SELECTED SOLUTIONS FROM THE HOMEWORK

ANDREW J. BLUMBERG

1. Solutions

1.2, 24 Explain why the span of the set of columns in an $m \times n$ matrix A is identical to the set $\{Ax : x \in \mathbb{R}^n\}$.

Proof. The matrix A has n columns $c_1, c_2, \ldots, c_n$, with each column a vector in $\mathbb{R}^m$. The set $\mathrm{Span}(\{c_1, c_2, \ldots, c_n\} \subset \mathbb{R}^m$ is the set of linear combinations $\{a_1c_1 + a_2c_2 + \ldots + a_nc_n \mid a_i \in \mathbb{R}\}$. On the other hand, observe that for an arbitrary vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then $Ax = x_1c_1 + x_2c_2 + \ldots x_nc_n$, by the definition of matrix-vector multiplication. As a consequence, for any x, we see that $Ax \in \mathrm{Span}(\{c_1, c_2, \ldots, c_n\})$ and so we can conclude that

$$\{Ax : x \in \mathbb{R}^n\} \subseteq \mathrm{Span}(\{c_1, c_2, \ldots, c_n\}).$$

The other direction is essentially the same. □

1.2, 28 If a system of equations $Ax = b$ is inconsistent, can we always restore consistency by changing one entry in vector b.

Proof. Definitely not; each zero row in the reduced form of A paired with a nonzero entry in b yields a violation of consistency. □

1.3, 9 Explain why the system $Ax = 0$ is always consistent.

Proof. The system $Ax = b$ is inconsistent when we obtain a zero row in the reduced row echelon form of A which corresponds to a nonzero entry in the correspondingly reduced form of b. There are never nonzero entries in the reduced form of 0, and so $Ax = 0$ is always consistent. □

1.3, 13 Establish that if the set $\{v_1, v_2, v_3\}$ is linearly independent, then so is $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$.

Proof. We'll do this using the contrapositive. Thus, assume that $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is linearly dependent. By definition, this means we can find $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_1) = 0$$

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and at least one of the a_i is nonzero. Expanding, we have

$$a_1v_1 + a_1v_2 + a_2v_2 + a_2v_3 + a_3v_3 + a_3v_1 = 0,$$

and factoring, we get

$$(a_1 + a_3)v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 = 0.$$

Since we cannot have $a_1+a_3 = a_1+a_2 = a_2+a_3 = 0$ (as that would imply that $a_1 = a_2 = a_3 = 0$), this is a witness that demonstrates that $\{v_1, v_2, v_3\}$ is linearly dependent. □

1.3, 20 Justify, without appealing to the reduced row echelon form, the assertion that if a system of equations $Ax = b$ has two solutions then it has infinitely many solutions.

Proof. If $Ax_1 = b$ and $Ax_2 = b$ for $x_1 \neq x_2$, then for any $t \in [0, 1]$ we have

$$\begin{aligned} A(tx_1 + (1-t)x_2) &= A(tx_1) + A((1-t)x_2) = tAx_1 + A(x_2 - tx_2) \\ &= tAx_1 + Ax_2 - tAx_2 = tb + b - tb = b. \end{aligned}$$

That is, we have produced an infinite number of distinct solutions. □

1.3, 23 Explain why two matrices that are row equivalent to each other must have the same rank.

Proof. If A is row equivalent to B, then A and B must have the same reduced-row echelon form (if we can reduce A to R, we can reduce B to R by first reducing B to A). □

1.3, 24 Explain why a set of n vectors in $\mathbb{R}^n$ is linearly independent if and only if the matrix having these vectors as its columns has rank n.

Proof. Let $\{v_1, v_2, \ldots v_n\}$ be the vectors in question, and denote by A the matrix formed by taking these vectors as the columns. Recall that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if and only if $Ax = 0$ has the unique solution $x = 0$. But $Ax = 0$ has a unique solution if and only if A has rank n. □

1.3, 38 If two matrices A and B are not row equivalent to each other, can they have the same reduced row echelon form?

Proof. No. Assume that A and B have the same reduced row echelon form R. That is, $A \sim R$ and $B \sim R$. By symmetry and transitivity (as row equivalence is an equivalence relation on matrices), we have that $A \sim R \sim B$ and so $A \sim B$. □

1.3, 48 If $\{v_1, v_2 + \lambda v_1\}$ is linearly independent for some nonzero $\lambda \in \mathbb{R}$, does it follow that $\{v_1, v_2\}$ is linearly independent?

Proof. We do this using the contrapositive. Assume that $\{v_1, v_2\}$ is linearly dependent. Then there exist $c_1, c_2 \in \mathbb{R}$ such that at least one of the c_i is nonzero and $c_1v_1 + c_2v_2 = 0$. Now consider the equation

$$a_1v_1 + a_2(v_2 + \lambda v_1) = 0.$$

Rewriting, we find

$$a_1v_1 + a_2(v_2 + \lambda v_1) = a_1v_1 + a_2v_2 + \lambda a_2v_1 = (a_1 + \lambda a_2)v_1 + a_2v_2.$$

Therefore, choosing $a_2 = c_2$ and $(a_1 + \lambda a_2) = c_1$ provides a candidate witness that $\{v_1, v_2 + \lambda v_1\}$ is linearly dependent. Solving, we see that $a_1 + \lambda c_2 = c_1$ implies that $a_1 = c_1 - \lambda c_2$. Finally, we need to check that a_1 and a_2 are not both zero. But if $a_2 = 0$, then $c_2 = 0$, which implies that $a_1 = c_1$, which is nonzero by hypothesis in this case. □

1.3, 63 Suppose that the equation $Ax = b$ has more than one solution. Explain why the equation $Ax = 0$ has infinitely many solutions.

Proof. If $Ax = b$ has more than one solution, then the reduced row echelon form of A must have rank smaller than the number of variables. This implies that $Ax = 0$ has infinitely many solutions. □

1.3, 79 Argue that a pair of vectors is linearly independent if and only if one of the vectors is a multiple of the other.

Proof. First, if $v_1 = kv_2$, then $v_1 - kv_2 = 0$, which provides a witness that $\{v_1, v_2\}$ are linearly dependent. On the other hand, assume that $\{v_1, v_2\}$ are linearly dependent. Then there exist $a_1, a_2 \in \mathbb{R}$ such that $a_1v_1 + a_2v_2 = 0$ and at least one of the a_i is nonzero. Without loss of generality, assume that $a_1 \neq 0$. Then $v_1 = -\frac{a_2}{a_1}v_2$. □

E-mail address: `blumberg@math.utexas.edu`