SELECTED SOLUTIONS FROM THE HOMEWORK

ANDREW J. BLUMBERG

1. Solutions

(2.2, 12) Prove that the following homogeneous system has a nontrivial solution if and only if $ad - bc = 0$:

$$ax_1 + bx_2 = 0$$
$$cx_1 + dx_2 = 0.$$

Proof. Suppose that $a = 0$. Then the first equation becomes $bx_2 = 0$, which implies that $x_2 = 0$. The second equation then becomes $cx_1 = 0$, which implies that $x_1 = 0$. Now consider the case in which $a \neq 0$. Dividing by a, the first row of the associated matrix becomes $[1, \frac{b}{a}]$. Next, we can remove the c in the first column by subtracting c times the first row from the second, which leaves $[0, d - \frac{bc}{a}]$. Clearly, $d - \frac{bc}{a} = 0$ is equivalent to $ad - bc = 0$. □

(2.2, 13) Suppose that $Ax = 0$ is a homogeneous system of n equations in n variables.
(a) If the system $A^2x = 0$ has a nontrivial solution, show that $Ax = 0$ also has a nontrivial solution.
(b) Generalize the result of part (a) to show that if the system $A^n x = 0$ has a nontrivial solution for some positive integer n, then $Ax = 0$.

Proof. Assume that $Ax = 0$ has only the trivial solution. For any vector z, if $A^2z = 0$, then $A(Az) = 0$. Thus, $Az = 0$, and so $z = 0$. Now we consider the general case. Assume that the result is true for $n \leq m$. So now we want to show that $A^{m+1}x = 0$ has only the trivial solution if $Ax = 0$ has only the trivial solution. For any vector z, if $A^{m+1}z = A(A^m z) = 0$, we know that $A^m z = 0$, which by the induction hypothesis implies that $z = 0$. □

(2.4, 9) (a) Give an example to show that $A + B$ can be singular if A and B are both nonsingular.
(b) Give an example to show that $A + B$ can be nonsingular if A and B are both singular.
(c) Give an example to show that even when A, B, and $A + B$ are all nonsingular, $(A + B)^{-1}$ is not necessarily equal to $A^{-1} + B^{-1}$.

Proof. For the first one, consider the matrices

$$A = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad \text{and} \qquad B = \begin{matrix} 0 & 1 \\ 1 & 0. \end{matrix}$$

For the second, take A and $-A$ for any singular matrix A. For the last, consider $A = B = I$. □

Date: March 9, 2015.

(2.4, 13) Let A be a symmetric nonsingular matrix. Prove that A^{-1} is symmetric.

Proof. We know that $A^T = A$ and A^{-1} exists. Applying the transpose to the equation $AA^{-1} = I$, we find that $(A^{-1})^T A^T = I$. Since $A^T = A$, we have $(A^{-1})^T A = I$, and now multiplying by A^{-1} on the right, we find that $(A^{-1})^T = A^{-1}$. □

E-mail address: blumberg@math.utexas.edu