $L := \mathcal{O}_{\mathbb{P}(V)}(-1)$. The homomorphism ι in (2.4) coincides with i in (2.2). Clearly, S is $V' := V \setminus \{0\}$, the space of nonzero vectors in V. For the homomorphism i in (2.2) it is clear that $\tau^*\Omega_P$ is the restriction of the symplectic form ω to V'.

We saw that the isomorphism of $L^{\otimes 2}$ with $K_{\mathbb{P}(V)}$ depends on the choice of ω. To show that the form Ω defined in (1.2) coincides with ω, first recall that the natural symplectic form Ω_0 on the total space K_X is exact. In fact,

$$\Omega_0 \,=\, d\theta\,,$$

where θ is the tautological 1-form.

Let $\mathbf{e}$ denote the Euler vector field on V. In other words, $\mathbf{e}(v) \,=\, v$ for all $v \in V$. For $\mathbb{P}(V)$, it is easy to see that the 1-form $\psi^*\theta$ on V' coincides with $i_{\mathbf{e}}\omega$, the contraction of ω by $\mathbf{e}$.

Using the identity $L_\nu \,=\, d\circ i_\nu + i_\nu \circ d$ for Lie derivative of forms the following

$$\psi^*\Omega \,=\, d\psi^*\theta \,=\, d\circ i_{\mathbf{e}}\omega \,=\, L_{\mathbf{e}}\omega$$

is obtained.

For a constant vector field $\mathbf{t}$ on V', we have $L_{\mathbf{e}}\mathbf{t} \,=\, [\mathbf{e}\,,\mathbf{t}] \,=\, -\mathbf{t}$. Now given another constant vector field $\mathbf{s}$ on V', from the identity

$$(L_{\mathbf{e}}\omega)(\mathbf{s},\mathbf{t}) \,=\, L_{\mathbf{e}}(\omega(\mathbf{s},\mathbf{t})) \,-\, \omega(L_{\mathbf{e}}\mathbf{s},\mathbf{t}) \,-\, \omega(\mathbf{s},L_{\mathbf{e}}\mathbf{t})$$

of Lie derivatives the equality $(L_{\mathbf{e}}\omega)(\mathbf{s},\mathbf{t}) \,=\, 2\omega(\mathbf{s},\mathbf{t})$ is obtained. Note that $\omega(\mathbf{s},\mathbf{t})$ is a constant function, and hence $L_{\mathbf{e}}(\omega(\mathbf{s},\mathbf{t})) = 0$.

Since constant vector fields span TV', the lemma is proved for $\mathbb{P}(V)$.

Now, given a projective structure P on X, let (U_α,ϕ_α) be a coordinate function as in (2.3) for P. Consider the identification of the open subset $U_\alpha \,\subset\, X$ with the open subset $\phi_\alpha(U_\alpha) \,\subset\, \mathbb{P}(V)$ given by the map ϕ_α. From the earlier construction of a theta characteristic using P it is immediate that this identification lifts to an identification of $\mathcal{L}\big|_{U_\alpha}$ with $L\big|_{\phi_\alpha(U_\alpha)}$. In this identification of total spaces of line bundles, the restrictions of the two 2-forms $\tau^*\Omega_P$ and Ω to $\mathcal{L}\big|_{U_\alpha}\setminus\{0\}$ coincide with the restrictions, to the subset $L\big|_{\phi_\alpha(U_\alpha)}\setminus\{0\}$, of the corresponding forms on $L\setminus\{0\}$. Here $\{0\}$ denotes the image of zero section. We already proved that $\tau^*\Omega_P = \Omega$ on $L\setminus\{0\}$. Therefore, this equality remains valid on $\mathcal{L}\big|_{U_\alpha}\setminus\{0\}$. This completes the proof of the lemma. $\square$

In the next section we will describe a quantization of the symplectic form Ω on S.

3. A quantization of symplectic structure

The vector space V equipped with the symplectic form ω has a natural quantization, known as the Moyal–Weyl algebra, which we will describe briefly. The details can be found in [7], [9].

