SOLUTIONS FOR THE PRACTICE EXAM FOR THE SECOND MIDTERM EXAM

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1. Notes

This practice exam has more problems than the real exam will. To make most effective use of this document, take the exam under conditions simulating the real exam — no book, no calculator.

(1) Short answer questions:
  (a) Explain when using the linear approximation is a good way to estimate change in a function.

  Proof. The linear approximation to f at p is a good way to describe the behavior of f near p; how close one has to be depends on f. □

  (b) Explain the relationship of the compound interest formula to the exponential function.

  Proof. The exponential function arises as the limit of the compound interest formula as the compounding period goes to 0. □

  (c) If we differentiate the equation $x^2 + y^2 = 1$ with respect to x, we get $2x+2yy' = 0$. If we differentiate with respect to t, we get $2xx'+2yy' = 0$. Why are these answers different?

  Proof. In the first case, we're treating y as an (implicitly defined) function of x — here we regard x^2 as an explicit function of x. In the second case, we're regarding both x and y as functions of t. The issue is in part that the prime notation is a bit ambiguous here. □

  (d) Explain in words why the derivative is zero at a local maximum.

  Proof. The point is that if you approximate the derivative from the left, all the secants have non-negative slope, and if you approximate from the right, all the secants have non-positive slope. This means the derivative is both $\geq$ and $\leq$ zero, and so must be zero. □

(2) Show how to derive the quotient rule from the product rule. (Hint: use the fact that $\frac{f(x)}{g(x)} = f(x)(g(x))^{-1}$.)

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Proof. Using the hint, we have

$$(f(x)(g(x))^{-1})' = f'(x)(g(x))^{-1} + f(x)(-(g(x))^{-2})g'(x)$$
$$= f'(x)g(x)(g(x))^{-2} - f(x)g'(x)(g(x))^{-2} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}.$$

□

(3) Compute the following derivatives:

(a) $f(x) = x^{x^2+3+4}$.

Proof. Using logarithmic differentiation, we differentiate

$$(x^2+7)\ln x$$

to get

$$(x^2+7)x^{-1} + (2x)\ln x,$$

which implies that

$$f' = ((x^2+7)x^{-1} + (2x)\ln x)(x^{x^2+7}).$$

□

(b) $f(x) = 19x^{56} + x^4 + \frac{x^2}{x^3+17}$.

Proof. The derivative is given by

$$f'(x) = (19)(56)x^{55} + 4x^3 + \frac{(x^3+17)(2x) + (x^3+17)(2x)}{(x^3+17)^2}.$$

□

(c) $f(x) = x^2 \sin e^{x^3+\ln(\cos x^2)}$.

Proof. Using the product rule and then many applications of the chain rule, we find that this is

$$(2x)(\sin e^{x^3+\ln(\cos x^2)}) + \cos e^{x^3+\ln(\cos x^2)}(e^{x^3+\ln(\cos x^2)})(3x^2 + \frac{1}{\cos x^2}(-\sin x^2)(2x)).$$

□

(d) $f(x) = \frac{\sqrt{x^3+x+1}}{(x^4+x^8-12x^2+3)^4}$.

Proof. Using logarithmic differentiation, we differentiate

$$\ln \frac{\sqrt{x^3+x+1}}{(x^4+x^8-12x^2+3)^4} = \frac{1}{2}\ln(x^3+x+1) - 4\ln(x^4+x^8-12x^2+3),$$

obtaining

$$g(x) = \frac{(3x^2+1)}{2(x^3+x+1)} - \frac{4(4x^3+8x^7-24x)}{x^4+x^8-12x^2+3}.$$

We conclude that $f'(x) = f(x)g(x)$. (Expansion of this formula is omitted.) □

(e) $f(x) = 10^{x^2+2}$.

Proof. Using the chain rule, we find that

$$f'(x) = 10^{x^2+2}(\ln 10)(2x).$$

□

(4) If \$1000 is invested at 4% interest, find how long it takes to double if we compound annually and continuously. Suppose we start with \$2000 — how long is the doubling time then?

