MATH 408C, SOLUTIONS FOR FIRST MIDTERM EXAM

ANDREW J. BLUMBERG

1. Problems

(1) Short answer questions:
   (a) Give as many different ways as you can think of for a function to fail to be differentiable.

   Proof. The function can fail to be continuous (e.g., a hole or removable discontinuity, a jump discontinuity, or a blow-up) or have a sharp corner (like $|x|$). □

   (b) When does a function have an inverse? Give an example of a function that has an inverse on the domain $[-1, 1]$ and an example of a function that doesn't.

   Proof. A function has an inverse when it is injective (1-1) and surjective (onto); when injective it always has an inverse on its image. We can test to see if a function is injective using the horizontal line test. A function that has an inverse on $[-1, 1]$ is x^3; a function that doesn't is x^2. □

   (c) Explain why the derivative computes the slope of the tangent. (Hint: use the definition of the derivative.)

   Proof. The derivative at a point a is defined as the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

   The expression in the limit is the slope of the line from $(x, f(x))$ to $(a, f(a))$ — the secant line joining these points. As the points get closer together, this slope converges to the slope of the tangent line as the secants approach the tangent. □

(2) Evaluate the following limits.
   (a)

$$\lim_{x \to \infty} \sin x.$$

   Proof. This limit does not exist; $\sin x$ oscillates between 1 and -1. □

   (b)

$$\lim_{x \to 3} x^2 + x - 1.$$

   Proof. The function is continuous, so we can evaluate the limit as $3^2 + 3 - 1 = 9 + 3 - 1 = 11$. □

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(c)

$$\lim_{x\to 2} \frac{x^2-6x+8}{(x-4)e^x}.$$

Proof. Once again, the function is continuous, so we can evaluate the limit to get

$$\frac{4-12+8}{(-2)e^2} = \frac{0}{-2e^2} = 0.$$

□

(d)

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}.$$

Proof. Writing the denominator as a difference of squares, we are computing

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

□

(e)

$$\lim_{x\to -\infty} \frac{4x^5-3x+2}{\sqrt{36x^{10}-x^8+x^3+2}}.$$

Proof. As $x \to -\infty$, the numerator is dominated by $4x^5$ and the denominator is dominated by $\sqrt{36x^{10}} = -6x^5$; the negative sign on the denominator comes from the fact that taking a negative number to the tenth power washes out the sign. Therefore, the limit is $-\frac{2}{3}$. □

(3) Solve the following equation:

$$e^{x \ln 16} = 4^{x+1}.$$

Proof. Since

$$e^{x \ln 16} = e^{\ln 16^x} = 16^x = 4^{2x},$$

the equation becomes

$$4^{x+1} = 4^{2x}.$$

Taking logs, this becomes $x+1 = 2x$ or $x = 1$. □

(4) (a) Where is the function $f(x) = \frac{e^{\sin x - x^3 + x^2 + \frac{4}{x}}}{\ln(x+2)}$ continuous? Use your answer to evaluate the limit at 1.

Proof. Since e^x, $\sin x$, x^3, and x^2 are continuous everywhere, the numerator is defined as long as $x \neq 0$. The denominator is defined and nonzero when $x > -2$ and $x \neq -1$. Therefore, the function is continuous on the domain $(-2,-1) \cup (-1,0) \cup (0,\infty)$. In particular, the function is continuous at 1, so we can evaluate to find that the limit is

$$\frac{e^{\sin 1 - 1 + 1 + 4}}{\ln 3} = \frac{e^{\sin 1 + 4}}{\ln 3}.$$

□

(b) Why isn't the function

$$f(x) = \frac{(x^2 - 4)}{(x + 2)}$$

continuous? Can you produce a continuous function $g(x)$ such that $g(x) = f(x)$ everywhere except the points of discontinuity?

Proof. The function $f(x)$ is not continuous on $\mathbb{R}$ since it is undefined at $x = -2$. On the other hand, this is a removable singularity; factoring the numerator as $(x^2 - 4) = (x + 2)(x - 2)$, we find that $g(x) = x - 2$ agrees with $f(x)$ everywhere except at $x = -2$. □

(5) Draw a graph of the weight of your pet rat over time, labeled with significant life events, as well as a graph of the derivative of the weight. Explain the connection between the two. (Note: there is no one correct answer; the only criterion is that the graphs you draw should make sense and be consistent with basic biological principles.)

Proof. Omitted. □

(6) Find the normal line to the curve $f(x) = x^3 - x + 2$ at the point $(3, f(3))$. Please use the limit definition of the derivative to produce your solution.

Proof. We compute the derivative as follows:

$$\begin{aligned} &\lim_{h \to 0} \frac{(x + h)^3 - (x + h) + 2 - (x^3 - x + 2)}{h} \\ &= \lim_{h \to 0} \frac{x^3 + 3x^2 h + 3xh^2 + h^3 - x - h + 2 - x^3 + x - 2}{h} \\ &= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3 - h}{h} \\ &= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 1 \\ &= 3x^2 - 1. \end{aligned}$$

Evaluating at $x = 3$, we find that the slope of the derivative is $3(3^2) - 1 = 26$. The point in question is $(3, 27 - 3 + 2) = (3, 26)$, so the normal line is given by the equation

$$y - 26 = -\frac{1}{26}(x - 3).$$

□

(7) (a) Use the intermediate value theorem to locate a root for the polynomial

$$p(x) = 4x^3 + 10x^2 + x - 2.$$

(Please find a range $[a, b]$ such that $b - a < 10$.)

Proof. Observe that $p(0) = -2$ and $p(1) = 4 + 10 + 1 - 2 = 13$. Therefore, since p is continuous the intermediate value theorem implies that p has a root on the interval $[0, 1]$. □

(b) Explain why the intermediate value theorem doesn't imply that the function $f(x) = \frac{1}{x}$ has a zero in the interval $[-1, 1]$.

Proof. The intermediate value theorem only applies to functions that are continuous on the interval in question; $f(x)$ is not continuous on $[-1, 1]$ since it is not continuous at $x = 0$. □

(8) Prove formally that the limit

$$\lim_{x\to\infty} 2^{-x}$$

is 0.

Proof. We need to show that $\forall\epsilon > 0, \exists M$ such that $x > M$ implies that $|2^{-x} - 0| < \epsilon$. Solving the latter equation for x, we find that

$$2^{-x} < \epsilon$$

(since $2^{-x} > 0$, we can drop the absolute value) and thus

$$-x < \log_2 \epsilon,$$

or

$$x > -\log_2 \epsilon.$$

Therefore, $\forall\epsilon$, if $x > -\log_2 \epsilon$ then $2^{-x} < \epsilon$ and so the limit is 0. □

(9) (Extra credit) Draw a cartoon involving the concept of continuity.

Proof. Omitted. □

E-mail address: `blumberg@math.utexas.edu`