We use the notation of Section 1. To describe a quantization it is enough the define the multiplication $f\star g \,\in\, \mathcal{A}(M_0)$, where $f,g \,\in\, C(M_0) \,\subset\, \mathcal{A}(M_0)$, satisfying the conditions. Indeed, using the bilinearity condition, the multiplication operation extends to $\mathcal{A}(M_0)$.

As in Section 2, consider the symplectic vector space V equipped with the symplectic form ω. Let $\Delta : V \longrightarrow V \times V$ be the diagonal embedding defined by $v \longmapsto (v, v)$. For $f, g \in C(V)$, the function on $V \times V$ defined by $(v, w) \longmapsto f(v)g(w)$ will be denoted by $f \otimes g$. There is a unique differential operator

$$D : C(V \times V) \longrightarrow C(V \times V)$$

with constant coefficients such that

$$\{f , g\} = \Delta^* D(f \otimes g)$$

for every $f, g \in C(V)$. The *Moyal–Weyl algebra* is defined by

$$f \star g = \Delta^* \exp(\sqrt{-1}hD/2)(f \otimes g) \in \mathcal{A}(V) \tag{3.1}$$

for all $f, g \in C(V)$. The multiplication is extended to $\mathcal{A}(V)$ using bilinearity.

From the definition of Moyal–Weyl algebra it is immediate that the algebra structure is invariant under the action of $\mathrm{SL}(V)$ on V. In other words, for every $T \in \mathrm{SL}(V)$ and every pair $f, g \in C(V)$, the identity

$$(f \circ T) \star (g \circ T) = (f \star g) \circ T \tag{3.2}$$

is valid.

We note that since the algebra structure in (3.1) is defined using differential operators, it is local. In other words, for functions f and g defined over an analytic open subset $U \subset V$, the product $f \star g$ is defined as an element of $\mathcal{A}(U)$.

Given a projective structure on X, using the Moyal–Weyl algebra a quantization of the holomorphic symplectic structure Ω on the algebraic surface S will be constructed.

Take a projective structure P on X. Since $\tau^*\Omega_P = \Omega$ (Lemma 2.6), it is equivalent to quantize $\tau^*\Omega_P$.

Let $\eta : S \longrightarrow X$ be the natural projection which also coincides with the restriction to S of π in (2.5). Take a point $y \in S$. The image $\eta(y)$ will be denoted by x.

Let $U \subset S$ be a simply connected analytic open subset containing y such that $\eta(U)$ is also simply connected. Therefore, the restriction of the connection ∇^P to $\eta(U)$ is trivial. Fix an isomorphism ρ of V with the fiber $\mathcal{V}_x$ at x that takes the symplectic structure ω on V to the symplectic structure $\overline{\omega}_x$ on $\mathcal{V}_x$ defined in (2.5).

Using ∇^P and ρ, the restriction of the vector bundle $\mathcal{V}$ to $\eta(U)$ is identified with the trivial vector bundle over U with fiber V. Let $\overline{p} : \mathcal{V}\big|_{\eta(U)} \longrightarrow V$ be the projection defined by this trivialization. Consider the holomorphic map

$$p : U \longrightarrow V$$

obtained by restricting $\overline{p}$ to $\mathcal{V}\big|_{\eta(U)} \bigcap U$.

The map p takes the symplectic form ω on V to the symplectic form Ω on U. Indeed, from Lemma 2.6 we know that $\tau^*\Omega_P$ is a symplectic form on S and it coincides with Ω. Clearly, $\tau^*\Omega_P = p^*\omega$, and hence p takes ω to Ω. This in particular implies that p

must be an immersion. Consequently, if we choose U to be sufficiently small, then p would be a biholomorphism of U with an open subset of V.