Proof. The formula for compound interest with interest rate r and period n is given by

$$P(t) = P_0(1 + \frac{r}{n})^{nt}.$$

The formula for compound interest with continuous compounding and interest rate r is given by

$$P(t) = P_0 e^{rt}.$$

Therefore, we're asking to solve the equations

$$2000 = 1000(1 + \frac{0.04}{1})^t$$

and

$$2000 = 1000e^{0.04t}.$$

For the first, we have

$$\ln 2 = t \ln(1 + \frac{0.04}{1})$$

or

$$t = \frac{\ln 2}{\ln(1 + \frac{0.04}{1})}.$$

For the second, we have

$$\ln 2 = (0.04)t,$$

or

$$t = \frac{\ln 2}{0.04}.$$

Notice that nothing here depended on the amount of money we started with. □

(5) Find the normal line to the curve $y^2 = 3x^3 + x + 1$ at the point(s) where $x = 0$.

Proof. Regarding y as an implicitly defined function of x, we differentiate and find

$$2yy' = 9x^2 + 1,$$

or

$$y' = \frac{9x^2 + 1}{2y}.$$

Solving for $x = 0$, we plug in and find

$$y^2 = 3(0)^3 + 0 + 1 = 1,$$

or $y = \pm 1$. Evaluating the derivative, we find that the slope of the tangent lines are $\frac{1}{2}$ and $\frac{-1}{2}$. Therefore the slope of the normal lines are -2 and 2. Thus, the normal lines are specified by the equations

$$(y-1) = -2x$$

and

$$(y+1) = 2x.$$

□

(6) Three people are running away from a sleeping bear. If Irving is running east at 10 mph, Maria is running west at 15 mph, and Olga is running north at 25 mph, find the rate of change of the sum of the distances between Irving and Olga and Irving and Maria after 15 minutes.

Proof. Let's denote the distance from Irving to the bear by x, Maria to the bear by y, and Olga to the bear by z. The problem is asking for the rate of change of $l_1 + l_2$, where $l_1 = x + y$ and $l_2 = \sqrt{x^2 + z^2}$. Differentiating individually, we find

$$\frac{dl_1}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

and

$$\frac{dl_2}{dt} = \frac{1}{2}(x^2 + z^2)^{-\frac{1}{2}}(2x\frac{dx}{dt} + 2z\frac{dz}{dt}).$$

We know that $\frac{dx}{dt} = 10$, $\frac{dy}{dt} = 15$, and $\frac{dz}{dt} = 25$. After 15 minutes, we also know that $x = \frac{10}{4} = \frac{5}{2}$ and $z = \frac{25}{4}$. Plugging in, we find that

$$\frac{dl_1}{dt} = 10 + 15 = 25$$

and

$$\frac{dl_2}{dt} = \frac{1}{2}(\frac{25}{4} + \frac{625}{16})^{-\frac{1}{2}}(5(10) + \frac{25}{2}(25)).$$

□

(7) The sides of a cube are expanding at a rate of 2 centimeters/ sec. Compute the rate of change of the volume and the surface area when the side length is 20 cm.

Proof. The volume of a cube is given by the expression $V = s^3$ and $S = 6s^2$. Differentiating, we find

$$V' = 3s^2 s'$$

and

$$S' = 12ss'.$$

Since we're given that $s' = 2$, we find that

$$V' = 3(20)^2(2)$$

and

$$S' = 12(20)(2).$$

□

(8) Use linear approximation to estimate $\log_2 2.1$.

Proof. The derivative of $\log_2 x$ is $\frac{1}{x \ln 2}$. Therefore, the linear approximation to $\log_2 x$ at 2 is given by

$$\ell(x) = (\log_2 2) + \frac{1}{2\ln 2}(x-2).$$

Evaluating at 2.1, we find

$$\ell(2.1) = (\log_2 2) + \frac{1}{2\ln 2}(.1) = 1 + \frac{1}{20 \ln 2}.$$

If we compute this, we find $\ell(2.1) = 1.072$, whereas $\log_2 2.1 = 1.07$. □

(9) Consider the linear approximation to $f(x) = x^2$ at $x = 0$. When estimating the change, when is the error in the linear approximation more than 50% of the actual value?

