SOLUTIONS TO THE PRACTICE EXAM FOR FIRST MIDTERM EXAM

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1. Solutions

(1) Short answer questions:
  (a) Do all functions that have an inverse have a derivative?

  Proof. No; consider any function which is 1-1 but not continuous. □

  (b) Give an example of a function which is continuous but not differentiable.

  Proof. Consider $f(x) = |x|$. □

  (c) Consider the function $f(x)$ defined to be 0 if $x \leq 0$ and 1 otherwise. Although $f(-1) = 0$ and $f(2) = 1$, f never takes on the value 0.5. Why doesn't this contradict the intermediate value theorem?

  Proof. The intermediate value theorem requires the function to be continuous on the interval in question; f is not continuous on $[-1, 2]$. □

  (d) Explain how to use continuity to evaluate limits.

  Proof. A function is continuous at p when the limit $\lim_{x\to p} f(x)$ exists and is equal to $f(p)$. For a continuous function, one can thus evaluate the limit $\lim_{x\to p} f(x)$ by simply calculating $f(p)$. □

  (e) Explain the horizontal line test.

  Proof. If you can draw a horizontal line through the graph of a function $f(x)$ which intersects the graph more than once, the function is not 1-1. □

(2) Solve the following equations for x.
  (a)
$$5^{3x} = 25^{x^2-x+2}.$$

  Proof. Rewriting, we have
$$5^{3x} = 5^{2x^2-2x+4}.$$
  Equating exponents, we find that solutions satisfy
$$3x = 2x^2 - 2x + 4,$$

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or

$$2x^2 - 5x + 4.$$

However, the discriminant of this function is

$$(-5)^2 - 4(2)(4) = 25 - 32 = -7,$$

so this quadratic has no real solutions. □

(b)

$$2^{2x-2} = 7^{x^2}$$

Proof. Taking logarithms of both sides, we find that

$$\log_2 2^{2x-2} = \log_2 7^{x^2},$$

which simplifies to

$$2x - 2 = (\log_2 7)x^2,$$

or

$$(\log_2 7)x^2 - 2x + 2 = 0.$$

Using the quadratic formula, we find that the solutions to this can be written

$$\frac{2 \pm \sqrt{4 - (4)(2)(\log_2 7)}}{2\log_2 7}.$$

(And since $\log_2 7$ is between 2 and 3, closer to 3, we see that again the discriminant is negative!) □

(c)

$$e^{2x} - 4e^x + 3 = 0.$$

Proof. Observe that we can write this function as

$$(e^x)^2 - 4e^x + 3 = 0.$$

Factoring, we obtain

$$(e^x - 3)(e^x - 1) = 0.$$

Therefore, the solutions are x such that $e^x = 3$ or $e^x = 1$. Solving, we find that $x = \ln 3$ or $x = 0$. □

(3) Find the inverse of x^4 on the domain $[0, 64]$, on the domain $[-64, 64]$, and on the domain $[-64, 0]$.

Proof. On the domain $[0, 64]$, the inverse of x^4 is $x^{\frac{1}{4}}$. On the domain $[-64, 0]$, the inverse of x^4 is $-x^{\frac{1}{4}}$. On the domain $[-64, 64]$, the function x^4 is not 1-1 (for instance, $f(-10) = f(10) = 10000$), and so has no inverse. □

(4) Find the inverses of the following functions algebraically:
(a) $f(x) = 5 - e^{-x}$.

Proof. We solve $y = 5 - e^{-x}$ for x. Thus, we have

$$e^{-x} = 5 - y,$$

or

$$-x = \ln(5 - y),$$

or finally

$$x = \ln(5 - y).$$

The inverse has domain $(-\infty, 5)$ and range $(-\infty, \infty)$. □

(b) $f(x) = \frac{1+x}{1-x}$.