We assume U to be such that p is a biholomorphism of it with $p(U)$. For functions $f, g \in C(U)$ define

$$f \star_1 g = p^*((f \circ p^{-1}) \star (g \circ p^{-1})) \in \mathcal{A}(U)\,, \tag{3.3}$$

where $\star$ is the Moyal–Weyl product on $p(U)$ defined in (3.1). (We noted earlier that the Moyal–Weyl product is actually locally defined.)

We first note that the operation $\star_1$ does not depend on the choice of the isomorphism ρ. Indeed, this is an immediate consequence of the identity (3.2).

If $U' \subset S$ is another such open set, then consider the corresponding operation $\star_1'$ on $C(U')$ defined as in (3.3). If p' is the corresponding map from U' to V, then $p' \circ p^{-1}$ is the restriction of the action of some element in $\mathrm{SL}(V)$. Therefore, from (3.3) it follows that $\star_1'$ coincides with $\star_1$ over $C(U) \bigcap C(U')$. This in turn implies that the locally defined operations $\star_1$ on S patch together compatibly to give a map

$$\star \,:\, C(S) \times C(S) \longrightarrow \mathcal{A}(S)\,. \tag{3.4}$$

Since the map p takes ω to Ω, from the construction of $\star$ in (3.4) it is immediate that $\star$ defines a quantization of the symplectic structure Ω on S.

Therefore, we have proved

Theorem 3.5. *The operation $\star$ on $C(S)$, constructed in (3.4), is associative and it defines a quantization of the symplectic structure Ω on S.*

From the definition $\star$ it is immediate that the involution of S defined by $v \longmapsto -v$ preserves it. Therefore, $\star$ defines a quantization of the quotient space for this involution. The quotient space is K_X', the complement of the zero section in K_X, equipped with the symplectic form $\Omega_0/2$.

This quantization of K_X' can be directly constructed as follows. Consider the map

$$\delta \,:\, \mathbb{C}^2 \setminus \{0\} \longrightarrow K'_{\mathbb{CP}^1} \,:=\, K_{\mathbb{CP}^1} \setminus \mathbb{CP}^1$$

defined by $(z, w) \longmapsto (zdw - wdz)/2$. This map commutes with the involution $(z, w) \longmapsto (-z, -w)$, and hence is a double cover that preserves the standard symplectic form $dz \wedge dw$ of $\mathbb{C}^2$. The descended symplectic form on $K'_{\mathbb{CP}^1}$, which coincides with the natural symplectic form $\Omega_0/2$ on the canonical bundle $K_{\mathbb{CP}^1}$, is left invariant by the action of $\mathrm{PSL}(2, \mathbb{C})$ on it. Therefore, the Moyal–Weyl quantization gives a quantization of the symplectic form $\Omega_0/2$ on $K'_{\mathbb{CP}^1}$. This construction can be found in [2] (page 137). Now using the identity (3.2) satisfied by the Moyal–Weyl algebra, it is easy to see that we have a quantization of the symplectic form $\Omega_0/2$ on K_X' for any Riemann surface X with a projective structure.

The quantization of $\Omega_0/2$ on K_X' does not extend to a quantization over K_X. This can be directly checked for $X = \mathbb{P}(V)$.

The factor $1/2$ in (1.2) is introduced only for notational convenience. Indeed, the automorphism of S defined by $v \longmapsto 2v$ takes Ω to 2Ω. The same is valid for Ω_0.

In the next section we will see how to recover the projective structure from the associated quantization.

4. Quantization determines projective structure

Given a projective structure P on X, there is a corresponding global differential operator

$$D_n^P \in H^0(X, \mathrm{Diff}_X^n(\mathcal{L}^{-n+1}, \mathcal{L}^{n+1})) \tag{4.1}$$

for every $n \geq 1$ with the constant function 1 as its symbol [4, Theorem 4.1]. We will recall a description of D_n^P.

Take a holomorphic coordinate function z on an open subset U of X which is compatible with the projective structure P. Let v be a holomorphic section of $\mathcal{L}$ over U such that $\psi_0(v \otimes v) = dz$, where ψ_0 is the isomorphism between $\mathcal{L}^{\otimes 2}$ and K_X. The operator D_n^P satisfies the identity

$$D_n^P(f v^{\otimes(1-n)}) = \frac{d^n f}{dz^n} v^{\otimes(n+1)},$$

where f is a holomorphic function on U.