Proof. The linear approximation to $f(x) = x^2$ at $x = 0$ is given by

$$\ell(x) = (0)^2 + 2(0)(x-0) = 0.$$

The error is given by the expression $|f(x) - \ell(x)| = x^2$. The error is always more than 50% of the value — it's the whole value (yes, this was sort of a stupid thing for me to ask.) Suppose instead that we had considered the point $x = 1$. Then the linear approximation is

$$\ell(x) = 1^2 + 2(1)(x-1) = 1 + 2(x-1) = 1 + 2x - 2 = 2x - 1.$$

The error is then given by

$$|x^2 - (2x-1)|,$$

and we can reasonably ask when

$$|x^2 - (2x-1)| \geq \frac{x^2}{2}.$$

(The solutions can be found by solving the associated quadratic.) □

(10) Find the absolute minimum and absolute maximum of the following functions on the given interval:

(a) $f(x) = xe^{\frac{-x^2}{8}}$ on [-1,4].

Proof. Differentiating, we find that

$$f'(x) = e^{\frac{-x^2}{8}} + xe^{\frac{-x^2}{8}}(-\frac{2x}{8})$$

or

$$f'(x) = e^{\frac{-x^2}{8}}(1 - \frac{x^2}{4}).$$

This is always defined, and solving for when it is zero amounts to solving

$$(1 - \frac{x^2}{4}) = 0$$

(since $e^{\frac{-x^2}{8}}$ is always positive). Thus, we have

$$4 = x^2,$$

or $x = \pm 2$. Since we're on the interval $[-1, 4]$, the only one of these that is relevant is $x = 2$. We now evaluate $f(x)$ at $x = -1$, $x = 2$, and $x = 4$. We find that $f(-1) = -e^{-\frac{1}{8}}$, $f(2) = 2e^{-\frac{1}{2}}$, and $f(4) = 4e^{-2}$.

Easy estimation tells us that $f(2) > f(4)$, and so we have that the global min is achieved at $f(-1)$ and the global max is achieved at $f(2)$.

□

(b) $f(x) = \frac{x}{x^2-x+1}$ on [0,3].

Proof. Differentiating, we find that

$$f'(x) = \frac{(x^2-x+1) - x(2x-1)}{(x^2-x+1)^2} = \frac{-x^2+1}{(x^2-x+1)^2}.$$

This is undefined when $(x^2-x+1) = 0$. However, this equation has no real roots, so the derivative is always defined. Solving for $f'(x) = 0$, we end up looking at

$$-x^2 = -1$$

or $x^2 = \pm 1$. Only the positive value is in the interval, so we end up evaluating f at $x = 0$, $x = 1$, and $x = 3$. When we do so, we find that $f(0) = 0$,

$$f(1) = \frac{1}{1} = 1$$

and

$$f(3) = \frac{3}{9-3+1} = \frac{3}{7}.$$

The global min is achieved at 0 and the global max at 1. □

(c) $f(x) = x^{\frac{1}{3}}(8-x)$ on [0,8].

Proof. Differentiating, we find that

$$f'(x) = x^{\frac{1}{3}}(-1) + (8-x)\frac{1}{3}x^{-\frac{2}{3}}.$$

Solving $f'(x) = 0$, multiplying through by $x^{\frac{2}{3}}$, we get

$$0 = -x + (8-x)$$

or

$$x = 4.$$

We now evaluate f at $x = 0$, $x = 4$, and $x = 8$. We find that $f(0) = f(8) = 0$ and $f(4) = 4^{\frac{1}{3}}(4)$. Therefore the global min is 0 (and achieved at 0 and 8) and the global max is achieved at $f(4)$. □

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