Proof. Again, we solve $y = \frac{1+x}{1-x}$ for y. Multiplying, we find

$$y(1 - x) = (1 + x),$$

or

$$y - yx = 1 + x.$$

Grouping the x terms, we get

$$y - 1 = x + yx,$$

or

$$y - 1 = x(y + 1),$$

or finally

$$x = \frac{y - 1}{y + 1}.$$

This is defined everywhere except when $y = -1$, and the range does not contain the point 1. □

(Please explain any restrictions on the domain or range.)

(5) Sketch the graph of the inverse of the function $y = \ln(x - 3)$.

Proof. Omitted. □

(6) The position of a particle at time t is given by $f(t) = t^3 - t + 1$.

(a) Find the average velocity over the interval $[0, 2]$.

Proof. The average velocity on this interval is

$$\frac{f(2) - f(0)}{2} = \frac{(2^3 - 2 + 1) - (0^3 - 0 + 1)}{2} = \frac{8 - 2 + 1 - 1}{2} = \frac{6}{2} = 3.$$

□

(b) Find the derivative using the definition, and find the average acceleration over the same interval.

Proof. To find the derivative, we evaluate the limit

$$\lim_{h \to 0} \frac{((t + h)^3 - (t + h) + 1) - (t^3 - t + 1)}{h}.$$

The numerator can be simplified as follows

$$t^3 + 3t^2 h + 3th^2 + h^3 - t - h + 1 - t^3 + t - 1,$$

which becomes

$$3t^2 h + 3th^2 - h.$$

Dividing by h, we find we're evaluating the limit

$$\lim_{h \to 0} 3t^2 + 3th - 1,$$

which is just $3t^2 - 1$. The average acceleration is now given by

$$\frac{(3(2)^2 - 1) - (3(0)^2 - 1)}{2} = \frac{12 - 1 + 1}{2} = 6.$$

□

(7) Describe the behavior of the slope of the tangents to $\sin x$ as x varies over $[0, 2\pi]$.

Proof. The slope starts positive, decreases until we get to 0 at $\frac{\pi}{2}$, then becomes increasingly negative as we go to π, then starts slowing down until it becomes 0 again at $\frac{3\pi}{2}$, then increases towards 2π. □

(8) Draw a graph of the position and a separate graph of the velocity of your car as you drive across the country. (Please assume realistic conditions; e.g., you need to sleep occasionally.)

Proof. Omitted. □

(9) Suppose that $\lim_{x \to 1}(f(x))^2 = 3$. What is $\lim_{x \to 1} f(x)$? (Does it have to exist?)

Proof. You can't tell. The limit could be $\sqrt{3}$. It could also be $-\sqrt{3}$. Or the limit might not exist, if the function oscillates between $\sqrt{3}$ and $-\sqrt{3}$. □

(10) Prove formally that $\lim_{x->\infty} 3e^x$ is ∞.

Proof. We want to show that for any $M > 0$ there exists $N > 0$ such that $x > N$ implies that $3e^x > M$. Solving the latter, we have

$$3e^x > M$$

is equivalent to

$$e^x > \frac{M}{3},$$

which is equivalent to

$$x > \ln \frac{M}{3}.$$

Therefore, given M, we choose any $N > \ln \frac{M}{3}$. □

(11) Find the following limits:
(a)

$$\lim_{x \to -\infty} \frac{x^5 + 3}{(x^{15} + 3x^9 - 4x^6 + 2)^{\frac{1}{3}}}.$$

Proof. Looking at the highest order terms, the numerator has x^5 and the denominator has $(x^15)^{\frac{1}{3}} = x^5$. So the limit is going to be either 1 or -1. As x becomes very negative, x^5 is negative. Similarly, x^15 will be negative and so $(x^{15})^{\frac{1}{3}}$ will be negative. Therefore, the limit is 1. □

(b)

$$\lim_{x\to 3}\frac{x^2-x-6}{x-3}.$$

Proof. Factoring, we find that

$$x^2-x-6=(x-3)(x+2).$$

Therefore, the limit is $3+2=5$. □

(12) Use the intermediate value theorem to:

(a) Locate a solution for the equation:

$$\log_2(x+1)+\log_4(x-1)=1.$$

Proof. Solving the given equation is equivalent to finding a zero for the function

$$p(x)=\log_2(x+1)+\log_4(x-1)-1.$$

The function $p(x)$ is continuous provided that $x>1$, so the intermediate value theorem applies. Now we need to evaluate at various points. Let's try $x=2$. Then we have

$$p(2)=\log_2 3+\log_4 1-1=\log_2 3+0-1.$$

Observe that $p(2)>0$ since $\log_2 3$ is between 1 and 2. On the other hand, let's take x such that $x-1=\frac{1}{64}$; i.e., $x=\frac{65}{64}$. In this case,

$$p(\frac{65}{64})=\log_2\frac{129}{64}+\log_4\frac{1}{64}-1=\log_2 129-8-3-1=(\log_2 129-8)-4.$$

Since $\log_2 129$ is very close to 8, p is negative here. Therefore, the intermediate value theorem tells us that the equation has a solution in the interval $[\frac{65}{64},2]$. □

(b) Show that $x^4=-1$ has no solutions.

Proof. Since x^4+1 is positive for any x, the equation has no solutions. (Strictly speaking, this doesn't depend on the intermediate value theorem.) □

(13) (a) Consider the following procedure: take a positive number x, round down to an integer, take the remainder when you divide by 12, add 1, and find the number of days in that month. Does this describe a continuous function?

Proof. No. On the interval $[0,1)$, the function outputs the number of days in the month of January. On the interval $[1,2)$, the function outputs the number of days in the month of February. □

(b) Where is $f(x)=e^{\sin(x)+x^{-10}\sqrt{x}}$ continuous?

Proof. The function e^x is continuous everywhere, as is $\sin(x)$. The function $x^{-10}=\frac{1}{x^{10}}$ is continuous as long as $x\neq 0$. The function $\sqrt{x}$ is only defined for $x\geq 0$. Therefore, f is continuous on $(0,\infty)$. □

(14) Let $f(x)=\frac{(x+3)^3}{(x-4)(x-1)(x-2)}$. Sketch the horizontal and vertical asymptotes of $f(x)$.

Proof. There are vertical asymptotes at $x = 4$, $x = 1$, and $x = 2$. The limits as $x \to \infty$ and $x \to -\infty$ are both 1, so the horizontal asymptote is $y = 1$. □

(15) Are the slopes of the tangents of $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 4x + 1$ ever parallel? (Please use the definition to find the derivatives in order to do this.)

Proof. First, we find the derivatives using the definition. For $f(x)$, this is given by:

$$\lim_{h\to 0} \frac{(x+h)^2 + 2 - x^2 - 2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0}(2x + h) = 2x.$$

For g, we have:

$$\lim_{h\to 0} \frac{(3(x+h)^3 - 4(x+h) + 1) - (3x^3 - 4x + 1)}{h}.$$

Expanding the numerator, we get

$$3(x^3 + 3x^2h + 3xh^2 + h^3) - 4x - 4h + 1 - 3x^3 + 4x - 1,$$

and collecting terms this is

$$9x^2h + 9xh^2 + 3h^3 - 4h.$$

Dividing by h, the limit becomes

$$\lim_{h\to 0} 9x^2 + 9xh + 3h^2 - 4 = 9x^2 - 4.$$

The tangent lines are parallel when the slopes are equal, which amounts to solving

$$2x = 9x^2 - 4,$$

or

$$9x^2 - 2x - 4 = 0.$$

Using the quadratic formula, we find that this has solutions

$$\frac{2 \pm \sqrt{4 - (4)(9)(-4)}}{18} = \frac{2 \pm \sqrt{148}}{18}.$$

□

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