The operator D_1^P is the exterior differentiation $d : \mathcal{O}_X \longrightarrow K_X$ and hence it is independent of P. The operator D_2^P is the *Liouville operator* associated to P. The local system obtained from the flat rank two bundle $(\mathcal{V}, \nabla^P)$ defined in Section 2 coincides with the local system defined by the (local) solutions of D_2^P. In other words, P is determined by D_2^P.

Let s be a holomorphic section of $\mathcal{L}^{-n+1}$ over an open subset U of X. For any $x \in U$ and any vector $v \in \eta^{-1}(x)$, where η denotes, as in Section 3, the natural projection of S to X, consider the natural pairing $\langle s(x), v^{n-1}\rangle \in \mathbb{C}$. This defines a holomorphic function on $\eta^{-1}(U)$. This function on $\eta^{-1}(U)$ will be denoted λ_s.

Let $\star$ denote the quantization of the symplectic structure Ω on S constructed from P in Section 3. For any holomorphic function g on $\eta^{-1}(U)$, let $\hat{g}$ denote the coefficient of h^n in the expansion of $\lambda_s \star g$. So, if

$$\lambda_s \star g = \sum_{j=1}^{\infty} h^j g_j,$$

then $\hat{g} = g_n$.

Let D_s denote the differential operator on $\eta^{-1}(U)$ that sends any function g to $k!2^n\hat{g}/\sqrt{-1}^n$. From the local expression of the Moyal-Weyl algebra (cf. [7, (2.2)]) it is immediate that D_s is a differential operator of order n.

Let σ_s denote the symbol of the differential operator D_s. It is easy to check that the function σ_s satisfies the identity

$$\sigma_s(cv) \;=\; c^{n+1}\sigma_s(v)$$

for every $c \in \mathbb{C} - \{0\}$ and $v \in S$. This identity implies that σ_s defines a section of $\mathcal{L}^{n+1}$ over U. Let $\hat{s}$ denote the section of $\mathcal{L}^{n+1}$ defined by σ_s.

From the earlier description of the operator D^P_n in terms of projective coordinates it is straight-forward to compute that the section $\hat{s}$ coincides with $D^P_n(s)$.

Acknowledgements. We wish to thank the referee for helpful comments. The second-named author acknowledges hospitality of the University of Chicago.

References

[1] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D.: Deformation theory and quantizations I, II, *Ann. of Phys.* **111** (1978), 61–151.

[2] Boutet de Monvel, L.: Complex star algebras, *Math. Phys. Anal. Geom.* **2** (1999), 113–139.

[3] Boutet de Monvel, L. and Guillemin, V.: *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies 99, Princeton University Press, Princeton, New Jersey, 1981.

[4] Biswas, I.: A remark on the jet bundles over the projective line, *Math. Res. Lett.* **3** (1996), 459–466.

[5] De Wilde, M. and Lecomte, P.B.A.: Star-products on cotangent bundles, *Lett. Math. Phys.* **7** (1983), 235–241.

[6] De Wilde, M. and Lecomte, P.B.A.: Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifold, *Lett. Math. Phys.* **7** (1983), 487–496.

[7] Fedosov, B.V.: A simple geometrical construction of deformation quantization, *Jour. Diff. Geom* **40** (1994), 213–238.

[8] Gunning, R.C.: *Lectures on Riemann surfaces*, Mathematical Notes 2. Princeton University Press, Princeton, New Jersey, 1966.

[9] Weinstein, A.: Deformation quantization, *Séminaire Bourbaki*, no. 789, 46ème année, 1994.

Department of Mathematics, The University of Chicago, 5734 University Avenue, Chicago, Illinois 60637, USA

E-mail address: benzvi@math.uchicago.edu

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

E-mail address: indranil@math.tifr.res.in