MATH 408C, REVIEW FOR THE FINAL EXAM

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1. Things you should know

The exam is cumulative, and will cover all parts of the class equally, **with the exception of ϵ-δ proofs**, which are omitted. In the book, sections 1.5, 1.6, all of 2, all of 3 except 3.7 and 3.11, 4.1-4.4, 4.7, all of 5 except 5.4, all of 6, and 7.1-7.4.

The format of the final will be very similar to the format of the midterms, as will the grading — there will be a curve, and I expect even the very good students in the class to miss a problem or two. I will give you a table of trigonometric identities (e.g., the double angle formula) and certain antiderivatives.

More specifically, here's a list of things you should know.

(1) What the graph of the exponential functions look like.
(2) Properties of exponents: $a^{x+y} = a^x a^y$, $(ab)^x = a^x b^x$, and $(a^x)^y = a^{xy}$.
(3) A function $f\colon X \to Y$ has an inverse if it is injective or 1-1 (equivalently, passes the horizontal line test) and surjective (hits all of Y). If it isn't injective, no inverse. If it isn't surjective, it has an inverse when you restrict the range to $f(X)$.
(4) Properties of inverse functions: $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$. (Remember that this notation is a bit abusive with regards to domains and ranges.)
(5) How to solve for the inverse, given an equation.
(6) The definition of the logarithm and what the graph looks like.
(7) Properties of logarithms: $\log_a(xy) = \log_a x + \log_a y$, $\log_a a = 1$, $\log_a 1 = 0$, $\log_a x^y = y \log_a x$, and $\log_a x = \frac{\log_b x}{\log_b a}$.
(8) The number e and the natural log (that is, ln).
(9) The definition of the inverse trigonometric functions.
(10) How to solve equations involving logs and exponentials (i.e., take exponentials or logs respectively and use the properties to simplify).
(11) The ϵ-δ definition of a limit.
(12) Limit laws for evaluating limits.
(13) Left and right limits.
(14) If $f \leq g$ near a and both limits exist, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

If $f \leq g \leq h$ near a, and all limits exist, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} g(x) = L$. (This was referred to as the "squeeze theorem" in class.)

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(15) Infinite limits and limits as $x \to \infty$ or $x \to -\infty$ (e.g., evaluate limits of rational functions by looking at the coefficients of the highest powers. Remember to be careful about signs!)
(16) The definition of continuity: a function f is continuous at a is $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$.
(17) Rules for combining continuous functions to make new ones: If f and g are continuous at a, then so are $f+g$, $f-g$, cf, fg, f/g (provided $g(a) \neq 0$). If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.
(18) Ways a function can fail to be continuous: e.g., removable discontinuity, jump discontinuity, blowup.
(19) The use of continuity to evaluate limits.
(20) The intermediate value theorem and its use in finding roots.
(21) The definition of the derivative: the derivative of f at a is the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

(22) How to compute derivatives from the definition (i.e., by evaluating limits).
(23) Ways a function can fail to be differentiable: e.g., point discontinuities, corners, vertical tangents. (Remember, differentiable implies continuous but not the other way around.)
(24) Interpretation of the derivative in terms of tangent lines (e.g., find the tangent at a point, find the normal at a point).
(25) Using the derivative to understand the local behavior of a function (e.g., determine where the function is increasing).
(26) Derivatives and rates of change (e.g., average speed computations).
(27) How to take derivatives of polynomials, rational functions, trigonometric functions, exponential and logarithmic functions, and combinations thereof. In particular, this means you must be comfortable with the product rule, quotient rule, and chain rule. Don't forget to use the chain rule:

$$\frac{d}{dx} f(g(x)) = \left(\frac{d}{dx} f\right)(g(x)) \frac{d}{dx} g(x)$$

or

$$(f \circ g)' = (f' \circ g)(g').$$

(28) How to use logarithmic differentiation to simplify the computation of derivatives (i.e., take logs of both sides, differentiate, solve). Remember that the key fact here is that

$$\frac{d}{dx} \ln f(x) = \frac{\frac{d}{dx} f(x)}{f(x)},$$

by the chain rule.
(29) How to use implicit differentiation to obtain the derivatives of functions defined by equations.
(30) How to compute the linear approximation to a function at a point and how to use this to approximate the change in a $f(x)$ when x changes a little. Remember that the linear approximation to f at $(a, f(a))$ is computed as

$$\tilde{f}(x) = f(a) + f'(a)(x-a).$$

You should also know how to decide if the linear approximation is a sensible thing to be doing.

(31) How to solve related rates problems. (Note that this is similar to implicit differentiation but slightly different; one is often differentiating through with respect to t rather than x.)
(32) The formulas for compound interest and how to compute doubling times.
(33) Absolute and local maxima and minima, and their relationship to the derivative.
(34) The extreme value theorem: if f is continuous on $[a, b]$, f takes on a global maximum and a global minimum.
(35) How to use critical points to find the global maximum or global minimum of a continuous function. When on an interval $[a, b]$,
    (a) Compute the value of $f(a)$ and $f(b)$.
    (b) Compute the values of f on the critical points.
    (c) The largest computed value of f is the global maximum, and the smallest computed value of f is the global minimum.

    When on all of $\mathbb{R}$, you aren't guaranteed existence, and you have to look at the limits of f as $x \to \infty$ and $x \to -\infty$.
(36) The first and second derivative tests for determining if critical points are local maxima or minima.
(37) The relationship between the second derivative and concavity.
(38) How to sketch a curve using derivative information.
(39) The statement of the mean value theorem and how to use it to infer the presence of roots or catch speeding teenagers.
(40) The fact that the mean value theorem implies that if $f' = g'$, the $f = g + C$.
(41) L'Hopital's rule, when it does and doesn't apply, and the various tricks for using it (e.g., to handle a limit involving a product $f(x)g(x)$ one rewrites as a quotient $\frac{f(x)}{\frac{1}{g(x)}}$, to handle a limit involving an exponential expression $f(x)^{g(x)}$ one takes the (natural) log and computes the limit of the log).
(42) How to solve constrained optimization problems. Recall that the trick here is always to manipulate the constraint to rewrite the expression you are optimizing as a function of one variable; then use the standard methods (i.e., take the derivative!) to find maxima and minima.
(43) The definition of area as the limits of the "outer" and "inner" approximation of areas with boxes (as the boxes get very small).
(44) The definition of the definite integral in terms of limits of Riemann sums.
(45) How to compute simple Riemann sums.
(46) Basic properties of the definite integral (e.g., if $f(x) \leq g(x)$, then $\int_a^b f \leq \int_a^b g$, and if $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\,\mathrm{dx} \leq M(b-a)$).
(47) The two forms of the fundamental theorem of calculus, and how they are related.
(48) How to compute the derivative of $g(x) = \int_a^{h(x)} f(x)\,\mathrm{dx}$.
(49) How to compute indefinite integrals, both directly and using the substitution rule.
(50) How to use indefinite integrals (antiderivatives) to compute definite integrals via the fundamental theorem of calculus.

(51) How to compute the area between curves via integration. (First, find the points of intersection of the curves to get the limits of integration, then integrate the difference of the bounding functions.)
(52) Compute volumes of revolution by integrating cross sections and cylinders. (Stack of disks and layers of cylindrical shells.)
(53) Compute indefinite and definite integrals using integration by parts. You should be able to handle situations where repeated integration by parts is needed (e.g., $\int x^2 e^x \,\mathrm{dx}$ and $\int \sin x e^x \,\mathrm{dx}$).
(54) Trigonometric integrals, using the trig identities ($\sin^2 x + \cos^2 x = 1$, the double-angle formulas, the angle addition formulas). Specifically, integrals of the form $\int \sin^k x \cos^\ell x \,\mathrm{dx}$ and $\int \sin(kx)\cos(mx)\,\mathrm{dx}$.
(55) Trigonometric substitutions; adroit use of the identities $\sin^2 + \cos^2 = 1$ and $\tan^2 + 1 = \sec^2$ as substitutions to handle expressions involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$. Know how to draw the associated triangle to handle evaluating the substitution (e.g., if $\sin\theta = \frac{x}{a}$, then the triangle has sides a, x, and $a^2 - x^2$). Know how to complete the square when necessary.
(56) Partial fractions — turn hard rational expressions into easy ones. Know how to use polynomial long division when the degree of the numerator is equal to or great than the degree of the denominator. Know the techniques to handle repeated linear factors and irreducible quadratics (e.g., $x^2 + 1$) in the denominator.

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